On generalized Powers-Størmer’s Inequality

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Plan of talk

1. Background (from Quantam information theory)
2. Formulation
3. Double piling structure of matrix monotone functions and matrix convex functions
4. Characterizations of the trace property
Background

1. Total error probability:

\[ \rho_1, \rho_2 : \text{hypothetic states on } \mathbb{C}^d \]

: density matrix on \( \mathbb{C}^d \), that is

\[ \rho_i \geq 0, \text{Tr}(\rho_i) = 1 \ (i = 1, 2) \]

\( E = \{E_1, E_2\} : \text{quantum multiple test} \)

: \( d \times d \) projections \( E_1 + E_2 = 1 \)

\[ \text{Succ}_i(E) := \text{Tr}(\rho_i E_i) \ (i = 1, 2) \]

\[ \text{Err}_i(E) := 1 - \text{Succ}_i(E) = \text{Tr}(\rho_i(1 - E_i)) \]

\[ \text{Err}(E) := \frac{1}{2} \text{Tr}(\rho_1 E_2) + \frac{1}{2} \text{Tr}(\rho_2 E_1) \]

\[ = \frac{1}{2} \{1 - \text{Tr}(E_1(\rho_1 - \rho_2))\} \]
2. Assymptotic error exponent for $\rho_1$ and $\rho_2$

$$\forall n \in \mathbb{N} \quad E_n : d^n \times d^n \text{ quantum multiple test}$$

$$\text{Err}_n(E_n) := \frac{1}{2} \{ 1 - \text{Tr}(E_n(\rho_1^\otimes n - \rho_2^\otimes n)) \}$$

If the limit $\lim_{n \to \infty} -\frac{1}{n} \log \text{Err}_n(E_n)$ exists, we refer to it as the asymptotic error exponent.

3. The quantum Chernoff bound for $\rho_1$ and $\rho_2$

$$\xi_{QCB}(\rho_1, \rho_2) := -\log \inf_{0 \leq s \leq 1} \text{Tr}(\rho_1^{1-s} \rho_2^s).$$

Let \( \{\rho_1, \rho_2\} \) be hypothetic states on \( C^d \) and \( E(n) \) be a support projections on \( (\rho_1^{\otimes n} - \rho_2^{\otimes n}) \). Then one has

\[
\xi_{QCB} = \lim_{n \to \infty} -\log \text{Err}_n(E(n))
\]

In the proof of Theorem 1 the following inequality played a key role.

**Theorem 2.** (K. M. R. Audenaert et al. 2011) For any positive matrices \( A \) and \( B \) on \( C^d \) we have

\[
\frac{1}{2}(\text{Tr}A + \text{Tr}B - \text{Tr}|A - B|) \leq \text{Tr}(A^{1-s}B^s) \quad (s \in [0, 1]).
\]

When \( s = \frac{1}{2} \), Powers and Størmer proved the inequality in 1970.
Formulation

If we consider a function \( f(t) = t^{1-s} \) and \( g(t) = t^s = \frac{1}{f(t)} \), then the previous inequality can be reformed by

\[
\frac{1}{2}(\text{Tr}A+\text{Tr}B-\text{Tr}|A-B|) \leq \text{Tr}(f(A)^{\frac{1-s}{2}}g(B)f(A)^{\frac{1-s}{2}})
\]

Problem 3. Let \( n \in \mathbb{N} \). When the inequality holds for any \( n \times n \) positive definite matrices \( A \) and \( B \)?

For \( 0 \leq s \leq 1 \) since the function \( t \mapsto t^s \) is operator monotone on \([0, \infty)\), we may hope that the inequality holds when \( f \) is operator monotone on \([0, \infty)\).
Definition 4.1. A function $f$ is said to be matrix convex of order $n$ or $n$-convex in short (resp. matrix concave of order $n$ or $n$-concave) whenever the inequality

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B), \; \lambda \in [0,1]$$

(resp. $f(\lambda A + (1-\lambda)B) \geq \lambda f(A) + (1-\lambda)f(B), \; \lambda \in [0,1]$) holds for every pair of selfadjoint matrices $A, B \in M_n$ such that all eigenvalues of $A$ and $B$ are contained in $I$.

2. A function $f$ is said to be Matrix monotone functions on $I$ are similarly defined as the inequality

$$A \leq B \implies f(A) \leq f(B)$$

for any pair of selfadjoint matrices $A, B \in M_n$ such that $A \leq B$ and all eigenvalues of $A$ and $B$ are contained in $I$.

We call a function $f$ operator convex (resp. operator concave) if for each $k \in \mathbb{N}$, $f$ is $k$-convex (resp. $k$-concave) and operator monotone if for each $k \in \mathbb{N}$ $f$ is $k$-monotone.
Example 5. Let $f(t) = t^2$ on $(0, \infty)$. It is well-known that $f$ is not 2-monotone. We now show that the function $f$ does not satisfy the inequality (1). Indeed, let us consider the following matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$ 

Then we have

$$AB^{-1}A = \frac{2}{3}A.$$

Set $\tilde{A} = A \oplus \text{diag}(1, \ldots, 1)$, $\tilde{B} = B \oplus \text{diag}(1, \ldots, 1)$ in $M_n$. Then, $\tilde{A} \leq \tilde{B}$ and for any positive linear function $\varphi$ on $M_n$
\[
\varphi(f(\tilde{A})^{\frac{1}{2}}g(\tilde{B})f(\tilde{A})^{\frac{1}{2}}) = \varphi(\tilde{A}\tilde{B}^{-1}\tilde{A}) \\
= \varphi\left(\frac{2}{3}A \oplus \text{diag}(1, \cdots, 1)\right) \\
< \varphi(A \oplus \text{diag}(1, \cdots, 1)) \\
= \varphi(\tilde{A}).
\]

On the contrary, since $\tilde{A} \leq \tilde{B}$, from the inequality (1) we have

\[
\varphi(\tilde{A}) + \varphi(\tilde{B}) - \varphi(\tilde{B} - \tilde{A}) \leq 2\varphi(f(\tilde{A})^{\frac{1}{2}}g(\tilde{B})f(\tilde{A})^{\frac{1}{2}}),
\]

or

\[
\varphi(\tilde{A}) \leq \varphi(f(\tilde{A})^{\frac{1}{2}}g(\tilde{B})f(\tilde{A})^{\frac{1}{2}}),
\]

and we have a contradiction. $\square$

Let $f$ be a $2n$-monotone function on $[0, \infty)$ such that $f((0, \infty)) \subset (0, \infty)$. Then for any pair of positive matrices $A, B \in M_n(\mathbb{C})$

$$\text{Tr}(A) + \text{Tr}(B) - \text{Tr}(|A - B|) \leq 2\text{Tr}(f(A)^{\frac{1}{2}}(A)g(B)f(A)^{\frac{1}{2}})$$

The point of the proof is the $n$-monotonicity of $g$. 
Double piling structure of matrix monotone functions and matrix convex functions

1. \( P_n(I) \): the spaces of \( n \)-monotone functions

2. \( P_\infty(I) \): the space of operator monotone functions

3. \( K_n(I) \): the space of \( n \)-convex functions

4. \( K_\infty(I) \): the space of operator convex functions

We have

\[
P_1(I) \supseteq \cdots \supseteq P_{n-1}(I) \supseteq P_n(I) \supseteq P_{n+1}(I) \supseteq \cdots \supseteq P_\infty(I)
\]
\[
K_1(I) \supseteq \cdots \supseteq K_{n-1}(I) \supseteq K_n(I) \supseteq K_{n+1}(I) \supseteq \cdots \supseteq K_\infty(I)
\]

\[
P_{n+1}(I) \not\subseteq P_n(I) \quad K_{n+1}(I) \not\subseteq K_n(I)
\]
\[
P_\infty = \cap_{n=1}^{\infty} P_n(I) \quad K_\infty = \cap_{n=1}^{\infty} K_n(I)
\]
Theorem 7. Let consider the following three assertions.

(i) $f(0) \leq 0$ and $f$ is $n$-convex in $[0, \alpha)$,

(ii) For each matrix $a$ with its spectrum in $[0, \alpha)$ and a contraction $c$ in the matrix algebra $M_n$,

$$f(c^*ac) \leq c^*f(a)c,$$

(iii) The function $\frac{f(t)}{t} (= g(t))$ is $n$-monotone in $(0, \alpha)$.

1. (Hansen-Pedersen:1985) Three assertions are equivalent if $f$ is operator convex. In this case a function $g$ is operator monotone.

2. (O-Tomiyama:2009)

$$(i)_{n+1} \prec (ii)_n \sim (iii)_n \prec (i)_{\lceil n/2 \rceil},$$

where denotation $(A)_m \prec (B)_n$ means that “if $(A)$ holds for the matrix algebra $M_m$, then $(B)$ holds for the matrix algebra $M_n$".
Using an idea in [Hansen-Pedersen:1985] we can show the following result.

**Proposition 8.** (D. T. Hoa-O-H. M. Toan 2012) Under the same condition in Theorem 7 consider the following assertions.

(iv) $f$ is $2n$-monotone.

(v) The function $\frac{t}{f(t)}$ is $n$-monotone in $(0, \alpha)$.

We have, then, $(iv)_{2n} \prec (v)_n$. 
Theorem 6: Let $f$ be a $2n$-monotone function on $[0, \infty)$ such that $f((0, \infty)) \subset (0, \infty)$. Then for any pair of positive matrices $A, B \in M_n(\mathbb{C})$

$$\text{Tr}(A)+\text{Tr}(B)-\text{Tr}(|A-B|) \leq 2\text{Tr}(f(A)^\frac{1}{2}(A)g(B)f(A)^\frac{1}{2})$$

Sketch of the proof:

$A, B$ : positive matrices
$A - B = (A - B)_+ - (A - B)_- = P - Q,$
$|A - B| = P + Q.$

We may show that

$$\text{Tr}(A) - \text{Tr}(f(A)^\frac{1}{2}(A)g(B)f(A)^\frac{1}{2}) \leq \text{Tr}(P)$$

holds.

$$\text{Tr}(A) - \text{Tr}(f(A)^\frac{1}{2}(A)g(B)f(A)^\frac{1}{2})$$

$$= \text{Tr}(f(A)^\frac{1}{2}g(A)f(A)^\frac{1}{2}) - \text{Tr}(f(A)^\frac{1}{2}(A)g(B)f(A)^\frac{1}{2})$$

$$\leq \text{Tr}(f(A)^\frac{1}{2}g(B + P)f(A)^\frac{1}{2}) - \text{Tr}(f(A)^\frac{1}{2}(A)g(B)f(A)^\frac{1}{2})$$

$$\leq \text{Tr}(f(B + P)^\frac{1}{2}(g(B + P) - g(B))f(B + P)^\frac{1}{2})$$

$$\leq \text{Tr}(f(B + P)^\frac{1}{2}g(B + P)f(B + P)^\frac{1}{2}) - \text{Tr}(f(B)^\frac{1}{2}g(B)f(B)^\frac{1}{2})$$

$$= \text{Tr}(P)$$
Since any C*-algebra can be realized as a closed selfadjoint *-algebra of $B(H)$ for some Hilbert space $H$. We can generalize Theorem 6 in the framework of C*-algebras.

**Theorem 9.** (D. T. Hoa-O-H. M. Toan 2012)

Let $\tau$ be a tracial functional on a $C^*$-algebra $\mathcal{A}$, $f$ be a strictly positive, operator monotone function on $[0, \infty)$. Then for any pair of positive elements $A, B \in \mathcal{A}$

$$\tau(A) + \tau(B) - \tau(|A - B|) \leq 2\tau(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}),$$

where $g(t) = tf(t)^{-1}$. 


Characterizations of the trace property

The generalized Powers-Størmer inequality implies the trace property for a positive linear functional on operator algebras.


Let $\varphi$ be a positive linear functional on $M_n$ and $f$ be a continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f((0, \infty)) \subset (0, \infty)$. If the following inequality

\begin{equation}
\varphi(A+B) - \varphi(|A-B|) \leq 2\varphi(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}})
\end{equation}

holds true for all $A, B \in M_n^+$, then $\varphi$ should be a positive scalar multiple of the canonical trace $\text{Tr}$ on $M_n$, where $g(t) = \begin{cases} t \frac{f(t)}{f(0)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$. 
Let $\varphi$ be a positive linear functional on $M_n$ and $s \in [0,1]$. From Lemma 10 it is clear that if the following inequality

\begin{equation}
\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(A^{\frac{1-s}{2}}B^sA^{\frac{1-s}{2}})
\end{equation}

holds true for any $A, B \in M_n^+$, then $\varphi$ is a tracial. In particular, when $s = 0$ the following inequality characterizes the trace property

\begin{equation}
\varphi(B) - \varphi(A) \leq \varphi(|A - B|) \ (A, B \in M_n^+).
\end{equation}

From this observation we have

**Corollary 11.** (Stolyarov-Tikhonov-Sherstnev:2005) Let $\varphi$ be a positive linear functional on $M_n$ and the following inequality

\begin{equation}
\varphi(|A + B|) \leq \varphi(|A|) + \varphi(|B|)
\end{equation}

holds true for any self-adjoint matrices $A, B \in M_n$. Then $\varphi$ is a tracial.
Corollary 12. (Gardner:1979) Let $\varphi$ be a positive linear functional on $M_n$ and the following inequality

\[(6) \quad |\varphi(A)| \leq \varphi(|A|)\]

holds true for any self-adjoint matrix $A \in M_n$. Then $\varphi$ is a tracial.

Let $\varphi$ be a positive normal linear functional on a von Neumann algebra $\mathcal{M}$ and $f$ be a continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f((0, \infty)) \subset (0, \infty)$. If the following inequality

\[\varphi(A) + \varphi(B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}})\]

holds true for any pair $A, B \in \mathcal{M}^+$, then $\varphi$ is a trace, where $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$. 


Let $\mathcal{A}$ be a von Neumann algebra and $\varphi$ be a positive linear functional on $\mathcal{A}$. In the case of the inequality (7) the set $P(\mathcal{A})$ is not enough as a testing set.

Indeed, let $p, q$ be arbitrary orthogonal projections from a von Neumann algebra $\mathcal{M}$. Since $q \geq p \wedge q$ it follows that $pqp \geq p(p \wedge q)p = p \wedge q$. So $pqp \geq p \wedge q$ holds for any pair of projections. From that it follows

$$
\varphi(p + q - |p - q|) = 2\varphi(p \wedge q) \leq 2\varphi(pqp) = 2\varphi(f(p)^{\frac{1}{2}}g(q)f(p)^{\frac{1}{2}})
$$
Corollary 14. Let $\varphi$ be a positive linear functional on a $C^*$-algebra $A$ and $f$ be a continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f((0, \infty)) \subset (0, \infty)$. If the following inequality

$$
\varphi(A) + \varphi(B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{1/2}g(B)f(A)^{1/2})
$$

holds true for any pair $A, B \in A^+$, then $\varphi$ is a tracial functional, where $g(t) = \begin{cases} 
\frac{t}{f(t)} & (t \in (0, \infty)) \\
0 & (t = 0)
\end{cases}$.

Take the universal representation $\pi$ of $A$ and consider enveloping von Neumann algebra $\mathcal{M} = \pi(A)''. \text{ Apply the previous Theorem to the normal positive functional } \hat{\varphi} \text{ on } \mathcal{M} \text{ such that } \hat{\varphi}(\pi(A)) = \varphi(A) \text{ for } A \in A.$
References


