The Spread of the Unicyclic Graphs

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Abstract

Let $G$ be a simple connected graph with $n$ vertices and $n$ edges which we call an unicyclic graph. In this paper, we first investigate the least eigenvalue $\lambda_n(G)$, then we present two sharp bounds of the spread $s(G)$ of $G$.

AMS classification: 05C50, 05C35
Key words: Unicyclic graph; Spread; Largest eigenvalue; Least eigenvalue.

1. Introduction

In this paper, all graphs are finite undirected connected graphs without loops and multiple edges. Let

$$\mathcal{A}_{n,n} = \{G | G \text{ is a connected graph with } n \text{ vertices and } n \text{ edges, } n \geq 3\}.$$ 

Obviously, there is an unique cycle in $G \in \mathcal{A}_{n,n}$. So, we call $G$ is an unicyclic graph and, $\mathcal{A}_{n,n}$ is the set of unicyclic graphs.

By eigenvalues of a graph $G$ we mean eigenvalues of its adjacency matrix $A$. The spectral spread (briefly the spread) $s(G)$ of $G$ is the spread $s(A)$ of its adjacency matrix $A$, i.e. $s(A) = s(G) = \lambda_1(G) - \lambda_n(G)$, where $\lambda_1(G)$ and $\lambda_n(G)$ are the largest and least eigenvalues of $G$, respectively. For definition and properties of the spread of matrix, one can consult [7].

As $s(G)$ is the diameter of the spectrum of $G$, it is very necessary to investigate the spread $s(G)$. For any connected graph $T$ with $n$ vertices and $n - 1$ edges (i.e. $T$ is a tree), since $\lambda_1(P_n) \leq \lambda_1(T) \leq \lambda_1(K_{1,n-1})$ [9] and $P_n, T, K_{1,n-1}$ are bipartite graphs, we can get that $s(P_n) \leq s(T) \leq s(K_{1,n-1})$. In this paper, we study the spread of a connected graph $G$ with $n$ vertices and $n$ edges (i.e. $G$ is an unicyclic

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graph). There is a considerable literature on the spread of the graphs with all kinds of propositions ([2] [3] [6] [8] [13]). Our results are due to the conjecture in [3].

Conjecture [3] Of all the graphs with \( n \) vertices and \( e \) edges, suppose \( G \) is one with maximum spread. If \( e \leq \left\lfloor \frac{n^2}{4} \right\rfloor \), then \( G \) must be bipartite.

We consider the spread \( s(G) \) of the unicyclic graphs by proving the lower bound of the least eigenvalue. We get that

\[
s(C_n) \leq s(G) \leq s(S^3_n),
\]

when \( G \in A_{n,n} - \{ S^5_6 \} \) and \( n \geq 6 \). Moreover, the left equality holds if and only if \( G \cong C_n \); the right equality holds if and only if \( G \cong S^3_n \), where \( S^k_n \) denote the graph which is obtained by joining one vertex of the cycle \( C_k \) to \((n-k)\) isolated vertices.

Obviously, the number of the edges of \( S^3_n \) is \( m = n \leq \left\lfloor \frac{n^2}{4} \right\rfloor \), but \( S^3_n \) is not a bipartite graph. This disproves the conjecture when \( G \) is restricted to be connected.

The terminology not defined here can be found in [1].

2. The Least Eigenvalues of \( G \)

In this section, we will investigate the least eigenvalue \( \lambda_n(G) \) of \( G \). We first present some basic lemmas in the following.

**Lemma 2.1** [9] Let \( G \) be a tree with \( n \) vertices. Then

\[
2\cos \frac{\pi}{n+1} \leq \lambda_1(G) \leq \sqrt{n-1}.
\]

The left-hand equality holds if and only if \( G \cong P_n \), the right-hand equality holds if and only if \( G \cong K_{1,n-1} \).

**Lemma 2.2** [5] Let \( G \) be a simple connected graph with \( n \) vertices. Then there exists a bipartite spanning subgraph \( G' \) of \( G \), satisfying \( \lambda_n(G) \geq \lambda_n(G') \).

**Lemma 2.3** [12] Let \( G \) be a simple graph, \( v \in V(G) \), \( C(v) \) be the set of all circuits including \( v \). Then

\[
P(G) = \lambda P(G - v) - \sum_{u \sim v} P(G - v - u) - 2 \sum_{z \in C(v)} P(G - V(z)).
\]

The following lemma has been given in [14]. Now, we give a new proof as below.

**Lemma 2.4** [14] Let \( \lambda_n(G) \) be the least eigenvalues of \( G \), \( G \in A_{n,n} \) and \( n \geq 12 \). Then

\[
\lambda_n(S^3_n) < \lambda_n(S^4_n).
\]
Fig.1.

**Proof.** By Lemma 2.3, we can get that
\[
P(S_3^n, \lambda) = P(P_2, \lambda)\lambda^{n-2} - (n - 3)P(P_2, \lambda)\lambda^{n-4} - 2\lambda^{n-2} - 2\lambda^{n-3} \\
= (\lambda^2 - 1)\lambda^{n-2} - (n - 3)(\lambda^2 - 1)\lambda^{n-4} - 2\lambda^{n-2} - 2\lambda^{n-3} \\
= \lambda^{n-4}(\lambda + 1)(\lambda^3 - \lambda^2 + (1 - n)\lambda + n - 3),
\]
\[
P(S_4^4, \lambda) = P(P_3, \lambda)\lambda^{n-3} - (n - 4)P(P_3, \lambda)\lambda^{n-5} - 2P(P_2, \lambda)\lambda^{n-4} - 2\lambda^{n-4} \\
= \lambda^{n-4}(\lambda^4 - n\lambda^2 + 2n - 8).
\]

By solving the equality \( \lambda^4 - n\lambda^2 + 2n - 8 = 0 \), we can get
\[
\lambda_{n}(S_3^4) = -\sqrt{n + \sqrt{n^2 - 8n + 32}}.
\]

Let \( f(\lambda) = \lambda^3 - \lambda^2 + (1 - n)\lambda + n - 3 \), we have
\[
f(-\sqrt{n - 1}) = -2 < 0, \quad f\left(-\sqrt{n - \frac{3}{2}}\right) = -\frac{3}{2} + 1\sqrt{n - \frac{3}{2}} > 0; \\
f(0) = n - 3 > 0, \quad f(1) = -2 < 0; \\
f(\sqrt{n - 1}) = -2 < 0, \quad f(\sqrt{n}) = \sqrt{n - 3} > 0,
\]
for \( n \geq 11 \). Thus, the cubic polynomial \( f(\lambda) \) has three real roots lying in the intervals \((-\sqrt{n - 1}, -\sqrt{n - \frac{3}{2}}), (0, 1) \) and \((\sqrt{n - 1}, \sqrt{n})\). That implies
\[
-\sqrt{n - 1} < \lambda_{n}(S_3^4) < -\sqrt{n - \frac{3}{2}}.
\]

It is evident that
\[
-\sqrt{n - \frac{3}{2}} < -\sqrt{n + \sqrt{n^2 - 8n + 32}}.
\]
when \( n \geq 12 \). Thus
\[
\lambda_{n}(S_3^4) < -\sqrt{n - \frac{3}{2}} < -\sqrt{n + \sqrt{n^2 - 8n + 32}} = \lambda_{n}(S_4^4),
\]
when \( n \geq 12 \).
That completes the proof. \( \square \)

By Lemma 2.4, we obtain two corollaries.

**Corollary 2.5** Let \( \lambda_n(G) \) be the least eigenvalues of \( G \in \mathcal{A}_{n,n} \). Then

1. \( -\sqrt{n-1} < \lambda_n(S_n^3) < -\sqrt{n-\frac{3}{2}} \) when \( n \geq 11 \);
2. \( -\sqrt{n-1} < \lambda_n(S_n^3) \leq -\sqrt{n-2} \) when \( 3 \leq n \leq 10 \).

**Proof.** By the proof of Lemma 2.4 and \( f(-\sqrt{n-2}) = -1 + \sqrt{n-2} \geq 0 \) when \( n \geq 3 \), we can easy to get the result. \( \square \)

**Corollary 2.6** Let \( \lambda_1(G) \) be the largest eigenvalues of \( G \in \mathcal{A}_{n,n} \). Then

1. \( \sqrt{n-1} < \lambda_1(S_n^3) \leq \sqrt{n} \) when \( n \geq 9 \);
2. \( \sqrt{n-1} < \lambda_1(S_n^3) \leq \sqrt{n+1} \) when \( 3 \leq n \leq 8 \).

**Proof.** From the proof of Lemma 2.4 and, \( f(\sqrt{n+1}) = 2\sqrt{n+1} - 4 \geq 0 \) when \( n \leq 3 \), we obtain this corollary. \( \square \)

**Lemma 2.7** [11] Let \( G \) be a tree with \( n \) vertices, \( n \geq 4 \) and \( G \not\sim K_{1,n-1} \). Then

\[
\lambda_1(G) \leq \sqrt{\frac{1}{2}(n-1 + \sqrt{n^2 - 6n + 13})}.
\]

The equality holds if and only if \( G \cong T_1 \) (in Fig.2).

**Lemma 2.8** [11] Let \( G \) be a tree with \( n \) vertices and \( G \not\sim K_{1,n-1}, G \not\sim T_1 \) (in Fig.2.). Then

\[
\lambda_1(G) \leq \sqrt{\frac{1}{2}(n-1 + \sqrt{n^2 - 10n + 33})}.
\]

The equality holds if and only if \( G \cong T_2 \) (in Fig.2).

![Fig.2.](image)

We denote the girth of \( G \) by \( g(G) \) or \( g \).

**Lemma 2.9** [4] Let \( G \) be an unicyclic graph with girth \( g(G) = k \). Then

\[
\lambda_1(G) \leq \lambda_1(S_n^k), \text{ and } \lambda_1(S_n^{k+1}) < \lambda_1(S_n^k),
\]

where \( 3 \leq k \leq n-1 \). Moreover, \( \lambda_1(G) = \lambda_1(S_n^k) \) if and only if \( G \cong S_n^k \).
We begin to consider the bound on the least eigenvalue $\lambda_n(G)$ of an unicyclic graph $G$ and give a new proof of the following result.

**Theorem 2.10** [14] For an arbitrary graph $G \in \mathcal{A}_{n,n}$ and $G \not\cong S_n^3$. Then

$$\lambda_n(S_n^4) \leq \lambda_n(G),$$

the equality holds if and only if $G \cong S_n^4$.

**Proof.** If $g = 3$ and $G \not\cong S_n^3$. By Lemma 2.2, we get that there exists a bipartite spanning subgraph $G'$ of $G$, satisfying $\lambda_n(G) \geq \lambda_n(G')$. So, $G'$ is a tree or a forest with $n$ vertices. Since $G \not\cong S_n^3$, then $G' \not\cong K_{1,n-1}$. By Lemma 2.7, it is easy to get that if $n \geq 6$,

$$\lambda_n(G) \geq \lambda_n(G') = -\lambda_1(G') \geq -\frac{1}{2}(n-1 + \sqrt{n^2 - 6n + 13}) \geq -\frac{1}{2}(n + \sqrt{n^2 - 8n + 32}) = \lambda_n(S_n^4).$$

If $n = 4$, there is not unicyclic graph satisfying $G \not\cong S_n^3$ and $g = 3$. If $n = 5$, there are two unicyclic graphs with $G \not\cong S_n^3$ and $g = 3$. It is easy to get that $\lambda_n$ of them greater then $\lambda_n(S_n^4)$.

If $g \geq 4$, we distinguish the following two cases.

**Case 1.** $g \equiv 0 \pmod{2}$.

By Lemma 2.9,

$$\lambda_1(G) \leq \lambda_1(S_n^g) \leq \lambda_1(S_n^{g-1}) \leq \cdots \leq \lambda_1(S_n^4).$$

Since the girth $g$ is an even number, $G$ and $S_n^4$ are bipartite graphs. Then $-\lambda_n(G) = \lambda_1(G)$ and $-\lambda_n(S_n^4) = \lambda_1(S_n^4)$. So $\lambda_n(G) \geq \lambda_n(S_n^4)$.

**Case 2.** $g \equiv 1 \pmod{2}$.

Since $|\lambda_n(G)| < \lambda_1(G)$ and $\lambda_n(G) < 0$, then $\lambda_n(G) > -\lambda_1(G)$. By Lemma 2.9, we have that $\lambda_n(G) > -\lambda_1(G) \geq -\lambda_1(S_n^g) \geq \cdots \geq -\lambda_1(S_n^4) = \lambda_n(S_n^4)$.

Obviously, from Lemma 2.7 and Lemma 2.9, the equality holds if and only if $G \cong S_n^4$.

This completes the proof. \[\square\]

By Theorem 2.10 and Lemma 2.4, it is easy to get the following corollary.

**Corollary 2.11** Let $G$ be an unicyclic graph with $n \geq 12$ vertices. Then

$$\lambda_n(S_n^3) \leq \lambda_n(G).$$
In the following theorem, we concentrate to determine $\lambda_n(G)$ of the unicyclic graph $G$ with $g = 3$ and $n \leq 11$.

**Theorem 2.12** Let $G$ be an unicyclic graph with $g = 3$, $n$ vertices and $6 \leq n \leq 11$. Then

$$\lambda_n(S_3^n) \leq \lambda_n(G).$$

**Proof.** By Lemma 2.2, there exists a bipartite spanning subgraph $G'$ satisfying: $\lambda_n(G) \geq \lambda_n(G')$. Since $g = 3$, $G'$ is a tree or a forest with $n$ vertices.

If $G'$ is a forest, there exists a component $T'$ (i.e. a tree) with $n' \leq n - 1$ vertices and $\lambda_{n'}(T') = \lambda_n(G')$. By Lemma 2.1 and Corollary 2.5, we have

$$\lambda_n(G) \geq \lambda_n(G') = \lambda_{n'}(T') = -\sqrt{n' - 1} \geq -\sqrt{n - 2} \geq \lambda_n(S_3^n).$$

If $G'$ is not a forest but a tree, we suppose $G \not\sim S_3^n$, $G \not\sim H$ and $G \not\sim H'$ (in Fig. 2). Then $G' \not\sim K_{1,n-1}$ and $G' \not\sim T_1$ (in Fig. 2). By Lemma 2.8 and Corollary 2.5, we can get that, for $n \geq 6$,

$$\lambda_n(G) \geq \lambda_n(G') = -\sqrt{\frac{1}{2}(n - 1 + \sqrt{n^2 - 10n + 33})} \geq -\sqrt{n - 2} \geq \lambda_n(S_3^n).$$

If $G \cong S_3^n$ or $G \cong H$ or $G \cong H'$ (in Fig. 2), we present the least eigenvalues of $G$ with vertices $6 \leq n \leq 11$ in the following table.

<table>
<thead>
<tr>
<th>n</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_n(S_3^n)$</td>
<td>-2.0861</td>
<td>-2.3234</td>
<td>-2.5366</td>
<td>-2.7321</td>
<td>-2.9136</td>
<td>-3.0839</td>
</tr>
<tr>
<td>$\lambda_n(H)$</td>
<td>-1.9202</td>
<td>-2.1542</td>
<td>-2.3713</td>
<td>-2.5726</td>
<td>-2.7603</td>
<td>-2.9365</td>
</tr>
<tr>
<td>$\lambda_n(H')$</td>
<td>-1.8723</td>
<td>-2.1220</td>
<td>-2.3489</td>
<td>-2.5562</td>
<td>-2.7478</td>
<td>-2.9267</td>
</tr>
</tbody>
</table>

From this table, we get that $\lambda_n(H) > \lambda_n(S_3^n)$ and $\lambda_n(H') > \lambda_n(S_3^n)$.

This completes the proof. □

**3. The Bounds of the Spread $s(G)$**

In this section, we discuss the upper bound and the lower bound of the spread $s(G)$ of the unicyclic graph. We first state some lemmas about $\lambda_1(G)$ and $s(G)$.
Lemma 3.1 [4] Let $G$ be a simple graph with $n$ vertices and, $\lambda_1(G)$ be the largest eigenvalue of an unicyclic graph. Then

$$\lambda_1(C_n) \leq \lambda_1(G) \leq \lambda_1(S_n^3).$$

The left-hand equality holds if $G \cong C_n$, the right-hand equality holds if $G \cong S_n^3$.

Lemma 3.2 [3] If $H$ is an induced subgraph of $G$. Then $s(G) \geq s(H)$ with strict inequality if $G$ is connected and $H$ is a proper induced subgraph of $G$.

Lemma 3.3 [3] If $H$ is a bipartite subgraph of a graph $G$, then $s(G) \geq s(H)$.

Lemma 3.4 [10] There are exactly 30 minimal connected graphs with respect to the property of having the spread greater than 4 and they are displayed in Fig.3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Fig.3.}
\end{figure}

Now, we give a sharp upper bounds on $s(G)$ of the unicyclic graphs.

Theorem 3.5 Let $G$ be an unicyclic graph with $n$ vertices and the grith $g$.

1. If $n \geq 6$, then $s(G) \leq s(S_n^3)$, the equality holds if and only if $G \cong S_n^3$.
2. If $n = 4$ or 5, then $s(G) \leq s(S_n^4)$, the equality holds if and only if $G \cong S_n^4$.

Proof. (1) If $n \geq 12$, by Corollary 2.11 and Lemma 3.1, we obtain

$$s(G) = \lambda_1(G) - \lambda_n(G) \leq \lambda_1(S_n^3) - \lambda_n(S_n^3) = s(S_n^3).$$

If $6 \leq n \leq 11$ and $g = 3$, by Theorem 2.12 and Lemma 3.1, we have

$$s(G) \leq \lambda_1(S_n^3) - \lambda_n(S_n^3) = s(S_n^3).$$

If $6 \leq n \leq 11$ and $g \geq 4$, then $G \not\cong S_n^3$. By Lemma 2.9 and Theorem 2.10, we get

$$s(G) \leq \lambda_1(S_n^4) - \lambda_n(S_n^4) = s(S_n^4).$$

7
From the following table, it is clear that \( s(S_n^4) < s(S_n^3) \). So \( s(G) < s(S_n^3) \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( s(S_n^4) )</th>
<th>( s(S_n^3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4.6002</td>
<td>4.5765</td>
</tr>
<tr>
<td>7</td>
<td>5.0047</td>
<td>4.8990</td>
</tr>
<tr>
<td>8</td>
<td>5.8670</td>
<td>5.2263</td>
</tr>
<tr>
<td>9</td>
<td>6.0647</td>
<td>5.5503</td>
</tr>
<tr>
<td>10</td>
<td>6.1745</td>
<td>5.5503</td>
</tr>
<tr>
<td>11</td>
<td>6.3810</td>
<td>5.8670</td>
</tr>
</tbody>
</table>

It is evident that the equality holds if and only if \( G \cong S_n^3 \).

(2) We present all the unicyclic graphs which has \( n = 5 \) vertices and their spread \( s(G) \) in Fig.4. So \( s(G) \leq s(S_5^4) \).

\[
\text{Fig.4.}
\]

Further, it is easy to get \( s(S_4^1) = s(C_4) = 4, s(S_4^2) = 3.6513 \). Thus \( s(S_4^1) > s(S_4^2) \).

This completes the proof. \( \square \)

Finally, we show a sharp lower bound on \( s(G) \) of an unicyclic graph as below.

**Theorem 3.6.** Let \( G \) be an unicyclic graph with \( n \) vertices. If \( n \geq 5 \), \( G \not\cong S_6^5 \), then \( s(G) \geq s(C_n) \). The equality holds if and only if \( G \cong C_n \).

**Proof.** As \( G \) is an unicyclic graph, without loss the generality, we suppose that the girth \( g \) of \( G \) is less than \( n \). So \( G \not\cong C_n \). Otherwise, \( G \cong C_n \) and \( s(G) = s(C_n) \). Clearly, the result is true.

If \( n=5 \), then \( s(G) \geq s(C_5) \approx 3.6180 \) by determining the spread of all unicyclic graph with 5 vertices (in Fig. 4). The equality holds iff \( G \cong C_5 \).

**Case 1.** \( g \equiv 0 \pmod{2} \). Obviously, \( C_g \) is a bipartite subgraph of \( G \). By Lemma 3.3, \( s(G) > s(C_g) = 4 \geq s(C_n) \).

**Case 2.** \( g \equiv 1 \pmod{2} \). We consider three subcases: (a) \( g \geq 7 \); (b) \( g = 5 \); (c) \( g = 3 \).

- **Subcase a.** As \( g \geq 7 \), the unicyclic graph \( G \) has the induced subgraph \( G_{20} \) or \( G_{29} \) which are showed in Lemma 3.4. By Lemma 3.2 and Lemma 3.4, we obtained \( s(G) \geq \min\{s(G_{20}), s(G_{29})\} > 4 \geq s(C_n) \).

- **Subcase b.** If \( g = 5 \) and \( n \geq 7 \), the unicyclic graph \( G \) has the induced subgraphs \( G_{15}, G_{16}, G_{18} \) and \( G_{23} \) which are showed in Lemma 3.4. By Lemma 3.4 and Lemma 3.2, it is easy to get \( s(G) \geq \min\{s(G_{15}), s(G_{16}), s(G_{18}), s(G_{23})\} > 4 \geq s(C_n) \).

- If \( g = 5 \) and \( n = 6 \), then there exists no graph nonisomophic \( S_6^5 \).

- **Subcase c.** As \( g = 3 \) and \( n \geq 6 \), the unicyclic graph \( G \) has the induced subgraphs \( G_5, G_7, G_8, G_{10}, \text{ or } G_{12} \) in Lemma 3.4. By Lemma 3.4 and Lemma 3.2, we can get that

\[
\min\{s(G_5), s(G_7), s(G_8), s(G_{10}), s(G_{12})\} > 4 \geq s(C_n).
\]
It is evident that the equality holds if and only if $G \cong C_n$.

Now, this completes the proof.  

Remark :

1. If $n=6$. Since $\lambda_1(S_6^5) \approx 2.114$, $\lambda_n(S_6^5) \approx -1.860$, $s(S_6^5) \approx 3.974 < 4 = s(C_6)$. Hence, $s(G) \geq s(S_6^5)$ and the equality holds iff $G \cong S_6^5$.

2. If $n=4$, then $s(G) \geq s(S_4^3) \approx 3.6513$. The equality holds iff $G \cong S_4^3$.

3. If $n=3$, then $s(G)=s(C_3)=3$. The equality holds iff $G \cong C_3$.

Acknowledgements

This paper was supported by National Natural Science Foundation of China (NO. 19671029 & 10271048) and Shanghai Priority Academic Discipline.

References


