Abstract. In this paper, a criterion on the semi-simplicity of the cyclotomic Brauer algebras is given. This result can be considered as a generalization of Wenzl's theorem on Brauer algebras [27], which was known as Hanlon-Wales conjecture before.

1. Introduction

Throughout the paper, we fix non-negative integers \( m \geq 1 \) and \( n \geq 1 \).

Let \( R \) be a commutative ring containing 1 and \( \delta_0, \delta_1, \cdots, \delta_{m-1} \), arbitrary elements of \( R \). In [17], Häring-Oldenburg introduced a class of associative algebras \( B_{m,n}(\delta_i) \) over \( R \), called the cyclotomic Brauer algebras. When \( m = 1 \), \( B_{m,n}(\delta_1) \) is a Brauer algebra \( B_n(\delta) \), which was introduced by Richard Brauer in [4].

A Brauer algebra \( B_n(\delta) \) over the complex field \( \mathbb{C} \) can be used to study knot invariants. A key is the existence of a non-degenerate Markov trace function defined on it. It is known that the Markov trace is non-degenerate if \( B_n(\delta) \) is semi-simple. It is natural to ask when \( B_n(\delta) \) is semi-simple. Having studied \( B_n(\delta) \) in the cases \( B_n(\delta) \) is not semi-simple in [13, 14, 15, 16], Hanlon and Wales proposed the following conjecture.

**Hanlon-Wales conjecture**: A Brauer algebra \( B_n(\delta) \) over \( \mathbb{C} \) is semi-simple if the parameter \( \delta \notin \mathbb{Z} \).

This conjecture was proved by Hans Wenzl in [27]. More explicitly, he proved that \( B_n(\delta) \) is not semi-simple for only finitely many integer parameter values. Motivated by Martin’s work on partition algebras[21], Doran, Wales and Hanlon studied \( B_n(\delta) \) in terms of Frobenius reciprocity and two functors \( F \) and \( G \), which are some kind of induction and restriction. This led them to give another proof of the conjecture in [7].

**Problem**: Is there a similar criterion on the semi-simplicity of \( B_{m,n}(\delta_1) \) over \( \mathbb{C} \)?

The main purpose of this paper is to answer this question. In fact, the following theorem will be proved.

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Theorem 1.1. Let $B_{m,n}(\delta_1)$ be a cyclotomic Brauer algebra over $\mathbb{C}$. Suppose $\xi$ is a primitive $m$-th root of unity. For any $0 \leq i \leq m - 1$, let

$$g_i(\delta_0, \delta_1, \cdots, \delta_{m-1}) = \sum_{s=0}^{m-1} \delta_s \xi^{si}.$$ 

Then $B_{m,n}(\delta_1)$ is semi-simple if $g_i(\delta_0, \delta_1, \cdots, \delta_{m-1}) \notin \mathbb{Z}$, for all $0 \leq i \leq m - 1$.

We will prove a more precise version of Theorem 1.1 in Theorems 8.6-8.7, implying that $B_{m,n}(\delta_1)$ is not semi-simple for only finitely many integer values of $g_i(\delta_0, \delta_1, \cdots, \delta_{m-1})$, $0 \leq i \leq m - 1$. However, there are infinitely many points $(\delta_0, \delta_1, \cdots, \delta_{m-1})$ in the hyperplane given by $\sum_{s=0}^{m-1} \xi^{si} x_s = c$ for any constant $c \in \mathbb{C}$ under the assumption $m > 1$. If $m = 1$, then $\xi = 1$ and $g_0(\delta_0) = \delta_0$. Therefore, in this case, Theorem 1.1 is Wenzl’s theorem on $B_n(\delta)$.

This paper is organized as follows. In section 2, the cyclotomic Brauer algebra $B_{m,n}(\delta_1)$ over a commutative ring [17] is introduced. In section 3, some basic results on complex representations of complex reflection groups $W_{m,n}$ are given. We prove the Littlewood-Richardson rule for $W_{m,n}$ in section 4. We also decompose certain complex induced modules, which will be used when we study the restriction of certain $B_{m,n}(\delta_1)$-modules in Theorem 6.2. Let $R$ be a commutative ring in which $x^m - 1 = \prod_{i=1}^{m} (x - u_i)$, $u_i \in R$, $1 \leq i \leq m$. In section 5, a cellular basis of $B_{m,n}(\delta_1)$ over $R$ is presented. Due to [12, 3.4], we can classify the simple $B_{m,n}(\delta_1)$-modules over a field. Using [12, 3.10] and [20, 3.1], we give a criterion on the quasi-heredity of $B_{m,n}(\delta_1)$. Section 6 is about the branching rule of cell modules for $B_{m,n}(\delta_1)$ over $\mathbb{C}$. In section 7, we study two functors $F$ and $G$, which will play a key role in the proof of Theorem 1.1 in section 8.

Remark 1.2. (a) Most of the results in §§6-8 will be proved by the method given in [7]. Since we consider labelled Brauer diagrams, we have to use the results on complex reflection groups of type $G(m,1,n)$ to replace those on symmetric groups.

(b) Although we focus on $B_{m,n}(\delta_1)$ over $\mathbb{C}$, it is possible to state a version of Theorem 1.1 over any field. See the remark at the end of section 8.

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2. Cyclotomic Brauer algebras.

In this section, the cyclotomic Brauer algebra $B_{m,n}(\delta_1)$ over a commutative ring $R$ [17] is defined. First, we give the definition of a labelled Brauer diagram.
Definition 2.1. A labelled Brauer diagram $D$ of type $G(m, 1, n)$ is a Brauer diagram with $2n$ vertices, in which each arc is labelled by $i$, $i \in \mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$.

Graphically, a labelled Brauer diagram $D$ of type $G(m, 1, n)$ may be presented in a rectangle of a plane, where there are $n$ numbers $\{1, 2, \ldots, n\}$ on the top row from left to right, and there are $n$ numbers $\{1, 2, \ldots, n\}$ on the bottom row from left to right. To indicate the label $i \in \mathbb{Z}_m$ on an arc, the arc is marked with a dot and a number $i$ is written in bracket above or below the dot. Sometimes, $i$ dots are drawn directly to indicate it has label $i$. For example, the following is a labelled Brauer diagram of type $G(m, 1, 6)$ with $m \geq 4$.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5 & 6
\end{array}
\]

An arc in a labelled Brauer diagram is called horizontal if both of its end points lie in the top row or in the bottom row. Otherwise, it is called vertical.

Given a horizontal arc $\{i, j\}$ with $i < j$, $i$ (resp. $j$) is called the left (resp. right) end point of $\{i, j\}$. It is assumed that all dots in a horizontal arc are marked at the left end point. A dot marked at the left (resp. right) end point of $\{i, j\}$ is called a left (resp. right) dot of the arc. For a vertical arc we do not define its left end point and its right end point.

The following rule defines equivalence classes of diagrams, which will be used to introduce the multiplication rule late.

The rule for movements of dots. Dots are allowed to move along an arc and may even move to another arc.

1. A left dot of a horizontal arc $\{i, j\}$ is equal to $m - 1$ right dots of $\{i, j\}$. Conversely, a right dot of $\{i, j\}$ is equal to $m - 1$ left dots of $\{i, j\}$.
2. A dot on a vertical arc can move freely to the end points of the arc.
3. Given two arcs $\{i, j\}$ and $\{j, k\}$. A dot in $\{i, j\}$ may cross $\{j, k\}$. Furthermore, a dot at the end point $j$ of the arc $\{i, j\}$ can be replaced by a dot at the end point $j$ of the arc $\{j, k\}$.

The rule for compositions. Given two labelled Brauer diagrams $D_1$ and $D_2$ of type $G(m, 1, n)$. A labelled Brauer diagram $D_1 \circ D_2$ is defined in the following way: First, $D_1$ and $D_2$ are composed in the same way as for Brauer diagrams. Then a new diagram $P$ is obtained.
(which is possibly not a labelled Brauer diagram). Second, we apply the rule for the movements of dots to relabel each arc in \( P \), and obtain a labelled Brauer diagram, denoted by \( D_1 \circ D_2 \).

**The rule for counting closed cycles.** For each closed cycle appearing in the above natural concatenation of \( D_1 \) and \( D_2 \), we apply the rule for movements of dots to relabel the cycle.

It should be noted that the number of dots on each cycle is considered to be in \( \mathbb{Z}/m\mathbb{Z} \). Let \( n(\bar{i}, D_1, D_2) \) be the number of relabelled closed cycles on which there are \( i \) dots. We recall an example in [24] to illustrate the above definition. Suppose \( m \geq 4 \).

\[
\begin{align*}
D_1 &= \, , \quad D_2 = \, ,
\end{align*}
\]

then we have a diagram

\[
D = \, .
\]

Thus the composition \( D_1 \circ D_2 \) of \( D_1 \) and \( D_2 \) is as follows:

\[
D_1 \circ D_2 = \begin{align*}
&= \, ,
\end{align*}
\]

Now we relabel the closed cycles in \( P \). By definition,

\[
\begin{align*}
&= \, ,
\end{align*}
\]

In this case, \( n(\bar{0}, D_1, D_2) = n(\bar{1}, D_1, D_2) = 0 \) and \( n(\bar{2}, D_1, D_2) = n(\bar{3}, D_1, D_2) = 1 \) for \( m \geq 4 \). The following definition has been introduced by Haering-Oldenburg [17].

**Definition 2.2.** [17] Let \( R \) be a commutative ring containing 1 and \( \delta_0, \ldots, \delta_{m-1} \). A cyclotomic Brauer algebra \( B_{m,n}(\delta_1) \) over \( R \) is an associative algebra. As a free \( R \)-module, it has a basis, which consists of all labelled Brauer diagrams of type \( G(m,1,n) \). The multiplication is defined
by

$$D_1 \cdot D_2 = \prod_{i=0}^{m-1} \sigma_i^{m(i_D_1, D_2)} D_1 \circ D_2.$$  

3. Representations of complex reflection groups $W_{m,n}$

In this section, we recall some of the notation in [8, 9] we shall use later on. We also recall some results on complex representations of $W_{m,n}$.

A complex reflection group $W_{m,n}$ of type $G(m, 1, n)$ [2] is generated by elements $s_0, s_1, \ldots, s_{n-1}$ subject to the relations

$$
\begin{align*}
s_0^m &= s_1^2 = 1, & \text{if } 1 \leq i \leq n - 1, \\
 s_0s_is_0 &= s_is_0s_i, \\
 s_is_{i+1}s_i &= s_{i+1}s_is_{i+1}, & \text{if } 1 \leq i \leq n - 2, \\
 s_is_j &= s_js_i, & \text{if } |i - j| \geq 2.
\end{align*}
$$  

(3.1)

The subgroup generated by elements $s_1, s_2, \ldots, s_{n-1}$ is isomorphic to the symmetric group $S_n$ if we identify $s_i$ with the transposition $(i, i+1)$. In order to state some results on representations of $W_{m,n}$, we need some combinatorics.

A composition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a sequence of nonnegative integers. Write $\lambda \models n$ if $|\lambda| = \sum \lambda_i = n$. If $\lambda_i \geq \lambda_{i+1}$ for all $i$, then $\lambda$ is called a partition. In this case, we write $\lambda \vdash n$.

An $m$-composition $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(m)})$ is an $m$-tuple of compositions. If $|\lambda| = \sum_{i=1}^{m} |\lambda^{(i)}|$, then it is denoted by $\lambda \models n$. $\lambda$ is called an $m$-partition if every $\lambda^{(i)}$ is a partition. In this case, write $\lambda \vdash n$.

For later use, let $\Lambda_n(m)$ (resp. $\Lambda_n^+(m)$) be the set of all $m$-compositions (resp. $m$-partitions) of $n$. If $m = 1$, we use $\Lambda_n$ (resp. $\Lambda_n^+$) instead of $\Lambda_n(1)$ (resp. $\Lambda_n^+(1)$).

For any $\lambda \vdash n$, let

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_r}$$

be the Young subgroup of $\mathfrak{S}_n$ associated to $\lambda$. If $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(m)}) \vdash n$, let

$$\tilde{\lambda} = \lambda^{(1)} \lor \lambda^{(2)} \lor \cdots \lor \lambda^{(m)}$$

be the composition of $n$ obtained from $\lambda$ by concatenation. For example, $\tilde{\lambda} = (1, 0, 2, 2, 1)$ if $\lambda = (((10), (221)), \mathfrak{S}_\lambda := \mathfrak{S}_{\tilde{\lambda}}$.

A Young diagram $Y(\lambda)$, $\lambda \in \Lambda_n$, consists of $n$ boxes placed at the matrix entries $\{(i, j)| 1 \leq j \leq \lambda_i\}$. If $\lambda_i = 0$, then there is no box in the $i$th row. A $\lambda$-tableau $\mathfrak{s}$ is a bijective map from the set of boxes in $Y(\lambda)$ to $\{1, 2, \ldots, n\}$. The $\lambda$-tableau $\mathfrak{s}$ is called row standard if the entries in $\mathfrak{s}$ are increasing from left to right in each row. When $\lambda \vdash n$, a row standard $\lambda$-tableau $\mathfrak{s}$ is
called standard if the entries in \( s \) are increasing from top to bottom in each column. The Young diagram \( Y(\lambda) \) of \( \lambda \in \Lambda_m(n) \) is an \( m \)-tuple of Young diagrams \((Y(\lambda^{(1)}), Y(\lambda^{(2)}), \ldots, Y(\lambda^{(m)}))\).

A \( \lambda \)-tableau \( t \) is a bijective map from the set of boxes in \( Y(\lambda) \) to \( \{1, 2, \ldots, n\} \). It should be noted that \( t \) has \( m \) components \( t_1, \ldots, t_m \). Each \( t_i \) is a \( \lambda^{(i)} \)-tableau though the numbers in the boxes are not necessarily in the set \( \{1, \ldots, |\lambda^{(i)}|\} \). If each \( t_i \) is row standard, \( t \) is called row standard. If \( \lambda \vdash n \) and each component is standard, \( t \) is called standard. Let \( T^n(\lambda) \) be the set of all standard \( \lambda \)-tableaux.

\( \mathfrak{S}_n \) acts on a \( \lambda \)-tableau (from right) by permuting its entries. Let \( t^\lambda \) be the \( \lambda \)-tableau in which the numbers \( 1, 2, \ldots, n \) appear in the natural ordering along the rows of the first component, and then along the rows of the second component, and so on. The row stabilizer of \( t^\lambda \) is the Young subgroup \( \mathfrak{S}_\lambda \). For \( \lambda \vdash n \), let \( t\lambda \) be the standard \( \lambda \)-tableau obtained from \( Y(\lambda) \) by putting the integers \( 1, 2, \ldots, n \) from top to bottom down successive columns of \( Y(\lambda^{(m)}) \), then the component \( Y(\lambda^{(m-1)}) \), and so on.

Let \( \Lambda[m, n] = \{[a_1, a_2, \ldots, a_m] \mid 0 \leq a_1 \leq a_2 \cdots \leq a_m = n\} \). For any \( a \in \Lambda[m, n] \), let \( w_a \in \mathfrak{S}_n \) be defined by setting

\[
(a_{i-1} + l)w_a = n - a_i + l \quad \text{for all } i \text{ with } a_{i-1} < a_i, 1 \leq l \leq a_i - a_{i-1}.
\]

For example, if \( a = [4, 8, 9] \), then

\[
w_a = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
6 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 1
\end{pmatrix}.
\]

For non-negative integers \( i \geq 1, j \geq 1, k \geq 0 \), let

\[
w_{i,j}^{(k)} = \begin{pmatrix}
    k + 1 & \cdots & k + i & k + i + 1 & \cdots & k + i + j \\
    k + j + 1 & \cdots & k + j + i & k + 1 & \cdots & k + j
\end{pmatrix}
\]

If either \( i = 0 \) or \( j = 0 \), set \( w_{i,j}^{(k)} = 1 \). In [8, 1.7], Du and the first author proved that, for any \( a \in \Lambda[m, n] \),

\[
w_a = w_{a_{m-1}, a_m-a_{m-1}, a_{m-2}, a_{m-1}-a_{m-2}, \ldots, a_1, a_2-a_1}^{(a_m-a_1)}.
\]

If \( k \in \mathbb{N} \), let

\[
w_a^{(k)} = w_{a_{m-1}, a_m-a_{m-1}, a_{m-2}, a_{m-1}-a_{m-2}, \ldots, a_1, a_2-a_1}^{(k+a_m-a_1)}.
\]

According to [12, 6], define

\[
\pi_0(x) = 1 \text{ and } \pi_a(x) = \prod_{i=1}^{a} (t_i - x), a \in \mathbb{N}, x \in \mathbb{C}.
\]
where \( t_1 = s_0 \) and \( t_i = s_{i-1} t_{i-1} s_{i-1} \). Then,

\[
t_i w = w t_{(i)w}, \forall w \in \mathfrak{S}_n, \text{ and } s_i \pi_a(x) = \pi_a(x) s_i \quad \text{if } i \neq a.
\]

For \( a \in \Lambda[m,n] \), let

\[
\pi_a = \prod_{i=1}^{m-1} \pi_{a_i}(u_{i+1}) \text{ and } \tilde{\pi}_a = \prod_{i=1}^{m-1} \pi_{a_i}(u_{m-i})
\]

where \( u_i = \xi^i \) and \( \xi \) is a primitive \( m \)th root of unity.

Let \( w_\lambda \in \mathfrak{S}_n \) be defined by setting

\[
t^\lambda w_\lambda = t_\lambda.
\]

Let \( w_\lambda \) be defined in (3.2), \( |\lambda| = [a_1, a_2, \ldots, a_m] \) with \( a_j = \sum_{j=1}^{\lambda(i)} i \). If \( t_i \) (resp. \( t_i \)) denotes the \( i \)-th sub-tableau of \( t^\lambda \) (resp. \( t_\lambda w_\lambda^{-1} \)), then \( w_{(i)} \in \mathfrak{S}_n \) is defined by setting

\[
t_i^i w_{(i)} = t_i.
\]

Suppose \( \lambda \in \Lambda^+_{m}(n) \). Define

\[
x_\lambda = \pi_\lambda x_\lambda \text{ and } y_\lambda = \tilde{\pi}_\lambda y_\lambda
\]

where \( \pi_\lambda = \pi_a \), \( \tilde{\pi}_\lambda = \tilde{\pi}_a \), \( |\lambda| = a = [a_1, a_2, \ldots, a_m] \) and \( a_i = \sum_{j=1}^{\lambda(i)} i \). The following result will be useful later on (see [8, 2.8, 3.1, 3.4]).

**Proposition 3.1.** Let \( H_{m,n} = \mathbb{C}[W_{m,n}] \) be the group algebra of \( W_{m,n} \). For any \( a \in \Lambda[m,n] \), let \( a' = [n - a_{m-1}, n - a_{m-2}, \ldots, n - a_1, n] \). Write \( a \leq b \) for \( a, b \in \Lambda[m,n] \) if \( a_i \leq b_i \), \( 1 \leq i \leq m \).

(a) For any \( a, b \in \Lambda[m,n] \), \( \pi_a H_{m,n} \tilde{\pi}_b = 0 \) unless \( a \leq b \). If \( a = b \), then \( \pi_a H_{m,n} \tilde{\pi}_a = \pi_a w_a \tilde{\pi}_a \mathbb{C}[\mathfrak{S}_n] = \mathbb{C}[\mathfrak{S}_n] \pi_a w_a \tilde{\pi}_a \), where \( \mathfrak{S}_n \) is the Young subgroup of \( \mathfrak{S}_n \) with respect to the composition \((a_1, a_2 - a_1, \ldots, a_m - a_{m-1})\).

(b) The set \( \left\{ \pi_a w_a \tilde{\pi}_a w | w \in \mathfrak{S}_n \right\} \) is a basis of \( \pi_a w_a \tilde{\pi}_a H_{m,n} \).

For later use, let \( v_a = \pi_a w_a \tilde{\pi}_a \). The following result can be found in [9, 2.2-2.3] and [8, 4.14].

**Proposition 3.2.** Let \( H_{m,n} \) be the group algebra of \( W_{m,n} \) over \( \mathbb{C} \).

(a) For any \( \lambda \in \Lambda^+_{m}(n) \), let \( \lambda' = (\lambda^{(m)})', \ldots, \lambda^{(1)'}) \) be the dual \( m \)-partition of \( \lambda \). Then

\[
z_\lambda = x_{\lambda} y_{\lambda} y_\lambda = Z_\lambda y_{\lambda}, \quad Z_\lambda = x_{\lambda^{(m)}} \cdots w_{(1)} \cdots w_{(m)} y_{\lambda^{(m)}}.
\]

(b) For any \( \lambda \in \Lambda^+_{m}(n) \), let \( S^\lambda = x_{\lambda} y_{\lambda} H_{m,n} \). Then \( \{ S^\lambda \mid \lambda \in \Lambda^+_{m}(n) \} \) forms a complete set of non-isomorphic irreducible \( H_{m,n} \)-modules.

(c) The functor \( F : H_{m,n}-\text{mod} \rightarrow \oplus_{a \in \Lambda[m,n]} \mathbb{C}[\mathfrak{S}_a]-\text{mod} \) defined by setting \( F(M) = M \epsilon \) is a Morita equivalence, where \( \epsilon = \sum_{a \in \Lambda[m,n]} e_a \) such that \( v_a H_{m,n} = e_a H_{m,n} \) for an idempotent \( e_a \) in \( H_{m,n} \).
Remark 3.3. (1) Propositions 3.1 and 3.2(a) also hold for a cyclotomic Hecke algebra $\mathbb{H}$ over a commutative ring true as well. When $\mathbb{H}$ is semi-simple over a field, 3.2(b) is true. 3.2(c) holds over certain fields with positive characteristic [8]. In particular, let $F$ be a splitting field of $x^m-1$. Write $x^m-1 = (x-u_1)(x-u_2)\cdots(x-u_m)$. Propositions 3.1-3.2 are still true if char$F
ot\mid r!$ and $u_i \neq u_j, 1 \leq i < j \leq m$. The reason is that the group algebra of $W_{m,n}$ over $F$ then is semi-simple (see e.g. [1] or [8]).

(2) The only fact about $e_a$ we are going to use is the equation $e_aH_{m,n} = e_aH_{m,n}$. For details, see [8, 9].

4. Decompositions of certain complex representations of $W_{m,n}$

In this section, we study the Littlewood-Richardson rule for $W_{m,n}$. We also decompose certain induced modules, which will be used in the proof of Theorem 6.2. All representations considered in this section are complex representations.

Let $W_{m,n_1} = \{\tilde{s}_0, \tilde{s}_1, \ldots, \tilde{s}_{n_1-1}\}$ (resp. $W_{m,n_2} = \{\tilde{s}_0, \tilde{s}_1, \ldots, \tilde{s}_{n_2-1}\}$) be the complex reflection group of type $G(m,1,n_1)$ (resp. $G(m,1,n_2)$). Both $W_{m,n_1}$ and $W_{m,n_2}$ can be embedded into $W_{m,n}$, where $n = n_1 + n_2$, in a natural way. The corresponding map $\phi$ is defined by setting $\phi(\tilde{s}_0) = s_0$, $\phi(\tilde{s}_0) = t_{n_1+1}$, $\phi(\tilde{s}_i) = s_i$ and $\phi(\tilde{s}_i) = s_{n_1+i}$.

Definition 4.1. Suppose $n, a \in \mathbb{N}$. For any $x \in \mathbb{C}$, let

$$\pi_a^{(n)}(x) = \prod_{i=1}^a (t_{i+n} - x).$$

If $a \in \Lambda[m,n]$ and $k \in \mathbb{N}$, set

$$\pi_a^{(k)} = \pi_{a_1}^{(k)}(u_2)\pi_{a_2}^{(k)}(u_3)\cdots\pi_{a_{m-1}}^{(k)}(u_m),$$

$$\tilde{\pi}_a^{(k)} = \pi_{a_1}^{(k)}(u_{m-1})\pi_{a_2}^{(k)}(u_{m-2})\cdots\pi_{a_{m-1}}^{(k)}(u_1).$$

It is always assumed that

1. $a_0 = b_0 = 0$, $s_{i,i} = 1$ and
2. $s_{i,j} = s_is_{i+1}\cdots s_{j-1}$ (resp. $s_{i,j} = s_{i-1}s_{i-2}\cdots s_j$) if $i < j$ (resp. $i > j$).

Lemma 4.2. Suppose $a = [a_1, \ldots, a_m] \in \Lambda[m,n_1]$ and $b = [b_1, \ldots, b_m] \in \Lambda[m,n_2]$. For any $k \geq n_1$, let $w^L_{a,b} = \prod_{i=1}^{m-1} w_{k,a_i,b_i-1}^{(a_i+1)}, w^R_{a,b} = \prod_{i=1}^{m-2} w_{b_i,a_i+1,b_i-1}^{(a_i+1)}$ and $a + b = [a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m]$. Then $\pi_a\pi_b^{(k)} = w^L_{a,b}\pi_{a+b}w^R_{a,b}$ and $\tilde{\pi}_a\tilde{\pi}_b^{(k)} = u^L_{a,b}\tilde{\pi}_{a+b}u^R_{a,b}$.

Proof. There is a quasi-linear automorphism of $H_{m,n}$ sending $u_i$ to $u_{m-i}$ and $s_i$ to $s_i, 0 \leq i \leq m-1$. This map sends $\pi_a$ to $\tilde{\pi}_a$. So, it is enough to prove the assertion involving $\pi$. We prove
it by induction on \(m\). Due to (3.6), the following equalities are obtained.

\[
\begin{align*}
\pi_{a_1}(u_2)\pi_{b_1}^{(k)}(u_2) &= \pi_{a_1}(u_2)(t_{k+1} - u_2)(t_{k+2} - u_2) \cdots (t_{k+b_1} - u_2) \\
&= \pi_{a_1}(u_2)(s_{k+1,a_1} + 1s_{a_1+1,k+1} - u_2)(t_{k+2} - u_2) \cdots (t_{k+b_1} - u_2) \\
&= s_{k+1,a_1+1} + 1(t_{k} - u_2) \cdots (t_{k+b_1} - u_2)s_{a_1+1,k+1} \\
&= \cdots \\
&= s_{k+1,a_1+1}s_{k+2,a_1+2} \cdots s_{k+b_1,a_1+b_1} \pi_{a_1+b_1}(u_2)s_{a_1+b_1,k+b_1} \\
&= w_{k-a_1,b_1}(u_2)(w_{k,a_1,b_1})^{-1} = w_{k-a_1,b_1}^{-1}(a_1+b_1)(u_2)w_{b_1,k-a_1}^{(a_1)}.
\end{align*}
\]

This proves the result in the case \(m = 2\). Let

\[
y_{a,b}^L = \prod_{i=1}^{m-2} w_{k-a_i+b_i-1}^{(a_i+b_i-1)} \quad \text{and} \quad y_{a,b}^R = \prod_{i=1}^{m-3} w_{b_i,a_i+1-b_i}^{(a_i)} w_{b_{m-2},k-a_{m-2}}^{(a_{m-2})}.
\]

We have

\[
\begin{align*}
w_{k-a_{m-2}}^{(a_{m-2})} w_{b_{m-2},k-a_{m-2}}^{(a_{m-2})} & = s_{a_{m-2}+b_{m-2},k+b_{m-2}} \cdots s_{a_{m-2}+1,k+1}s_{a_{m-1}+1,k} \cdots s_{k+b_{m-1},a_{m-1}+b_{m-1}} \\
& = s_{a_{m-2}+b_{m-2},k+b_{m-2}} \cdots s_{a_{m-2}+2,k+2}s_{a_{m-2}+1,k} \cdots s_{k+b_{m-1},a_{m-1}+b_{m-1}} \\
& = s_{a_{m-2}+b_{m-2},k+b_{m-2}} \cdots s_{a_{m-2}+2,k+2} + s_{a_{m-2}+1,k+1} \cdots s_{k+b_{m-1},a_{m-1}+b_{m-1}} \\
& = \cdots \\
& = s_{k+b_{m-2}+1,a_{m-1}+b_{m-2}+1} \cdots s_{k+b_{m-2},a_{m-1}+b_{m-2}} + s_{a_{m-2}+2,k+2} + s_{a_{m-2}+1,k+1} \cdots s_{k+b_{m-1},a_{m-1}+b_{m-1}} \\
& = w_{k-a_{m-1},b_{m-1}}^{(a_{m-1}+b_{m-2})} w_{b_{m-2},a_{m-1}-a_{m-2}}^{(a_{m-2})}.
\end{align*}
\]

So,

\[
\begin{align*}
\pi_{a,b}^{(k)} &= y_{a,b}^L \pi_{a+b_1}(u_2) \cdots \pi_{a_{m-2}+b_{m-2}}(u_{m-1}) \prod_{i=1}^{m-3} w_{b_i,a_i+1-b_i}^{(a_i)} w_{b_{m-2},k-a_{m-2}}^{(a_{m-2})} w_{k-a_{m-1},b_{m-1}}^{(a_{m-1})} \\
&= y_{a,b}^L \pi_{a+b_1}(u_2) \cdots \pi_{a_{m-2}+b_{m-2}}(u_{m-1}) \prod_{i=1}^{m-3} w_{b_i,a_i+1-b_i}^{(a_i+b_{m-2})} \\
&= w_{b_{m-2},a_{m-1}-a_{m-2}} \pi_{a_{m-1}+b_{m-1}} w_{b_{m-1},k-a_{m-1}}^{(a_{m-1})} \\
&= w_{a,b}^L \pi_{a+b} w_{a,b}^R \quad \text{by (3.6)}.
\end{align*}
\]

\[\square\]

**Definition 4.3.** For any \(\lambda \vdash n_1, \mu \vdash n_2\) and \(\nu \vdash n_1 + n_2\), let \(L_{\lambda,\mu}^{\nu} \in \mathbb{N}\) be defined by setting

\[
\text{Ind}_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}}^{\mathfrak{S}_{n_1+n_2}} S_{\lambda} \otimes S_{\mu} \cong \bigoplus_{\nu} L_{\lambda,\mu}^{\nu} S_{\nu},
\]

where \(\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}\) is the wreath product of symmetric groups.
Then $L_{\lambda,\mu}^\nu \in \mathbb{N}$ is called a Littlewood-Richardson coefficient (see, e.g. [18, §2.8]). If $\lambda \in \Lambda_m^+(n_1), \mu \in \Lambda_m^+(n_2)$ and $\nu \in \Lambda_m^+(n_1 + n_2)$, define $L_{\lambda,\mu}^\nu = \prod_{i=1}^{n_1} L_{\lambda_i,\mu_i}^{\nu_i}$.

**Theorem 4.4.** Suppose $n = n_1 + n_2$. If $\lambda \in \Lambda_m^+(n_1)$ and $\mu \in \Lambda_m^+(n_2)$, then

$$
\text{Ind}_{W_{m,n_1} \times W_{m,n_2}}^{W_{m,n}} S^\lambda \otimes S^\mu \cong \bigoplus_{\nu \in \Lambda_m^+(n)} L_{\lambda,\mu}^\nu S^\nu.
$$

**Proof.** Due to Proposition 3.2(c),

$$
L_{\lambda,\mu}^\nu = \dim_c \text{Hom}_{H_{m,n}}(\text{Ind}_{H_{m_1,n_1} \times H_{m_2,n_2}}^{H_{m,n}} S^\lambda \otimes S^\mu, S^\nu)
$$

$$
= \dim_c \text{Hom}_{H_{m,n}}(S^\lambda \otimes S^\mu \otimes H_{m,n} \epsilon, S^\nu \epsilon)
$$

where $\epsilon = \sum a \in \Lambda_{m,n} e_a$ and $e_a H_{m,n} = v_a H_{m,n}$. According to 3.1(a) and 3.2(a), the following isomorphism is obtained.

$$
S^\nu \epsilon = S^\nu e_{|\nu|} \cong S^{|\nu|} \otimes \cdots \otimes S^{|\nu|}.
$$

Since $e_a e_b = \delta_{a,b} e_a$ (see [8, 3.10]),

$$
\text{Hom}_{H_{m,n}}(S^\lambda \otimes S^\mu \otimes H_{m,n} \epsilon, S^\nu \epsilon) = \text{Hom}_{H_{m,n}}(S^\lambda \otimes S^\mu \otimes H_{m,n} \epsilon_{|\nu|}, S^\nu e_{|\nu|})
$$

If we denote by $H_{m,n_2} \subset H_{m,n}$, the group algebra of $W_{m,n_2}$, then $S^\mu = z_{\mu}^{(n_2)} H_{m,n_2}$ where $z_{\mu}^{(n_1)}$ is obtained from $z_{\mu}$ by replacing $s_i$ (resp. $s_0$) by $s_{i+n_1}$ (resp. $t_{i+n_1-1}$). (see Proposition 3.2 for the definition of $z_{\mu}$). Therefore,

$$
S^\lambda \otimes S^\mu \otimes H_{m,n} \epsilon_{|\nu|} = S^\lambda S^\mu H_{m,n} \epsilon_{|\nu|} = Z_{\lambda \mu}^{(n_1)} v_{|\lambda|}^{(n_1)} H_{m,n} v_{|\nu|},
$$

Write $a = [\lambda], b = [\mu]$ and $c = [\nu]$. Due to Lemma 4.2,

$$
v_{a}^{(n_1)} H_{m,n} v_{c} = \pi_{a}^{(n_1)} w_{a} \pi_{b}^{(n_1)} \tilde{\pi}_{b}^{(n_1)} H_{m,n} \tilde{\pi}_{c} w_{c} \tilde{\pi}_{c'}
$$

$$
= w_{a,b} R \pi_{a+b} \pi_{a}^{(n_1)} w_{a} \pi_{b}^{(n_1)} \tilde{\pi}_{b}^{(n_1)} \pi_{a+b} \pi_{a}^{(n_1)} w_{a} \pi_{b}^{(n_1)} \tilde{\pi}_{b}^{(n_1)} H_{m,n} \pi_{c} w_{c} \tilde{\pi}_{c'}.
$$

It should be noted that $a' + b' = (a + b)'$. If $v_{a}^{(n_1)} H_{m,n} v_{c} \neq 0$, then, according to Proposition 3.1(a), $c = a + b$. In this case, write

$$
w_{a,b} R \pi_{a+b} \pi_{a}^{(n_1)} w_{a} \pi_{b}^{(n_1)} \tilde{\pi}_{b}^{(n_1)} H_{m,n} \pi_{c} w_{c} \tilde{\pi}_{c'} = x h y,
$$

with $x \in \mathcal{G}_c$, $y \in \mathcal{G}_{c'}$ and $h \in D_{c,c'}$, where $D_{c,c'}$ is the set of double coset representatives in $\mathcal{G}_c \backslash \mathcal{G}_n / \mathcal{G}_{c'}$. Then,

$$
w_{a,b} \pi_{a+b} \pi_{a}^{(n_1)} w_{a} \pi_{b}^{(n_1)} \tilde{\pi}_{b}^{(n_1)} H_{m,n} \pi_{c} w_{c} \tilde{\pi}_{c'} = w_{a,b} \pi_{a}^{(n_1)} H_{m,n} \pi_{c} w_{c} \tilde{\pi}_{c'} \neq 0.
$$

Due to [8, 3.1], $\pi_{c} w_{c} \tilde{\pi}_{c'} \neq 0$ only if $h = w_c$. Thus

$$
w_{a,b} \pi_{a}^{(n_1)} H_{m,n} \pi_{c} w_{c} \tilde{\pi}_{c'} = w_{a,b} \pi_{a}^{(n_1)} H_{m,n} \pi_{c} w_{c} \tilde{\pi}_{c'}
$$

$$
= w_{a,b} x \pi_{c} H_{m,n} \pi_{c} w_{c} \tilde{\pi}_{c'} - w_{a,b} \pi_{c} H_{m,n} \pi_{c} w_{c} \tilde{\pi}_{c'} = w_{a,b} \pi_{c} H_{m,n} \pi_{c} w_{c} \tilde{\pi}_{c'} = w_{a,b} \pi_{c} H_{m,n} \pi_{c} w_{c} \tilde{\pi}_{c'}.
$$

For some $\tilde{x} \in \mathcal{G}_c$ (see [8, 3.1, 3.10])

$$
w_{a,b} \pi_{c} w_{c} \tilde{\pi}_{c'} = w_{a,b} \pi_{c} w_{c} \tilde{\pi}_{c'}.
$$
By the definition of $w_{a,b}^L$,

$$Z_{\lambda} z_{\mu}^{(n_1)} w_{a,b}^L = w_{a,b}^L x_{\lambda(1)\vee \mu(1)} w(1) \bar{w}(1) y_{\lambda(1)\vee \mu(1)}' \cdots x_{\lambda(m)\vee \mu(m)} w(m) \bar{w}(m) y_{\lambda(m)\vee \mu(m)}'.$$

A direct computation shows that the $\mathbb{C}$-linear map $\phi : \mathbb{C}[\mathfrak{S}_c] \to \mathbb{C}[\mathfrak{S}_c]v_c w_c^{-1}$ defined by setting $\phi(x) = xv_c w_c^{-1}$ is a right $\mathbb{C}[\mathfrak{S}_c]$-module isomorphism. So, there is a right $\mathbb{C}[\mathfrak{S}_c]$-module isomorphism

$$S^\lambda S^\mu H_{m,n} c_e \cong \frac{w_{a,b}^L x_{\lambda(1)\vee \mu(1)} w(1) \bar{w}(1) y_{\lambda(1)\vee \mu(1)}'}{\mathbb{C}[\mathfrak{S}_c]} \cdots x_{\lambda(m)\vee \mu(m)} w(m) \bar{w}(m) y_{\lambda(m)\vee \mu(m)}'.

\begin{align*}
&\cong \text{Ind}_{\mathfrak{S}_e \times \mathfrak{S}_e} \left( S^{\lambda(1)} \otimes S^{\mu(1)} \right) \otimes \cdots \\
&\otimes \text{Ind}_{\mathfrak{S}_{(c_{m-1}+1, \ldots, c_m)} \times \mathfrak{S}_{(b_{m-1}+1, \ldots, b_m)}} \left( S^{\lambda(m)} \otimes S^{\mu(m)} \right).
\end{align*}

According to the Littlewood-Richardson rule for symmetric groups (see, e.g. [18, §2.8]), the following equality is obtained.

$$\dim \text{Hom}_{H_{m,n_1+n_2}} (\text{Ind}_{H_{m,n_1} \times H_{m,n_2}} S^\lambda \otimes S^\mu, S^\nu) = \prod_{i=1}^{m} L_{\lambda(i) \mu(i)}^{\nu(i)} = L_{\lambda(\mu)}^{\nu}.$$

Theorem 6.2. The following result can be verified easily.

**Lemma 4.5.** Suppose $W_{m,2k} = (s_0, s_1, \ldots, s_{2k-1})$. Write $\tilde{t}_i = t_{i-1} t_i$, $1 \leq i \leq k$, $\tilde{s}_i = s_{2i} s_{2i-1} s_{2i+1} s_{2i}$, $1 \leq i \leq k - 1$ and $\tilde{s}_k = s_{2k-1}$. Then,

(a) $\tilde{t}_i \tilde{t}_j = \tilde{t}_j \tilde{t}_i$, $1 \leq i, j \leq k$ and $\tilde{t}_1^n = 1$, $1 \leq i \leq k$.

(b) $\tilde{s}_i^2 = 1$ for $1 \leq i \leq k$.

(c) $\tilde{s}_i \tilde{s}_{i+1} \tilde{s}_i = \tilde{s}_{i+1} \tilde{s}_i \tilde{s}_{i+1}$ for $1 \leq i \leq k - 2$.

(d) $\tilde{s}_k \tilde{s}_{k-1} \tilde{s}_k \tilde{s}_{k-1} \tilde{s}_k \tilde{s}_{k-1} \tilde{s}_k$.

(e) $\tilde{s}_i \tilde{s}_j = \tilde{s}_j \tilde{s}_i$ if $1 \leq i < j - 1 \leq k - 2$.

(f) $\tilde{s}_i \tilde{t}_i \tilde{s}_i = \tilde{t}_i \tilde{s}_i$ for any $i \neq k$.

(g) $\tilde{s}_k \tilde{t}_i = \tilde{t}_i \tilde{s}_k$ for any $1 \leq i \leq k$.

(h) $\tilde{s}_j \tilde{t}_i = \tilde{t}_i \tilde{s}_j$ for any $j \neq k$. 

In the remaining part of this section, a complex induced representation of $W_{m,2k}$ is decomposed. This result will be used when studying certain complex representations of $B_{m,n}(\delta_1)$ in Theorem 6.2. The following result can be verified easily.
The subgroup of $W_{m,2k}$ generated by $\{\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_k, \tilde{s}_1, \cdots, \tilde{s}_{k-1}, \tilde{s}_k\}$ is denoted by $\mathbb{Z}_m \wr B_k$.

It is isomorphic to the group generated by $S = \{s_{-1}, s_0, s_1, \cdots, s_{k-1}\}$ subject to the relations:

$$s_{-1}^2 = 1, \quad s_i^2 = 1,$$

$$s_{-1}s_1s_{-1} = s_1s_{-1}s_1,$$

$$s_0s_1s_0s_1 = s_1s_0s_1s_0,$$

$$s_{-1}s_i = s_is_{-1},$$

$$s_is_j = s_js_i,$$

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1},$$

$$s_0s_{i-1} \cdots s_{i}s_{i-1}s_{i-1} = s_{i-1} \cdots s_1s_{-1}s_1 \cdots s_{i-1}s_0,$$

$$s_0s_{-1} = s_{-1}s_0.$$  

It is not difficult to see that $\mathbb{Z}_m \wr B_k$ can be realized as a quotient of a toroidal symmetric group [26]. The modular representation theory of $\mathbb{Z}_m \wr B_k$ and its associated endomorphism algebra, called a double cyclotomic Schur algebra, have been studied in [23].

**Theorem 4.6.** For any positive integers $k, l$, let $G$ be the subgroup of $W_{m,2k+l} \times W_{m,l}$ defined by

$$G = \{ (\pi, \sigma \tau_1^{e_1} \cdots t_i^{e_i} \cdots t_l^{e_l}\sigma) \mid \pi \in \mathbb{Z}_m \wr B_k, \sigma \in G, 0 \leq e_j \leq m - 1, 1 \leq j \leq l \}.$$  

Denote by $1$ the trivial representation of $G$ over $\mathbb{C}$ on which $t_i^{e_i} \cdots t_l^{e_l}\sigma$ acts on the right and the others act on the left. Then we have

$$\text{Ind}_{G}^{W_{m,2k+l} \times W_{m,l}} 1 \cong \bigoplus_{\lambda \in \Lambda_m^+(2k+l)} m_{\lambda, \nu} S^\nu \otimes S^\lambda,$$

where $m_{\lambda, \nu} = \sum_{\eta \in \Lambda_m^+(2k+l)} c_\eta L^\nu_{\eta, \lambda}$, $[\eta] = [a_1, a_2, \cdots, a_m]$, $a_0 = 0$ and

$$c_\eta = \sum_{\mu \in \Lambda_m^+(2k+l)} \sum_{0 \leq \nu_m = \mu \leq m - 1} \prod_{j=1}^{m} L_{\nu_{j-1}, \eta(j)}^\nu.$$  

**Proof.** By transitivity of the tensor functors and the Wedderburn-Artin Theorem for finite dimensional semi-simple algebras, the following is obtained.

$$\text{Ind}_{G}^{W_{m,2k+l} \times W_{m,l}} 1 \cong \bigoplus_{\lambda \in \Lambda_m^+(l)} \text{Ind}_{W_{m,2k+l} \times W_{m,l}}^{W_{m,2k+l} \times W_{m,l}} 1 \otimes \text{Ind}_{G}^{\mathbb{Z}_m \wr B_k \times W_{m,l}} 1 \otimes S^\lambda \otimes S^\lambda.$$  

In order to compute

$$\text{Ind}_{\mathbb{Z}_m \wr B_k}^{W_{m,2k+l}} 1 \cong \sum_{w \in \mathbb{Z}_m \wr B_k} wH_{m,2k},$$
we need determine $\text{Hom}_{H_{m,2k}}(\sum_{w \in Z_{m,l}B_k} wH_{m,2k}, S^\eta)$ for any $\eta \in \Lambda_+^+(2k)$. It should be noted that $e_a e_b = \delta_{a,b} e_a$ for any $a, b \in \Lambda[m, 2k]$. Thus $S^\eta e_a \neq 0$ only if $a = [\eta]$. Due to 3.1(a) and 3.2(c),

$$\text{Hom}_{H_{m,2k}}(\sum_{w \in Z_{m,l}B_k} wH_{m,2k}, S^\eta) \cong \text{Hom}_{H_{m,2k}^a}(\sum_{w \in Z_{m,l}B_k} wH_{m,2k}^a, S^\eta e_a)$$

By Proposition 3.1(b), there is a $C$-module isomorphism

$$(4.3) \sum_{w \in Z_{m,l}B_k} wH_{m,2k} e_a = \sum_{w \in Z_{m,l}B_k} wH_{m,2k} e_a = \text{wC}[\mathfrak{S}_a] e_a \cong \text{Ind}_{B_k}^{\mathfrak{S}_2k} 1.$$ 

As $C[\mathfrak{S}_a]$-modules, $S^\eta e_a \cong S^{(1)} \otimes \cdots \otimes S^{q(m)}$. So,

$$\text{Hom}_C[\mathfrak{S}_a](\sum_{w \in Z_{m,l}B_k} wH_{m,2k} e_a, S^\eta e_a) \cong \text{Hom}_C[\mathfrak{S}_a](\text{Ind}_{B_k}^{\mathfrak{S}_2k} 1, S^{(1)} \otimes \cdots \otimes S^{q(m)})$$

$$\cong \text{Hom}_C[\mathfrak{S}_a](\bigoplus_{\mu \in \Lambda^+(2k) \text{ even}} S^\mu, S^{q(1)} \otimes \cdots \otimes S^{q(m)}) \quad (\text{see } [18, 5.4.23])$$

$$\cong \bigoplus_{\mu \in \Lambda^+(2k) \text{ even}} \text{Hom}_C[\mathfrak{S}_a](S^\mu, \text{Ind}_{\mathfrak{S}_2k}^{\mathfrak{S}_a} S^{q(1)} \otimes \cdots \otimes S^{q(m)})$$

Using transitivity of the induced functors and the ordinary Littlewood-Richardson rule for symmetric groups [18, §2.8] we obtain

$$\text{dim}_C \text{Hom}_C[\mathfrak{S}_a](S^\mu, \text{Ind}_{\mathfrak{S}_2k}^{\mathfrak{S}_a} S^{q(1)} \otimes \cdots \otimes S^{q(m)}) = \sum_{\mu \in \Lambda(\mathfrak{S}_a), j=1}^m \prod_{0 \leq i \leq m-1 \nu_i} L_{\nu_{j-1}, \eta(1), \nu_j}^{\nu_i}.$$ 

Furthermore, if $L_{0,\eta(1)}^{\nu_1} \neq 0$, then $\eta(1) = \nu_1$ and $L_{0,\nu_1}^{\nu_1} = 1$. So,

$$(4.4) \text{Ind}_{Z_{m,l}B_k}^{\mathfrak{S}_2k} 1 \cong \bigoplus_{\eta \in \Lambda_+^+(2k)} e_{\eta} S^\eta.$$ 

The decomposition given in (4.1) follows from Theorem 4.4 and (4.4). \hfill $\square$

## 5. REPRESENTATIONS OF THE CYCLOTOMIC BRAUER ALGEBRAS

The main purpose of this section is to show that $B_{m,n}(\delta_1)$ over a commutative ring $R$ is a cellular algebra. Due to the results in [12, §3], we can classify irreducible $B_{m,n}(\delta_1)$-modules over a field. Furthermore, a criterion on the quasi-heredity of $B_{m,n}(\delta_1)$ is obtained.

First, let $R$ be a commutative ring containing 1, and $\delta_i, 0 \leq i \leq m - 1$. Assume

$$x^m - 1 = (x - u_1)(x - u_2) \cdots (x - u_m)$$

for some $u_i \in R, 1 \leq i \leq m$. \hfill $\square$


Suppose \( n, k \in \mathbb{N} \) with \( 0 \leq k \leq \lfloor n/2 \rfloor \). An \((n, k)\)-labelled parenthesis diagram is a diagram, which consists of \( n - 2k \) free vertices and \( k \) horizontal arcs on which there are \( i \) dots, \( i \in \mathbb{Z}_m \). Let \( V(n, k) \) be the free \( R \)-module generated by \( P(n, k) \), the set of all \((n, k)\)-labelled parenthesis diagrams.

Each element \( w \in W_{m,n} \) corresponds to a labelled Brauer diagram and the multiplication in \( W_{m,n} \) is compatible with that in \( B_{m,n}(\delta_1) \). The following are labelled Brauer diagrams with respect to \( s_0, s_i, 1 \leq i \leq n - 1 \).

Recall that a Brauer diagram with \( k \) horizontal arcs is determined by a pair of parenthesis diagrams and a \( w \in S_{n - 2k} \) (see, e.g [12]). Similarly, a labelled Brauer diagram \( D \) with \( k \) horizontal arcs is determined by a \( w \in W_{m,n - 2k} \) and a pair of \((n, k)\)-labelled parenthesis diagrams \( D_1, D_2 \in P(n, k) \) as follows.

Cutting off all the vertical lines of \( D \), we get \( D_1, D_2 \in P(n, k) \). Removing all horizontal arcs in the top and bottom rows of \( D \) and labelling the remaining vertices in the top (resp. bottom) row as \( 1, 2, \cdots, n - 2k \) from left to right, we obtain a unique diagram which corresponds to a \( w \in W_{m,n - 2k} \). Conversely, a pair of \((n, k)\)-labelled parenthesis diagrams \( (D_1, D_2) \) and a \( w \in W_{m,n - 2k} \) determine a unique \((n, k)\)-labelled Brauer diagram. More explicitly, put \( D_1 \) above \( D_2 \) and join the \( i \)th free vertex of \( D_1 \) to the \( j \)th free vertex of \( D_2 \) to get a vertical arc if \( \{i, j\} \) is a vertical arc in the labelled Brauer diagram of \( w \). Therefore, each labelled Brauer diagram \( D \) can be determined uniquely by a triple pair \((D_1, w, D_2)\). For later use, write \( D = D_1 \otimes w \otimes D_2 \).

The following result can be proved easily.

**Lemma 5.1.** Let \( D = D_1 \otimes w \otimes D_2 \) be a \((n, k)\)-labelled Brauer diagram. Then there are \( w_l, w_r \in W_{m,n} \) such that

\[
D_1 \otimes w \otimes D_2 = (D_1 \otimes id_{n - 2k} \otimes D_2)w_r = w_l(D_1 \otimes id_{n - 2k} \otimes D_2).
\]

The following lemma follows from Definition 2.2.

**Lemma 5.2.** Let \( M_{n,k} \) be the free \( R \)-module generated by all \((n, k)\)-labelled Brauer diagrams. For \( i = 1, 2 \), let \( D_i = D_i' \otimes w_i \otimes D_i'' \) with \( D_i', D_i'' \in P(n, k_i) \), \( w_1 \in W_{m,n - 2k_1} \) and \( w_2 = id \in W_{m,n - 2k_2} \).
If $D_1 \diamond D_2 \in M_{n,k_2}$, then $k_1 \leq k_2$. Moreover,

$$D_1 \cdot D_2 = \left\{ \prod_{i=0}^{m-1} \delta_i^{n_i(D_1,D_2)} \right\} D \otimes \pi(D_1,D'_2) \otimes D'_2$$

for some $D \in \mathcal{P}(n,k_2)$ and $\pi(D_1,D'_2) \in \mathcal{W}_{m,n-2k_2}$ such that $D$, $\pi(D_1,D'_2)$ and $\prod_{i=0}^{m-1} \delta_i^{n_i(D_1,D_2)}$ do not depend on $D'_2$.

Since $x^m - 1 = (x - u_1)(x - u_2) \cdots (x - u_m)$ for some $u_i \in R, 1 \leq i \leq m$, the group algebra $H_{m,n} := \mathcal{R}[W_{m,n}]$ is the cyclotomic Hecke algebra of type $G(m,1,n)$ with parameters $q = 1$, $u_i$, $1 \leq i \leq m$. For $s \in T^s(\lambda)$, $\lambda \in \Lambda_{m}^{+}(n)$, let $d(s) \in \mathfrak{S}_n$ be defined by setting $s = t\lambda d(s)$. Write $x^\lambda_{st} = d(s)^{-1}x^\lambda d(t)$. Then $\cup_{\lambda \in \Lambda_{m}^{+}(n)} \mathcal{B}^\lambda = \{ x^\lambda_{st} | s, t \in T^s(\lambda) \}$ is a cellular basis of $H_{m,n}$ [6, 3.26] in the sense of [12, 1.1]. In particular, let $\leq$ be the dominance order on $\Lambda_{m}^{+}(n)$. For any $\pi \in H_{m,n},$

$$(5.1) \quad \pi x^\lambda_{st} \equiv \sum_{s_1} f_{\lambda,s}(\pi,s_1)x^\lambda_{s't} \mod H_{m,n}^\lambda$$

where $H_{m,n}^\lambda$ is the free $\mathcal{R}$-submodule generated by $\mathcal{B}^\mu, \mu \triangleright \lambda$. Moreover, the coefficient $f_{\lambda,s}(\pi,s_1)$ depends on $\pi, s_1$ and $s$ only.

**Definition 5.3.** Let $\Lambda = \{(k, \lambda) | 0 \leq k \leq \lfloor n/2 \rfloor, \lambda \in \Lambda_{m}^{+}(n-2k)\}$. Define $(k, \lambda) \leq (l, \mu)$ if either $k < l$ (usual total order on $\mathbb{Z}$) or $k = l$ and $\lambda \leq \mu$. Then $(\Lambda, \leq)$ is a poset.

**Theorem 5.5.** Let $R$ be a commutative ring containing $1, u_1, u_2, \ldots, u_m$ and parameters $\delta_i, 0 \leq i \leq m-1$ such that $x^m - 1 = (x - u_1)(x - u_2) \cdots (x - u_m)$, $u_i \in R, 1 \leq i \leq m$.

(a) For any $s, t \in T^s(\lambda), \lambda \in \Lambda_{m}^{+}(n)$, define

$$C_{(D_1,s),(D_2,t)}^{(k,\lambda)} = D_1 \otimes x^\lambda_{st} \otimes D_2,$$

and $I(k, \lambda) = \{(D,s) \in \mathcal{P}(n,k) \times T^s(\lambda) \}$ and $\mathcal{B}^{(k,\lambda)} = \{ D_1 \otimes x^\lambda_{st} \otimes D_2 | D_1, D_2 \in \mathcal{P}(n,k), s, t \in T^s(\lambda) \}$. Then $\mathcal{B} = \cup_{(k, \lambda) \in \Lambda} \mathcal{B}^{(k, \lambda)}$ is a basis of $B_{m,n}(\delta_i)$.

(b) There is an $\mathcal{R}$-linear anti-involution $\sigma$ on $B_{m,n}(\delta_i)$ such that $\sigma(C_{(D_1,s),(D_2,t)}^{(k,\lambda)}) = C_{(D_2,t),(D_1,s)}^{(k,\lambda)}$.

(c) For any $D_1 = D'_1 \otimes w \otimes D''_1 \in B_{m,n}(\delta_i)$ and $D_2 = D'_2 \otimes id \otimes D''_2 \in M_{n,k}$,

$$(D'_1 \otimes w \otimes D''_1)C_{(D'_2,s),(D''_2,t)}^{(k,\lambda)} \equiv \sum_{s_1 \in T^s(\lambda)} \prod_{i=0}^{m-1} \delta_i^{n_i(D_1,D_2)}f_{\lambda,s}(\pi,s_1) C_{(D_1,s_1),(D''_2,t)}^{(k,\lambda)} \mod R^{(k,\lambda)},$$
where $R^{(k,\lambda)}$ is the free $R$-module generated by $B^{(i,\mu)}$, $(i,\mu) > (k,\lambda)$ and $D \in P(n,k)$. Moreover, the coefficient $\prod_{i=0}^{m-1} \delta_{i}^{n(i, D_{1}, D_{2})} f_{\lambda, s}(\pi, s_{1})$ does not depend on $D''_{2}$ and $t$. Therefore, $B$ is a cellular basis in the sense of [12, 1.1].

Proof. (a) follows from the statement above Lemma 5.1. Let $\iota: H_{m,n} \to H_{m,n}$ be the $R$-linear anti-involution such that $\iota(s_{i}) = s_{i}$, $0 \leq i \leq n - 1$. Then $\iota(x_{\lambda_{s_{t}}}) = x_{\lambda_{t_{s}}}$ for any $s, t \in T^{s}(\lambda)$, and $\lambda \in \Lambda_{m}^{+}(n - 2k), 0 \leq k \leq \lfloor n/2 \rfloor$ [6, 3.14]. The $R$-linear map sending $D_{1} \otimes w \otimes D_{2}$ to $D_{2} \otimes \iota(w) \otimes D_{1}$ is the anti-involution $\sigma$ we need in (b). (c) is verified as follows. According to Lemma 5.1,

$$C^{(k,\lambda)}_{(D'_{2}, s), (D''_{2}, t)} = (D'_{2} \otimes id \otimes D''_{2}) \cdot y$$

for some $y \in H_{m,n}$. If $(D'_{1} \otimes w \otimes D'_{1}) \circ (D'_{2} \otimes id \otimes D''_{2}) \not\in M_{n,k}$, then there is nothing to be proved since

$$D_{1} \cdot C^{(k,\lambda)}_{(D'_{2}, s), (D''_{2}, t)} \equiv 0 \mod R^{(k,\lambda)}.$$ 

Suppose $(D'_{1} \otimes w \otimes D'_{1}) \circ (D'_{2} \otimes id \otimes D''_{2}) \in M_{n,k}$. Due to Lemmas 5.1-5.2,

$$(D'_{1} \otimes w \otimes D'_{1})(D'_{2} \otimes x_{\lambda_{s_{t}}}) \otimes D''_{2} = \{ \prod_{i=0}^{m-1} \delta_{i}^{n(i, D_{1}, D_{2})} \} D \otimes \pi x_{\lambda_{s_{t}}} \otimes D''_{2}$$

for some $D \in P(n, k)$ and $\pi \in W_{m,n-2k}$. Furthermore, $\prod_{i=0}^{m-1} \delta_{i}^{n(i, D_{1}, D_{2})}$, $D$ and $\pi$ are independent of $D''_{2}$ and hence,

$$(D'_{1} \otimes w \otimes D''_{2})C^{(k,\lambda)}_{(D'_{2}, s), (D''_{2}, t)} \equiv \sum_{s_{1} \in T^{s}(\lambda)} \{ \prod_{i=0}^{m-1} \delta_{i}^{n(i, D_{1}, D_{2})} f_{\lambda, s}(\pi, s_{1}) \} C^{(k,\lambda)}_{(D'_{1}, s_{1}), (D''_{1}, t_{1})} \mod R^{(k,\lambda)}.$$ 

According to (5.1), $f_{\lambda, s}(\pi, s_{1})$ do not depend on $t$. $\square$

In the remaining part of this section, it is assumed that $R = F$ is a field. For each $(k, \lambda) \in \Lambda$, let $f_{k, \lambda} : I(k, \lambda) \times I(k, \lambda) \to F$ be a map defined by

$$C^{(k,\lambda)}_{(D'_{1}, s_{1}), (D''_{2}, s_{2})} C^{(k,\lambda)}_{(D'_{2}, t_{1}), (D''_{2}, t_{2})} \equiv f_{k, \lambda}((D'_{1}, s_{2}), (D'_{2}, t_{1}), (D''_{1}, s_{1}), (D''_{2}, t_{2})) C^{(k,\lambda)}_{(D''_{2}, s_{1}), (D''_{2}, t_{2})} \mod F^{(k,\lambda)}.$$ 

Due to [12, 2.4], $f_{k, \lambda}$ induces an associative, symmetric bilinear form. The following lemma follows from [12, 3.4]

**Lemma 5.5.** Let $B_{m,n}(\delta_{1})$ be a cyclotomic Brauer algebra over a field $F$. The isomorphism classes of simple $B_{m,n}(\delta_{1})$-modules are indexed by $\{(k, \lambda) \in \Lambda \mid f_{k, \lambda} \neq 0 \}$.

For any $\lambda \in \Lambda_{m}^{+}(n - 2k)$, let $\phi_{\lambda} : T^{s}(\lambda) \times T^{s}(\lambda) \to F$ be the bilinear form defined by

$$x_{ss_{1}}^{\lambda} x_{tt}^{\lambda} = \phi_{\lambda}(s, t)x_{ss_{1}}^{\lambda} \mod H^{\lambda}_{m,n}.$$
The following Lemma has been proved in [12] in the case \( m = 1 \). The general case can be proved similarly. We denote \( \lambda \) by 0 if \( \lambda \vdash 0 \).

**Lemma 5.6.** Keep the set up above.

(a) If \( \phi_\lambda = 0 \) for \( \lambda \in \Lambda_+^m(n - 2k), 0 \leq k \leq \lfloor n/2 \rfloor \), then \( f_{k, \lambda} = 0 \).

(b) Assume \( \phi_\lambda \neq 0 \), \( \lambda \in \Lambda^+_m(n - 2k) \). If \( k \neq n/2 \) and \( n \) is even, then \( f_{k, \lambda} \neq 0 \).

(c) Suppose \( k = n/2 \) and \( n \) is even. If \( \delta_0 \neq 0 \) for some \( 0 \leq i_0 \leq m - 1 \), then \( f_{k, 0} \neq 0 \). If \( \delta_i = 0 \) for all \( 0 \leq i \leq m - 1 \), then \( f_{k, 0} = 0 \).

**Proof.** Suppose \( f_{k, \lambda} \neq 0 \). Then there are \((D'_1, s), (D'_2, t)\) \( \in I(k, \lambda) \) such that

\[
\phi_{D'_1, s}(D'_2, t) \neq 0.
\]

Write \( D_i = D'_i \otimes \text{id} \otimes D''_i, i = 1, 2 \). If \( D_1 \circ D_2 \notin \mathcal{M}_{n,k} \), then \( D_1 \circ D_2 \in F_{>}(k, \lambda) \).

Since \( F_{>}(k, \lambda) \) is a two-sided ideal,

\[
\text{C}^{(k, \lambda)}_{(D'_1, s), (D'_2, t)} \equiv 0 \mod F_{>}(k, \lambda).
\]

In this case, \( f_{k, \lambda}((D'_1, s), (D'_2, t)) = 0 \), a contradiction. Therefore, \( D_1 \circ D_2 \in \mathcal{M}_{n,k} \) and \( D_1 \cdot D_2 = D'_1 \otimes w \otimes D''_2 \) for some \( w \in H_{m,n-2k} \). Since \( \phi_\lambda = 0 \), \( x_{ss}^\lambda w x_{tt}^\lambda \equiv 0 \mod H^\lambda_{m,n} \). Due to Lemma 5.1,

\[
\phi_{D'_1, s}(D'_2, t) = D'_1 \otimes x_{ss}^\lambda w x_{tt}^\lambda \otimes D''_2 \equiv 0 \mod F_{>}(k, \lambda).
\]

Thus \( f_{k, \lambda}((D'_1, s), (D'_2, t)) = 0 \), contradicts to our assumption. So, (a) must be true. Let

\[
X_{ij} = D \otimes \text{id} \otimes D
\]

where \( D \in P(n, 1) \) contains a unique horizontal arc \( \{i, j\} \). Take \( D_1 = X_1 X_3 \cdots X_{2k-1,2k} \) and \( D_2 = X_{23} X_{45} \cdots X_{2k,2k+1} \). Write \( D_i = D'_i \otimes \text{id} \otimes D''_i, i = 1, 2 \). Then \( D_1 D_2 = D'_1 \otimes \text{id} \otimes D''_2 \).

By Lemma 5.1,

\[
\text{C}^{(k, \lambda)}_{(D'_1, s), (D'_2, t)} \equiv \phi_{\lambda}(s, t) D'_1 \otimes x_{ss}^\lambda \equiv 0 \mod F_{>}(k, \lambda)
\]

Therefore, \( f_{k, \lambda}((D'_1, s), (D'_2, t)) = \phi_{\lambda}(s, t) \), forcing \( f_{k, \lambda} \neq 0 \).

(c) Suppose \( \delta_i \neq 0 \) for some \( 0 \leq i \leq m - 1 \). Let \( D_1 = X_{12} X_{23} \cdots X_{n-1,n} t_1^i t_2^i \cdots t_{n-1}^i \) and \( D_2 = X_{12} X_{23} \cdots X_{n-1,n} \). Obviously, \( D_1 D_2 = \delta_i D_2 \). Write \( D_i = D'_i \otimes \text{id} \otimes D''_i, i = 1, 2 \). Then

\[
f_{k, 0}((D'_1, 0), (D'_2, 0)) = \delta_i^k \neq 0.
\]

Suppose \( \delta_i = 0 \) for all \( 0 \leq i \leq m - 1 \). Since \( k = n/2 \) and \( n \) is even, an \((n, k)\)-labelled Brauer diagram \( D \) has no free vertex in the top and bottom rows. Therefore, the composition of two \((n, k)\)-labelled Brauer diagrams must contain at least one interior cycle. Each interior cycle will provide a factor \( \delta_i \) for some \( 0 \leq i \leq m - 1 \) if there are \( i \) dots on it. Since \( \delta_i = 0 \) for any \( 0 \leq i \leq m - 1, D_1 D_2 = 0 \). So, \( f_{k,0} = 0 \).
Let $e = \text{char} F$ (resp. $+\infty$) if $\text{char} F \neq 0$ (resp. $\text{char} F = 0$). A partition $\lambda = (n^{m_n}, \ldots, 1^{m_1}) \in \Lambda^+(n)$ is called $e$-\textbf{regular} if $m_i < e$ for all $i$. $\lambda$ is called $e$-\textbf{restricted} if $\lambda'$, the dual of $\lambda$, is $e$-regular.

**Definition 5.7.** An $m$-partition $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})$ is called $e$-\textbf{admissible} with respect to $u_1, u_2, \ldots, u_m$ if

(a) each partition $\lambda^{(i)}, 1 \leq i \leq m$ is $e$-restricted,

(b) $\lambda^{(i)} = \emptyset$ whenever $u_j = u_i$ and $1 \leq j < i \leq n$.

The following theorem has been proven by Mathas [22].

**Theorem 5.8.** [22, 3.7] Let $H_{m,n}$ be the group algebra of $W_{m,n}$ over a splitting field $F$ of $x^m - 1$. Write $x^m - 1 = (x - u_1)(x - u_2) \cdots (x - u_m)$, $u_i \in F, 1 \leq i \leq m$. Then $\phi_{\lambda} \neq 0$ if and only if $\lambda$ is $e$-admissible.

**Theorem 5.9.** Let $B_{m,n}(\delta_1)$ be the cyclotomic Brauer algebra over $F$, a splitting field of $x^m - 1$, which contains $\delta_i, 0 \leq i \leq m - 1$. Write $x^m - 1 = (x - u_1)(x - u_2) \cdots (x - u_m)$, $u_i \in F, 1 \leq i \leq m$.

(a) Suppose $\delta_i \neq 0$ for some $0 \leq i \leq m - 1$ or $\delta_i = 0$ for all $0 \leq i \leq m - 1$ and $n$ is odd. Then the isomorphism classes of simple $B_{m,n}(\delta_1)$-modules are indexed by the set

$$\{(k, \lambda) \mid 0 \leq k \leq [n/2], \lambda \in \Lambda^+_{m}(n - 2k) \text{ is } e\text{-admissible.}\}$$

(b) If $\delta_i = 0$ for all $0 \leq i \leq m - 1$ and $n$ is even, then the isomorphism classes of simple $B_{m,n}(\delta_1)$-modules are indexed by the set

$$\{(k, \lambda) \mid 0 \leq k < n/2, \lambda \in \Lambda^+_{m}(n - 2k) \text{ is } e\text{-admissible.}\}$$

(c) If $n$ is odd, then $B_{m,n}(\delta_1)$ is quasi-hereditary in the sense of [5] if and only if $e > n$ and $u_i \neq u_j$ for all $1 \leq i < j \leq m$.

(d) If $n$ is even and $\delta_i \neq 0$ for some $0 \leq i \leq m - 1$, then $B_{m,n}(\delta_1)$ is quasi-hereditary if and only if $e > n$ and $u_i \neq u_j$, $1 \leq i < j \leq m$.

(e) If $n$ is even and $\delta_i = 0$ for all $0 \leq i \leq m - 1$, then $B_{m,n}(\delta_1)$ is not quasi-hereditary.

**Proof.** (a) and (b) follows immediately from Lemma 5.6 and Theorem 5.8. Due to [12, 3.10] and [20, 3.1], $B_{m,n}(\delta_1)$ is quasi-hereditary if and only if $f_{k,\lambda} \neq 0$ for all $(k, \lambda) \in \Lambda$.

It follows from Lemma 5.6(c) that $f_{k,0} = 0$ under the assumptions in (e). Therefore, $B_{m,n}(\delta_1)$ is not quasi-hereditary. Under the assumptions given in (c) or (d), due to Lemma 5.6(a)-(b), $f_{k,\lambda} \neq 0$ if and only if $\phi_{\lambda} \neq 0$, which is equivalent to $\lambda$ being $e$-admissible (see Theorem 5.8). However, all $m$-partitions of $n - 2k$, $0 \leq k \leq n/2$, are $e$-admissible with respect to $u_1, u_2, \ldots, u_m$ if and only if $e > n$ and $u_i \neq u_j, 1 \leq i < j \leq m$. Thus, (c) and (d) follow. □
Remark 5.10. The case \( m = 1 \) was dealt with in [10, 12, 13, 19, 27], etc. See also [28] for its \( q \)-analogue.

Finally, we give another cellular basis of \( B_{m,n}(\delta_1) \), which will be used later on. Define \( y_{st}^\lambda = d(s)^{-1}y_\lambda d(t) \) for all \( s, t \in T^s(\lambda) \). Then \( \{ y_{st}^\lambda | \lambda \in \Lambda_m^+(r), s, t \in T^s(\lambda) \} \) is another cellular basis of \( H_{m,n} \) [9]. The cell module of \( H_{m,n} \) with respect to \( \lambda \) is denoted by \( \Delta(\lambda) \). It is proved in [9, 2.9] that

\[
\Delta(\lambda) \cong S^{\lambda'},
\]

where \( S^{\lambda'} \) is given in Proposition 3.2(b). The proof of the following result is similar to that of Theorem 5.4.

Theorem 5.11. Let \( R \) be a commutative ring containing \( 1, u_1, u_2, \cdots, u_m \) and parameters \( \delta_i, 0 \leq i \leq m-1 \) such that \( x^m - 1 = (x - u_1)(x - u_2) \cdots (x - u_m), \ u_i \in R, 1 \leq i \leq m. \) For any \( s, t \in T^s(\lambda), \lambda \in \Lambda_m^+(n), \) define

\[
\tilde{C}_{(D_1,s),(D_2,t)}^{(k,\lambda)} = D_1 \otimes y_{st}^\lambda \otimes D_2,
\]

and \( I(k, \lambda) = \{(D, s) \in P(n, k) \times T^s(\lambda)\} \) and \( \tilde{B}^{(k,\lambda)} = \{ D_1 \otimes y_{st}^\lambda \otimes D_2 | D_1, D_2 \in P(n, k), s, t \in T^s(\lambda) \} \). Then \( \tilde{B} = \cup_{(k, \lambda) \in \Lambda} \tilde{B}^{(k,\lambda)} \) is a cellular basis of \( B_{m,n}(\delta_1) \). When \( R \) is a field, a cell module of \( B_{m,n}(\delta_1) \) with respect to this cellular basis is denoted by \( S^{(k,\lambda)}, (k, \lambda) \in \Lambda \).


The branching rule of \( S^{(k,\lambda)} \) over \( \mathbb{C} \) is presented in this section. It should be noted that the proofs of Theorems 6.1-6.2 depend only on the ordinary branching rule of \( H_{m,n} \). Therefore, they are still true if we replace \( \mathbb{C} \) by the field \( F \) in Remark 3.3. Finally, all \( B_{m,n}(\delta_1) \)-modules considered in this section are left modules.

Due to Theorem 5.11 and [12, 2.1],

\[
S^{(k,\lambda)} \cong V(n, k) \otimes \Delta(\lambda) \otimes D_0, \ D_0 \in P(n, k).
\]

According to (5.3),

\[
S^{(k,\lambda)} \cong V(n, k) \otimes S^{\lambda'} \otimes D_0.
\]

For later use, let \( D_0 = \text{top}(X_{n-2k+1}X_{n-2k+3} \cdots X_{n-1}) \). The following is the labelled parenthesis diagram with respect to \( D_0 \).

\[ D_0 = \begin{array}{cccccc}
\circ & \circ & \cdots & \circ & \circ & \circ \\
\circ & \circ & \cdots & \circ & \circ & \circ \\
\circ & \circ & \cdots & \circ & \circ & \circ \\
\end{array} \]
For any $D \in P(n, k)$, let

$$f(D) = D \otimes id \otimes D_0.$$  

Suppose $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)}) \in \Lambda^+_m(n)$. The node $(j, \lambda^{(i)}_j)$ of $Y(\lambda)$ is called a removable node if $Y(\lambda) \setminus (j, \lambda^{(i)}_j)$ corresponds to another $m$-partition $\mu \in \Lambda^+_m(n - 1)$. In this case, we write either $\mu \rightarrow \lambda$ or $\lambda \leftarrow \mu$. The following result can be proved by arguments similar to those in [7, §4].

**Theorem 6.1.** Assume that $n = l + 2k, l \leq n$ and $\lambda \in \Lambda^+_m(l)$. There is a short exact sequence

$$0 \rightarrow \bigoplus_{\mu, \mu \rightarrow \lambda} S^{(k, \mu)} \rightarrow S^{(k, \lambda)} \downarrow \bigoplus_{\mu, \lambda \leftarrow \mu} S^{(k-1, \mu)} \rightarrow 0$$

where $M \downarrow$ is the restriction of a $B_{m,n}(\delta_1)$-module to a $B_{m,n-1}(\delta_1)$-module.

**Proof.** Let $I^{\geq k}_{m,n}$ (resp. $I^{> k}_{m,n}$) be the vector space generated by all $(n, l)$-labelled Brauer diagrams with $l \geq k$ (resp. $l > k$). Let $I^{k}_{m,n}(\delta_1) = I^{\geq k}_{m,n} / I^{> k}_{m,n}$. Then $I^{k}_{m,n}(\delta_1)$ is a $B_{m,n}(\delta_1)$-module. Let $I^{k'}_{m,n}(\delta_1) \subset I^{k}_{m,n}(\delta_1)$ be the free $R$-module generated by $\{ D \otimes w \otimes D_0 \mid D \in P(n, k), w \in W_{m,n-2k} \}$.

Then,

$$S^{(k, \lambda)} \cong I^{k'}_{m,n}(\delta_1) \otimes_{W_{m,l}} \Delta(\lambda)$$

As stated before, $\{ y_{s,t_0}^{\lambda} \mid s \in T^*(\lambda) \}$ with any fixed $t_0 \in T^*(\lambda)$ is a basis of $\Delta(\lambda)$ (see [6]). So, $\{ f(D) \otimes_{W_{m,l}} y_{s,t_0}^{\lambda} \mid D \in P(n, k), s \in T^*(\lambda) \}$ is a basis of $I^{k'}_{m,n} \otimes_{W_{m,l}} \Delta(\lambda)$.

Divide $P(n, k)$ into two subsets $A$ and $B$ such that the $n$th vertex in $D \in A$ (resp. $D \in B$) is free (resp. fixed.)

Let $W \subset V = S^{(k, \lambda)}$ be the subspace generated by $\{ f(D) \otimes_{W_{m,l}} y_{s,t_0}^{\lambda} \mid D \in A, s \in T^*(\lambda) \}$. Then

$$\dim W = \# A \cdot \dim \Delta(\lambda) = \dim I^{k'}_{m,n-1}(\delta_1) \dim \Delta(\lambda).$$

$W$ is a $B_{m,n-1}(\delta_1)$-module since the $n$th vertex of $D$ is fixed by $B_{m,n-1}(\delta_1)$.

For any $D \in A$, let $\bar{D} \in P(n - 1, k)$ be obtained from $D$ by removing the vertex $n$. The $\mathbb{C}$-linear map $\alpha$ is defined by setting

$$\alpha : W \rightarrow I^{k'}_{m,n-1}(\delta_1) \otimes_{W_{m,l-1}} \Delta(\lambda) \downarrow f(D) \otimes y_{s,t_0}^{\lambda} \rightarrow f(\bar{D}) \otimes y_{s,t_0}^{\lambda}.$$

It is surjective. Due to (6.4), it must be a linear isomorphism. Since the $n$th vertex of $D$ is fixed by $B_{m,n-1}(\delta_1)$, $\alpha$ is a $B_{m,n-1}(\delta_1)$-homomorphism. By the ordinary branching rule for $H_{m,l}$ (see e.g. [3] or [9]),

$$\Delta(\lambda) \downarrow \cong \bigoplus_{\mu, \mu \rightarrow \lambda} \Delta(\mu).$$
So,
\[ W \cong \bigoplus_{\mu, \mu - \lambda} I_{m,n-1}^k(\delta_i) \otimes_{W_{m,l}} \Delta(\mu) \cong \bigoplus_{\mu, \mu - \lambda} S^{(k, \mu)} \]

Suppose \( D \in B \). Let \( D' \in P(n-1, k-1) \) be obtained from \( D \) by removing the vertex \( n \) and the horizontal arc, say \( \{i, n\} \). The surjective map from \( \phi : B \rightarrow P(n-1, k-1) \) sending \( D \) to \( D' \) is not injective. More explicitly, there are \((l+1)m\) elements in \( \phi^{-1}(D') \).

Suppose there are \( j - 1 \) free vertices in \( D \), which are left to the vertex \( i \). If there are \( a \) dots on \( \{i, n\} \in D \), then we define
\[ \sigma_D = \sigma^a_j = s_j s_{j+1} \cdots s_{l+1} t^a_{l+1}, 0 \leq a \leq m - 1, 1 \leq j \leq l + 1. \]
In particular, \( \sigma^a_{l+1} = t^a_{l+1} \). For example:

It can be verified easily that \( \{\sigma^a_j \mid 0 \leq a \leq m - 1, 1 \leq j \leq l + 1\} \) is a set of left coset representatives in \( W_{m,l+1}/W_{m,l} \). So,
\[ \Delta(\lambda)^{W_{m,l+1}}_{W_{m,l}} = \sum_{0 \leq a \leq m-1} \sigma^a_j \cdot \Delta(\lambda) \]

Define \( \mathbb{C} \)-linear map \( \beta \) by setting
\[ (6.7) \quad \beta : S^{(k, \lambda)}/W \rightarrow I_{m,n-1}^{k-1'}(\delta_i) \otimes_{W_{m,l+1}} \Delta(\lambda) \]
\[ f(D) \otimes y_{\lambda_{st}} \mapsto f(D) \otimes \sigma^a_j y_{\lambda_{st}} \]

Since \( \# \phi^{-1}(D) = (l+1)m \) and \( \dim \Delta(\lambda) \uparrow = (l+1)m \dim \Delta(\lambda) \),
\[ \dim S^{(k, \lambda)}/W = \dim I_{m,n-1}^{k-1'}(\delta_i) \otimes_{W_{m,l+1}} \Delta(\lambda) \uparrow. \]

Obviously, \( \beta \) is a surjective map (see (6.6)). So, \( \beta \) is a linear isomorphism. We claim \( \beta \) is a \( B_{m,n-1}(\delta_i) \)-module isomorphism.

According to (6.7), \( \beta \) preserves the action of \( w \in W_{m,n-1} \). Since \( B_{m,n-1}(\delta_i) \) is generated by \( W_{m,n-1} \) and \( X_{i,j}(1 \leq i < j \leq n-1) \), we need to check whether \( \beta \) commutes with \( X_{i,j} \).

Let \( \{j', n\} \in D \) be a horizontal arc. If \( j' \not\in \{i, j\} \), \( \beta \) commutes with \( X_{i,j} \). Suppose \( j' = i \). There are two cases.

If \( j \in D \) is free, then the vertex \( n \) in \( X_{i,j}(f(D) \otimes y_{\lambda_{st}}) \) is free. So, \( X_{i,j}(f(D) \otimes y_{\lambda_{st}}) \in W \) and \( X_{i,j}(f(D) \otimes y_{\lambda_{st}}) = 0 \) in \( V/W \). On the other hand, the vertices \( i, j \) in \( D \) are free. This implies
that $X_{i,j}f(D) = 0$ in $I^{k-1}_{m,n-1}(\delta_1)$. So,

$$X_{i,j} \beta(f(D) \otimes y_{st_0}^\lambda) = \beta(X_{i,j}(f(D) \otimes y_{st_0}^\lambda)) = 0.$$  

If $\{j, h\} \in D$ is a horizontal arc, then there is a $w \in W_{m,n-1}$ such that $X_{ij}f(D) = wf(D)$ (see (3) in the proof of Proposition 8.2). Therefore,

$$\beta(X_{i,j}(f(D) \otimes y_{st_0}^\lambda)) = \beta(w(f(D) \otimes y_{st_0}^\lambda))$$

$$= w\beta(f(D) \otimes y_{st_0}^\lambda)$$

$$= X_{i,j}^\beta(f(D) \otimes y_{st_0}^\lambda).$$

This completes the proof of the claim. Due to (6.5) and Frobenius reciprocity,

$$S^{(k, \lambda)}/W \cong I^{k-1}_{m,n-1}(\delta_1) \otimes_{W_{m,l+1}} \Delta(\lambda) \uparrow_{\nu, \lambda \rightarrow \nu} \cong \bigoplus_{\mu, \lambda \rightarrow \lambda} S^{(k-1, \mu)},$$

which has been proved by Haering-Oldenburg via Jones basic construction in [17].

Since $H_{m,n} \subset B_{m,n}(\delta_1)$, every cell module $S^{(k, \lambda)}$ can be viewed as $H_{m,n}$-module. If $k = 0$, then there is a $H_{m,n}$-module isomorphism

$$S^{(0, \lambda)} \cong \Delta(\lambda) \cong S^\lambda.$$  

A cell module $S^{(k, \lambda)}$ can be decomposed into a direct sum of $\Delta(\mu), \mu \in \Lambda_m^+(n)$. For any $H_{m,n}$-modules $M$ and $N$, let

$$\langle M, N \rangle_{H_{m,n}} = \dim_{\mathbb{C}} \text{Hom}_{H_{m,n}}(M, N).$$

**Theorem 6.2.** For any $\mu \in \Lambda_m^+(n-2k)$, $\lambda \in \Lambda_m^+(n)$ and $2k = |\lambda| - |\mu|$, let $m_{\mu, \lambda} \in \mathbb{N}$ be given as in Theorem 4.6. Then $S^{(k, \mu)} \cong \bigoplus_{\lambda \in \Lambda_m^+(n)} m_{\mu, \lambda} S^{\lambda, \lambda} \cong \bigoplus_{\lambda \in \Lambda_m^+(n)} m_{\mu, \lambda} \Delta^{(\lambda)}$.

**Proof.** Suppose $l = n - 2k$. The vector space $I^{k'}_{m,n}(\delta_1)$ introduced in the proof of Theorem 6.1 is a cyclic $W_{m,n} \times W_{m,l}$-module in which $W_{m,n}$ acts on the left and $W_{m,l}$ acts on the right. Let $D_1 = \text{top}(X_1X_2 \cdots X_{2k-1}) \in P(n, k)$. Then

$$I^{k'}_{m,n}(\delta_1) \cong \text{Ind}_G^{W_{m,n} \times W_{m,l}}(f(D_1)).$$

where $G = \text{stab}_{W_{m,n} \times W_{m,l}}(f(D_1))$. A direct computation shows that

$$G = \{(\pi, \sigma t^1_i \cdots t^l_i, t^1_i \cdots t^l_i \sigma) \mid \pi \in \mathbb{Z}_m, B_k, \sigma \in S_l, 0 \leq e_j \leq m - 1, 1 \leq j \leq n - 2k\}.$$
Therefore, there are two functors $F, G$. A direct computation shows that

\[
\langle S^\lambda, S^{(k,\mu)} \rangle_{H_{m,n}} = \langle S^\lambda \otimes_C S^{\mu'}, S^{(k,\mu)} \otimes_C S^{\mu'} \rangle_{H_{m,n} \times H_{m,t}} = \langle S^\lambda \otimes_C S^{\mu'}, I_{m,n}^k(\delta_1) \otimes_{H_{m,n}-2k} (S^{\mu'} \otimes_C S^{\mu'}) \rangle_{H_{m,n} \times H_{m,t}} = \langle S^\lambda \otimes_C S^{\mu'}, I_{m,n}^k(\delta_1) \rangle_{H_{m,n} \times H_{m,t}} = m_{\mu',\lambda}
\]

\[\square\]

7. Induction and Restriction

In this section, we discuss $B_{m,n}(\delta_1)$ over $\mathbb{C}$. For simplicity, we always assume $\delta_i \neq 0$ for some $i, 0 \leq i \leq m-1$. We will not need the results on the cases $\delta_i = 0$ for all $0 \leq i \leq m-1$ when we prove the main theorem.

Following [21, 7], we study two functors $F$ and $G$. The main purpose of this section is to set up a relation between $\langle S^{(0,\lambda)}, S^{(k,\mu)} \rangle$ and $\langle S^{(0,\lambda)}, S^{(1,\mu)} \rangle$ for any $k \in \mathbb{N}$. It should be noted that most of the results in this section can be proved by arguments similar to those in [7]. We only point out the difference if necessary. Let

\[(7.1)\]

\[X_{i,j} = e^{s,t} X_{i,j} t_{i}^{t}\]

where $X_{i,j}$ is defined in (5.2). The following is the definitions of two functors $F$ and $G$.

**Definition 7.1.** Suppose $\delta_i \neq 0$ for some $0 \leq i \leq m-1$. Define $e = \delta_i^{-1} X_{n-1,n}^{-1}$. Then $e^2 = e$.

A direct computation shows that

\[eB_{m,n}(\delta_1)e \cong B_{m,n-2}(\delta_1)\].

Therefore, there are two functors $F$ and $G$ defined by setting

\[F : B_{m,n}(\delta_1) \text{-mod} \to B_{m,n-2}(\delta_1) \text{-mod}, \text{ with } F(M) = eM\]

\[G : B_{m,n-2}(\delta_1) \text{-mod} \to B_{m,n}(\delta_1) \text{-mod}, \text{ with } G(M) = B_{m,n}(\delta_1)e \otimes B_{m,n-2}(\delta_1) M\].

By a general result in [11, §6.2], $FG = 1$.

**Proposition 7.2.** Suppose $\delta_i \neq 0$ for some $0 \leq i \leq m-1$.

(a) $G(\phi) \neq 0$ for any non-zero $B_{m,n-2}(\delta_1)$-homomorphism $\phi$.

(b) $G(S^{(k-1,\lambda)}) = S^{(k,\lambda)}$, $G(S^{(k-1,\lambda)} *) = S^{(k,\lambda)} *$ where $\lambda \in \Lambda^+_m(n-2k), k \geq 1$. 
In particular, for 
\[ F(S^{(k, \lambda)}_D) = S^{(k-1, \lambda)}_D, \]
\[ F(S^{(k, \lambda)}_D) = S^{(k-1, \lambda)}_D, \]
where \( \lambda \in \Lambda_m^+(n-2k) \), \( k \geq 1 \).

Proof. (a) follows from a general result in [11, §6.2]. Since \( FG = 1 \), (c) follows from (b) by applying the functor \( F \). We prove (b) as follows. We claim that there is a \((B_{m,n}(\delta_1), W_{m,n-2k})\)-bimodule isomorphism

\[ (7.2) \quad I_{m,n}(\delta_1) \cong B_{m,n}(\delta_1)e \otimes B_{m,n-2}(\delta_1) I_{m,n-2}(\delta_1). \]

In fact, for any \( D_1 e \otimes D_2 \in B_{m,n}(\delta_1)e \otimes B_{m,n-2}(\delta_1) I_{m,n-2}(\delta_1) \) with \( D_1 \in B_{m,n}(\delta_1), D_2 \in I_{m,n-2}(\delta_1) \), let \( e^i = X_{l+1,l+2}^{0} X_{l+3,l+4}^{0} \cdots X_{n-3,n-2}^{0}, \ l = n - 2k \). Then \( e^i \in I_{m,n-2}(\delta_1) \) and there is a \( w \in W_{m,n-2} \), such that \( D_2 = we^i \). So,

\[
D_1 e \otimes D_2 = \delta_i^{-1} D_1 we \otimes (X_{l+1,l+2}^{0} X_{l+3,l+4}^{0} \cdots X_{n-3,n-2}^{0}) e^i
\]

Assume \( D_3 = D_1 w X_{l+1,l+2}^{0} \cdots X_{n-3,n-2}^{0} X_{n-1,n}^{0} \). Then \( D_3 \in I_{m,n}(\delta_1) \). By a direct computation, one can verify that the map \( \phi : B_{m,n}(\delta_1)e \otimes B_{m,n-2}(\delta_1) I_{m,n-2}(\delta_1) \rightarrow I_{m,n}(\delta_1) \) defined by setting \( \phi(D_3 e \otimes e^i) = D_3 \) is the desired isomorphism.

So,

\[
G(S^{(k-1, \lambda)}_D) = B_{m,n}(\delta_1)e \otimes B_{m,n-2}(\delta_1) \left( I_{m,n-2}^{-1}(\delta_1) \otimes W_{m,n-2k} \Delta(\lambda) \right)
\]

\[
= \left( B_{m,n}(\delta_1)e \otimes B_{m,n-2}(\delta_1) I_{m,n-2}(\delta_1) \right) \otimes W_{m,n-2k} \Delta(\lambda)
\]

\[
= I_{m,n}(\delta_1) \otimes W_{m,n-2k} \Delta(\lambda) = S^{(k, \lambda)}. \]

The second isomorphism can be proved similarly. □

Definition 7.3. For any \( B_{m,n}(\delta_1) \)-modules \( M \) and \( N \), let

\[
\langle M, N \rangle_{B_{m,n}(\delta_1)} = \dim_{\mathbb{C}} \text{Hom}_{B_{m,n}(\delta_1)}(M, N).
\]

For simplicity, we replace \( \langle M, N \rangle_{B_{m,n}(\delta_1)} \) by \( \langle M, N \rangle_n \) if there is no confusion.

Theorem 7.4. Suppose \( \delta_i \neq 0 \) for some \( 0 \leq i \leq m-1 \). For any \( \mu \in \Lambda_m^+(n), \mu_1 \in \Lambda_m^+(n-2k) \), \( k_0 \in \mathbb{N}, \langle S^{(k_0, \mu), S^{(k+k_0, \mu_1)}}, S^{(k+k_0, \mu_1)} \rangle_{n+2k_0} = 0 \) if and only if \( \langle S^{(0, \mu), S^{(k, \mu_1)}}, S^{(k, \mu_1)} \rangle_n = 0 \).

Proof. This result can be proved similarly as [7, 5.4]. The only difference is to replace \((1/Q)X_{n+n_0-1,n+n_0} \) in the proof of [7, 5.4] by \( \delta_i^{-1} X_{n+2k_0-1,n+2k_0} \). Interested readers can turn to [7] for details. □

By the similar argument as in the proof of [7, 6.1], the following result can be proved.

Proposition 7.5. Let \( M \) be any \( B_{m,n}(\delta_1) \)-module. Then \( M \uparrow \cong G(M) \downarrow \) as \( B_{m,n+1}(\delta_1) \)-modules. In particular, for \( \| \lambda \| \leq n, S^{(k, \lambda)} \uparrow \cong S^{(k+1, \lambda)} \downarrow \).
Corollary 7.6. Let $\lambda \in \Lambda_m^+(n)$. Then for all $\lambda \rightarrow \nu$, \( \langle S^{(0,\lambda)} \uparrow, S^{(0,\nu)} \rangle_{n+1} \neq 0 \).

**Proof.** By Theorem 6.1, there is a submodule $W \subset S^{(1,\lambda)} \downarrow$ such that
\[
S^{(1,\lambda)} \downarrow / W \cong \bigoplus_{\lambda \rightarrow \nu} S^{(0,\nu)}.
\]
So, \( \langle S^{(1,\lambda)} \downarrow / W, S^{(0,\nu)} \rangle_{n+1} \neq 0 \) and \( \langle S^{(1,\lambda)} \downarrow, S^{(0,\nu)} \rangle_{n+1} \neq 0 \) for all $\lambda \rightarrow \nu$. Via Proposition 7.5, the result follows. \( \square \)

The following result sets up a relation between $k = 1$ and arbitrary $k$. Though the proof is similar to that in [7, 7.1], it is presented as it is a key step in the proof of Theorem 1.1.

**Proposition 7.7.** Suppose $\lambda \in \Lambda_m^+(n), \mu \in \Lambda_m^+(n-2k)$ and \( \langle S^{(0,\lambda)}, S^{(k,\mu)} \rangle_n \neq 0 \). Then for every $\lambda^0 \rightarrow \lambda$, \( \langle S^{(0,\lambda^0)}, S^{(k,\mu)} \rangle_{n-1} \neq 0 \). Furthermore, we have either \( \langle S^{(0,\lambda^0)}, S^{(k-1,\mu^0)} \rangle_{n-1} \neq 0 \) for some $\mu^0 \leftarrow \mu$ or \( \langle S^{(0,\lambda^0)}, S^{(k,\mu^1)} \rangle_{n-1} \neq 0 \) for $\mu^1 \rightarrow \mu$.

**Proof.** Due to Corollary 7.6, \( \langle S^{(0,\lambda^0)} \uparrow, S^{(0,\lambda)} \rangle_n \neq 0 \) for any $\lambda^0 \rightarrow \lambda$. Since $S^{(0,\lambda)} \cong \Delta(\lambda)$ as $H_{m,n}$-modules, $S^{(0,\lambda)}$ must be an irreducible $B_{m,n}(\delta_1)$-module. By assumption, \( \langle S^{(0,\lambda)}, S^{(k,\mu)} \rangle_n \neq 0 \). Hence \( \langle S^{(0,\lambda^0)} \uparrow, S^{(k,\mu)} \rangle_n \neq 0 \). According to Frobenius reciprocity, \( \langle S^{(0,\lambda^0)} \uparrow, S^{(k,\mu)} \rangle_{n-1} \neq 0 \).

Let $V = S^{(k,\mu)} \downarrow$. By Theorem 6.1, there is a submodule $W \subset V$ such that
\[
W \cong \bigoplus_{\mu^1, \mu^1 \leftarrow \mu} S^{(k,\mu^1)}.
\]
Let $S$ be the non-zero image of $S^{(0,\lambda^0)}$ in $V$. Then $S \cong S^{(0,\lambda^0)}$. If $S \subset W$, then there is a $S^{(k,\mu^1)} \supset S$. So, \( \langle S^{(0,\lambda^0)}, S^{(k,\mu^1)} \rangle_{n-1} \neq 0 \). If $S \not\subset W$, then $(S \oplus W)/W \cong S$ is an irreducible submodule of $V/W$. Due to Theorem 6.1,
\[
V/W \cong \bigoplus_{\mu^0, \mu^0 \rightarrow \mu} S^{(k-1,\mu^0)}.
\]
Therefore, $(S \oplus W)/W \subset S^{(k-1,\mu^0)}$ for some $\mu^0 \leftarrow \mu$. Hence \( \langle S^{(0,\lambda^0)}, S^{(k-1,\mu^0)} \rangle_{n-1} \neq 0 \). \( \square \)

8. **Proof of Theorem 1.1**

In this section, we deal with $B_{m,n}(\delta_1)$ over $\mathbb{C}$. First, we introduce $T \in B_{m,n}(\delta_1)$ which is similar to that defined in [7, p655].

**Definition 8.1.** For any $1 \leq i < j \leq n$, let $T = \sum_{1 \leq i < j \leq n} \sum_{s=0}^{m-1} X_{i,j}^s$, where $X_{i,j}^s = X_{i,j}^{s,m-s}$.

The following result generalizes [7, 3.2].
Proposition 8.2. Let $B_{m,n}(\delta_1)$ be a cyclotomic Brauer algebra over $\mathbb{C}$. Let $i_l$, $1 \leq l \leq k$, be the left end point of the $l$-th horizontal arc of $D$, $D \in P(n, k)$. Write $y = f(D) = D \otimes id \otimes D_0$. Then, in $I_{m,n}^l(\delta_1)$,

$$Ty = \left[ \sum_{l=1}^{k} \sum_{s=0}^{m-1} (t_{i_l}^{m-s} \delta_s - t_{i_l}^{m-2s}) + \sum_{1 \leq i < j \leq n} t_{i}^{s} t_{j}^{m-s}(i, j) - \sum_{1 \leq a < b \leq n-2k} t_{a}^{s} t_{b}^{m-s}(a, b) \right] \cdot y$$

where $\sum_{1 \leq a < b \leq n-2k} t_{a}^{s} t_{b}^{m-s}(a, b)$ act on the right of $y$ and the other terms act on the left of $y$.

Proof. This result can be proved similarly as [7, 3.2]. The only difference is that dots on some arcs must be considered. There are four cases about the element $X_{i,j}^s$. We only list the results which can be verified easily.

1. If $\{i, j\}$ is a horizontal arc in which there are $t$ dots, then $X_{i,j}^s y = \delta_{1-s} t_{i}^{s-t} y$.
2. If $i$ and $j$ are free vertices in $D$, then $X_{i,j}^s y = 0$ in $I_{m,n}^l(\delta_1)$.
3. If the vertex $i$ is free in $D$ and $\{j, l\}$ is an arc for some $l \in D$, on which there are $t$ dots, then $X_{i,j}^s y = t_{i}^{s-t} t_{l}^{t-s}(i, l) y$. For example:

4. Suppose $i$ and $j$ are fixed vertices such that $\{i, j\}$ is not an arc. Then there are $l, h \in D$ such that $\{i, l\}$ (resp. $\{j, h\}$) is a horizontal arc in which there are $p$ (resp. $q$) dots. Due to $i < j$, there are three cases.

(a) If $i < l < h < j$, then $X_{i,j}^s y = t_{i}^{s-q} t_{h}^{q-s}(i, h) y$.
(b) If $l < i < j < h$, then $X_{i,j}^s y = t_{i}^{s+p} t_{h}^{-q}(i, h) y$.
(c) If $i < h < l < j$, then $X_{i,j}^s y = t_{i}^{s-q} t_{h}^{q-s}(i, h) y = t_{l}^{s-p} t_{j}^{q-p}(l, j) y = X_{i,j}^s y$

Consider the action of $T$ on $f(D)$. Divide $\{(i, j) \mid 1 \leq i < j \leq n\}$ into four subsets as above.

In the first case,

$$\sum_{(i,j) \in (1)} \sum_{s=0}^{m-1} X_{i,j}^s y = \sum_{1 \leq i < j \leq n} \delta_{s} t_{i}^{m-s} y$$
Lemma 8.3. Let \( \lambda \in \Lambda_m^+(n) \) be an arbitrary partition. Then the central element \( w = \sum_{0 \leq s \leq m-1} t^s t_j^{m-s}(i,j) \) acts on the Specht module \( S^\lambda \) as the scalar \( C_\lambda \),

\[ C_\lambda = m \cdot \sum_{s=1}^{m} \omega_{\lambda^{(s)}(1,2)} \]
Proof. We consider a right Specht module \( S^\lambda \). For left modules, similar results are obtained since there is an anti-involution on \( H_{m,n} \) sending \( s_i \) to \( s_i \), \( 0 \leq i \leq n-1 \). Recall that \( S^\lambda \) is a \( \mathbb{C} \)-free module with a basis (see [9, 2.9])

\[
\{ z_\lambda d \mid d \in \mathfrak{S}_n, \lambda \text{ is standard} \}
\]

where \( z_\lambda = Z_\lambda^m(\lambda) = Z_\lambda^m(\lambda) \) with \( a = [\lambda] = [a_1, a_2, \cdots, a_m] \). Since \( w \) is in the center of \( H_{m,n} \), we only need to consider the action of \( w \) on \( z_\lambda \) (i.e. \( d = 1 \)). Let \( a' = [\lambda'] \) where \( \lambda' \) is the dual partition of \( \lambda \). Then there is a unique \( k, 0 \leq k \leq m-1 \) such that \( a'_k < i \leq a'_{k+1} \). Here \( a'_i = n - a_{m-i} \). Write \( t_i = s_{i-1} \cdots s_{a'_k+1} s_{a'_k+1} s_{a'_k+2} \cdots s_{i-1} \). By (3.6),

\[
(8.3) \pi(a' \cdot (t_i - u_{m-k}) = s_{i-1} \cdots s_{a'_k+1} \pi b' \cdot s_{a'_k+1} \cdots s_{i-1}
\]

with \( b = [a_1, \cdots, a_k-1, \cdots, a_m] \in \Lambda[m, n] \). Obviously, \( a > b \). Due to 3.1(a), \( z_\lambda \cdot (t_i - u_{m-k}) = 0 \) and \( z_\lambda \cdot i = u_{m-k} z_\lambda \).

Consider the action of \( \sum_{s=0}^{m-1} t_i^{s_j m-s} \) for a fixed pair \((i, j)\). There are two cases.

(a) if \( a'_k < i < j < a'_{k+1} \) for some \( k, 0 \leq k \leq m-1 \), then \((i, j) \in \mathfrak{S}_m(a'_k+1, \cdots, a'_{k+1}) \subset \mathfrak{S}_m \) and

\[
z_\lambda \cdot (\sum_{s=0}^{m-1} t_i^{s_j m-s}) = \sum_{s=0}^{m-1} u_{m-k} u_{m-k} \cdot z_\lambda = m \cdot z_\lambda.
\]

(b) Suppose that there are two integers \( 0 \leq k_1 < k_2 \leq m-1 \) such that \( a'_{k_1} < i \leq a'_{k_1+1} < a'_{k_2} + 1 \leq j < a'_{k_2+1} \). Since \( u_i = x_i^{1}, 1 \leq i \leq m, \)

\[
z_\lambda \cdot (\sum_{s=0}^{m-1} t_i^{s_j m-s}) = \sum_{s=0}^{m-1} (x_i^{k_2-k_1}) s \cdot z_\lambda = 0
\]

Thus,

\[
z_\lambda \cdot w = m \cdot Z_\lambda \sum_{(i, j) \in \mathfrak{S}_a} (i, j) v[\lambda]
\]

\[
= m \sum_{s=1}^{m-1} [ x_\lambda(i) v(1) y_\lambda(i)^{x_\lambda(i)} \cdots x_\lambda(m-1) v(m-1) y_\lambda(m)^{x_\lambda(m-1)} ]
\]

\[
\sum_{(i, j) \in \mathfrak{S}_{a_{k_1+1}, \cdots, a_{k_2}}} (i, j) \cdots [ x_\lambda(m) v(m) y_\lambda(m)^{x_\lambda(m)} ]
\]

\[
= m \cdot \sum_{s=1}^{m} \omega_\lambda(s) (1, 2) z_\lambda, \text{ by } [18, 7.3.5].
\]

Suppose \( \mu \) and \( \lambda \) are two \( m \)-partitions. We write \( \mu \subseteq \lambda \) if \( \mu_j^{(k)} \leq \lambda_j^{(k)} \) for any \( j \) and \( k, 1 \leq k \leq m \).

Definition 8.5. For any \( m \)-partitions \( \lambda, \mu \) with \( \mu \subseteq \lambda \), let

\[
C_{\lambda/\mu} = C_{\lambda} - C_{\mu} = m \cdot \sum_{s=1}^{m} \omega_\lambda(s) / \mu(s) (1, 2). \]
For any \(i, 0 \leq i \leq m - 1\), let \(\gamma_i(\delta_0, \cdots, \delta_{m-1}) = \delta_{m-1}u_i^{m-1} + \cdots + \delta_1u_i + \delta_0\) with \(u_i = \xi^i\).

**Theorem 8.6.** If \(\langle S^{(0, \lambda')}, S^{(1, \mu')} \rangle_n \neq 0\) for \(\lambda \in \Lambda^+_m(n), \mu \in \Lambda^+_m(n - 2)\), then either \(\gamma_0(\delta_0, \cdots, \delta_{m-1}) + C_{\lambda/\mu} - m = 0\) or there is an \(i, 1 \leq i \leq m - 1\) such that \(\gamma_i(\delta_0, \cdots, \delta_{m-1}) + C_{\lambda/\mu} = 0\).

**Proof.** Since \(\langle S^{(0, \lambda')}, S^{(1, \mu')} \rangle_n \neq 0\), there is a non-zero \(\phi \in \text{Hom}_{B_m}(S^{(0, \lambda')}, S^{(1, \mu')})\). Take a \(v \in S^{(0, \lambda)}\) with \(\phi(v) \neq 0\). Since \(T = \sum_{1 \leq i < j \leq n} \sum_{s=0}^{m-1} X_{i,j}^s \in M_{n,1}\) (see Lemma 5.2 for \(M_{n,1}\)), \(T\phi(v) = 0\). Let \(v_j\) be the basis elements of \(S^\mu\). Write

\[
\phi(v) = \sum a_{D^j,v_j} f(D^j) \otimes v_j
\]

where \(D^j \in P(n, 1)\) is a labelled parenthesis diagram with one horizontal arc in which there are \(t\) dots. Due to Proposition 8.2,

\[
T\phi(v) = \sum a_{D^j,v_j}(T \cdot f(D^j)) \otimes v_j
\]

\[
= \sum_{1 \leq i < j \leq n} \sum_{s=0}^{m-1} (t_l^m, \delta_s - t_l^{m-2s}) f(D^j) \otimes v_j \quad (\text{the left end point of the arc in } D^j)
\]

\[
+ \sum a_{D^j,v_j}\left[ (\sum_{1 \leq i' < j' \leq n} t_{i'}^{m-s}(i', j')) f(D^j) \otimes v_j - f(D^j) \cdot (\sum_{1 \leq a < b \leq n-2} \sum_{1 \leq s \leq m-1} a_{ab} t_{i,j}^m \phi(v)) \right]
\]

By Theorem 6.2, there is an \(H_{m,n}\)-module isomorphism

\[
S^{(1, \mu')} \cong \bigoplus_{\nu \in \Lambda^+_n(n)} m_{\mu, \nu} S^\nu.
\]

So, \(m_{\mu, \lambda} \neq 0\) and \(\lambda \subseteq \mu\). Since \(S^{(0, \lambda')}\) is irreducible, the image of \(v\) must be in the \(S^\lambda\)-component of \(S^{(1, \mu')}\). Then

\[
\sum a_{D^j,v_j}\left[ (\sum_{1 \leq i' < j' \leq n} t_{i'}^{m-s}(i', j')) f(D^j) \otimes v_j \right]
\]

\[
= \left( \sum_{1 \leq i' < j' \leq n} t_{i'}^{m-s}(i', j') \right) \cdot \left[ \sum a_{D^j,v_j} f(D^j) \otimes v_j \right]
\]

\[
= \left( \sum_{1 \leq i' < j' \leq n} t_{i'}^{m-s}(i', j') \right) \cdot \phi(v)
\]

\[
= C_{\lambda} \phi(v), \quad \text{by Proposition } 8.4
\]

\[
= C_{\lambda} \cdot \left[ \sum a_{D^j,v_j} f(D^j) \otimes v_j \right].
\]

Using Proposition 8.4 again, we have

\[
\left( \sum_{1 \leq a < b \leq n-2} \sum_{1 \leq s \leq m-1} a_{ab} t_{i,j}^m \phi(v) \right) \cdot v_j = C_{\mu} v_j
\]

for any \(v_j \in S^\mu\). Thus

\[
(T\phi(v) = \sum a_{D^j,v_j}\left[ (\sum_{s=0}^{m-1} (t_l^m, \delta_s - t_l^{m-2s}) f(D^j) \otimes v_j + \sum a_{D^j,v_j} C_{\lambda/\mu} f(D^j) \otimes v_j.\right.
\]

\[
(8.4) \quad T\phi(v) = \sum a_{D^j,v_j}\left[ (\sum_{s=0}^{m-1} (t_l^m, \delta_s - t_l^{m-2s}) f(D^j) \otimes v_j + \sum a_{D^j,v_j} C_{\lambda/\mu} f(D^j) \otimes v_j.\right.
\]
Consider the coefficient of \( f(D^t) \otimes v_j \) in (8.4). Suppose \( C = \sum_{s=0}^{m-1} (t_{t-s}^m - t_{t-s}^{-2}) \). Then
\[
C = \begin{cases} 
\sum_{s=0}^{m-1} (\delta_s - 1)t_{t-s}^m & \text{if } m \text{ is odd,} \\
\sum_{s=1}^{m/2} (\delta_{s-1}t_{t-2s+1}^m + (\delta_{2s-2} - 2)t_{t-2s+2}^{-2}) & \text{otherwise.}
\end{cases}
\]

If \( m \) is odd, then the coefficient of \( f(D^t) \otimes v_j \) is
\[
\sum_{h=0}^{m-1} a_{D^h,j} \cdot (\delta_{h-t} - 1) + C_{\lambda/\mu} \cdot a_{D^t,j}, \text{ for all } t
\]
where \( D^h \in P(n, 1) \) can be obtained from \( D^t \) by replacing \( t \) dots in the horizontal arc by \( h \) dots.

Since \( T \phi(v) = 0 \), the coefficient of \( f(D^t) \otimes v_j \) must be zero for all \( D^t \) and \( j \). Therefore, we get a system of equations for \( \{a_{D^h,j}\} \) with the coefficient matrix \( \text{diag}(D_m, D_m, \ldots, D_m) \), where
\[
D_m = \begin{pmatrix}
\delta_0 + C_{\lambda/\mu} - 1 & \delta_1 - 1 & \delta_2 - 1 & \cdots & \delta_{m-1} - 1 \\
\delta_{m-1} - 1 & \delta_0 + C_{\lambda/\mu} - 1 & \delta_1 - 1 & \cdots & \delta_{m-2} - 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\delta_{l-1} - 1 & \delta_{l-2} - 1 & \delta_{l-3} - 1 & \cdots & \delta_0 + C_{\lambda/\mu} - 1
\end{pmatrix}
\]

It is a circulative matrix. Let
\[
h(x) = (\delta_{m-1} - 1)x^{-m} + \cdots + (\delta_1 - 1)x + \delta_0 + C_{\lambda/\mu} - 1.
\]

Then \( |D_m| = h(u_0)h(u_1) \cdots h(u_{m-1}) \), \( u_i = \xi^i \). The following equalities can be verified easily.
\[
h(u_i) = \begin{cases} 
g_i(\delta_0, \ldots, \delta_{m-1}) + C_{\lambda/\mu}, & 1 \leq i \leq m - 1, \\
g_0(\delta_0, \ldots, \delta_{m-1}) + C_{\lambda/\mu} - m, & i = 0.
\end{cases}
\]

Since \( \phi(v) \neq 0 \), \( a_{D^t,j} \neq 0 \) for some \( D^t \) and \( j \). Thus \( |D_m| = 0 \) and \( h(u_i) = 0 \) for some \( 0 \leq i \leq m - 1 \).

If \( m \) is even, then the coefficient matrix is \( \text{diag}(D_m, D_m, \ldots, D_m) \) with
\[
D_m = \begin{pmatrix}
\delta_0 + C_{\lambda/\mu} - 2 & \delta_1 - 2 & \delta_2 - 2 & \cdots & \delta_{m-1} - 2 \\
\delta_{m-1} - 2 & \delta_0 + C_{\lambda/\mu} - 2 & \delta_1 - 2 & \cdots & \delta_{m-2} - 2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\delta_{l-1} - 2 & \delta_{l-2} - 2 & \delta_{l-3} - 2 & \cdots & \delta_0 + C_{\lambda/\mu} - 2
\end{pmatrix}
\]

In this case, \( |D_m| = f(u_0)f(u_1) \cdots f(u_{m-1}) \), \( u_i = \xi^i \), where
\[
f(x) = \delta_{m-1}x^{-m} + (\delta_{m-2} - 2)x^{-m} + \cdots + (\delta_2 - 2)x^2 + \delta_1x + \delta_0 + C_{\lambda/\mu} - 2.
\]

Furthermore, \( f(u_i) = h(u_i) \), \( 0 \leq i \leq m - 1 \). Since \( \phi(v) \neq 0 \), \( |D_m| = 0 \) and hence \( f(u_i) = 0 \) for some \( 0 \leq i \leq m - 1 \). \( \square \)

The following is Theorem 1.1, the main result of this paper. The case \( m = 1 \) is due to Wenzl in [27]. See also [7].
Theorem 8.7. Let $B_{m,n}(\delta_1)$ be a cyclotomic Brauer algebra over $\mathbb{C}$. If $g_i(\delta_0, \cdots, \delta_{m-1}) \notin \mathbb{Z}$ for all $0 \leq i \leq m - 1$, then $B_{m,n}(\delta_1)$ is semi-simple.

Proof. Due to Theorem 5.11, $B_{m,n}(\delta_1)$ is a cellular algebra. It is proved in [12, 3.8] that a cellular algebra over a field is semi-simple if and only if all cell modules are non-isomorphic irreducible.

Suppose $B_{m,n}(\delta_1)$ is not semi-simple. There is at least one non-simple cell module, say $S^{(k_2, \mu)}$. The length of its composition series is strictly greater than 1. Therefore, a simple submodule can be found, say $D^{(k_1, \lambda)} \subset S^{(k_2, \mu)}$. By [12, 3.4], $D^{(k_1, \lambda)}$ is a simple head of $S^{(k_2, \mu)}$. If $(k_1, \lambda) = (k_2, \mu)$, then the multiplicity of $D^{(k_1, \lambda)}$ in $S^{(k_2, \mu)}$ is at least 2, contradicting to [12, 3.6]. So, $(k_1, \lambda) \neq (k_2, \mu)$ and $\langle S^{(k_1, \lambda)}, S^{(k_2, \mu)} \rangle_{|\lambda|+\lambda_1} \neq 0$. By [12, 2.6(i)], $(k_1, \lambda) < (k_2, \mu)$. Due to (5.3), either $k_1 < k_2$ or $k_1 = k_2$ and $\lambda \triangleleft \mu$.

If $k_1 < k_2$, then $|\mu| < |\lambda|$. Since $\sum_{i=0}^{m-1} \delta_i \notin \mathbb{Z}$, there is at least $i$, $0 \leq i \leq m - 1$ such that $\delta_i \neq 0$. Due to Theorem 7.4, we can assume $\lambda \in \Lambda_m^+(n)$. In other words, $\langle S^{(0, \lambda)}, S^{(k, \mu)} \rangle_{|\lambda|} \neq 0$ for some positive integers $k$. Applying Proposition 7.7 repeatedly, one can assume $k = 1$. This contradicts the assumption by Theorem 8.6.

Suppose $k_1 = k_2$. By Theorem 7.4, one can assume

$$\langle S^{(0, \lambda)}, S^{(0, \mu)} \rangle_{|\lambda|} \neq 0$$

This is a contradiction since both $S^{(0, \lambda)}$ and $S^{(0, \mu)}$ are irreducible and $S^{(0, \lambda)} \neq S^{(0, \mu)}$. \qed

Remark 8.8. (a) From the proof of Theorems 8.6-8.7, one may see that $B_{m,n}(\delta_1)$ over $\mathbb{C}$ is not semi-simple for finitely many integer values of $g_i(\delta_0, \delta_1, \cdots, \delta_{m-1})$. If $m = 1$, then $g_0(\delta_0) = \delta_0$. In this case, Theorem 8.7 is Wenzl’s Theorem.

(b) Due to Theorem 5.11, $B_{m,n}(\delta_1)$ over $F$, a splitting field of $x^m - 1$, is a cellular algebra. If $B_{m,n}(\delta_1)$ is semi-simple, then all cell modules $S^{(0, \lambda)}$, $\lambda \in \Lambda_m^+(r)$, are irreducible [12, 3.8], which is equivalent to $H_{m,n}$ being semi-simple over $F$. Therefore, char$F \nmid m!$ and $u_i \neq u_j$, $1 \leq i < j \leq m$, i.e., $F$ contains a primitive $m$-th root of unity, say $\xi$. This is a necessary condition for $B_{m,n}(\delta_1)$ to be semi-simple. Let $F$ be such a field. It should be noted that the results in sections 3-7 hold true. Also, [18, 7.3.5] holds for arbitrary field. Therefore, one can state Theorem 1.1 via (8.5). However, it is difficult to describe explicitly the values given in (8.5). Therefore, it is difficult for us to get some helpful information via Theorems 8.6-8.7 if $F$ is a finite field.

Motivated by [24, 25], we propose the following problem.

**Problem:** Assume $m > 1$. Are the following conditions equivalent?

(a) $B_{m,n}(\delta_1)$ over a field is semi-simple.
(b) All cell modules $S^{(k,\lambda)}$, $k = 0, 1$, are simple.

If there is a positive answer, it is possible to give a necessary and sufficient condition for $B_{m,n}(\delta_i)$ to be semi-simple over a field $F$. Finally, we point out there is a positive answer for a cyclotomic Temperley-Lieb algebra of type $G(m,1,n)$ with $m > 1$. For details, see [25, 4.6].

References


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