LEIBNIZ CENTRAL EXTENSIONS ON SOME INFINITE
DIMENSIONAL LIE ALGEBRAS

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ABSTRACT

In this paper, all one-dimensional Leibniz central extensions on the algebras of differential
operators over \( \mathbb{C}[t, t^{-1}] \) and \( \mathbb{C}((t)) \), as well as on the quantum 2-torus, the Virasoro-like algebra
and its \( q \)-analog are studied. We determine all nontrivial Leibniz 2-cocycles on these infinite
dimensional Lie algebras.

Key Words: Leibniz 2-cocycle; Witt algebra; Quantum 2-torus; Differential operators algebra;
Virasoro-like algebra; \( q \)-Analog.

1. INTRODUCTION

The concept of Leibniz algebras was first introduced by Loday [11] in the study of the so-called
Leibniz homology of Lie algebras as a “noncommutative” analog of Lie algebra homology initially
found by Cuvier [1] and Loday [12] respectively. Loday-Pirashvili [13] established the concept of
universal enveloping algebras of Leibniz algebras and interpreted the Leibniz (co)homology \( HL_\ast \)
(resp. \( HL^\ast \)) as a Tor-functor (resp. Ext-functor). By definition Lie algebras are naturally Leibniz
algebras. Recently, Gao [2] obtained an interesting criterion that one may distinguish affine or
non-affine Kac-Moody algebras merely via judging their corresponding second Leibniz homology
groups being nonvanishing or not.

As in the Lie algebras case, Leibniz central extensions also play an important role in the theory
of Leibniz algebras. Loday-Pirashvili [13] proved that the Virasoro algebra \( Vir \) is a universal central
extension of \( \text{Der}(\mathbb{C}[t, t^{-1}]) \) both in the Lie framework and in the Leibniz framework. The aim
of this paper is to determine all nontrivial Leibniz 2-cocycles on the infinite dimensional Lie algebras
of differential operators over \( \mathbb{C}[t, t^{-1}] \) and over \( \mathbb{C}((t)) \), as well as on the recently-interested algebraic
objects: the quantum 2-torus, the Virasoro-like algebra and its \( q \)-analog.

First, we introduce these infinite dimensional Lie algebras for the later use.

Denote by \( \mathbb{Z}, \mathbb{N} \) and \( \mathbb{C} \) the ring of integers, the set of non-negative integers and the field
of complex numbers, respectively. Let \( \mathbb{C}[t, t^{-1}] \) be the algebra of Laurent polynomials over \( \mathbb{C} \). It is
well known that the Witt algebra \( W \) over \( \mathbb{C}[t, t^{-1}] \) is the derivation algebra of \( \mathbb{C}[t, t^{-1}] \) and the
Virasoro algebra \( Vir \) is its universal central extension. As a vector space over \( \mathbb{C} \), \( W = \text{Span}_\mathbb{C}\{e_m =
$t^{m+1}\partial \mid m \in \mathbb{Z}$}, its Lie bracket is given by

\[ [\epsilon_m, \epsilon_n] = t^{m+1}\partial(t^{n+1})\partial - t^{n+1}\partial(t^{m+1})\partial = (n-m)\epsilon_{m+n}. \]

Denote $D = \text{Diff } \mathbb{C}[t, t^{-1}]$ the algebra of differential operators over $\mathbb{C}[t, t^{-1}]$. As a vector space over $\mathbb{C}$, $D = \text{Span}_{\mathbb{C}}\{t^m\partial^n \mid m \in \mathbb{Z}, \ n \in \mathbb{N}\}$, its Lie bracket is given by

\[ [t^m\partial^n, t^{m_1}\partial^{n_1}] = t^m\partial^n(t^{m_1}\partial^{n_1}) - t^{m_1}\partial^{n_1}(t^m\partial^n) \]

\[ = \sum_{i=0}^{n} \binom{n}{i} t^m \partial^i(t^{m_1})\partial^{n-i} - \sum_{j=0}^{n_1} \binom{n_1}{j} t^{m_1} \partial^j(t^m)\partial^{n_1-j}, \]

where $\partial = d/dt$.

Let $\mathbb{C}((t))$ be the algebra of formal Laurent series over the complex numbers field $\mathbb{C}$, i.e.,

\[ \mathbb{C}((t)) = \left\{ \sum_{i \in \mathbb{Z}, i \geq n_0} a_i t^i \mid a_i \in \mathbb{C}, n_0 \in \mathbb{Z} \right\}. \]

The Witt algebra $W$ over $\mathbb{C}((t))$ is the derivation algebra of $\mathbb{C}((t))$. For $f(t) = \sum a_i t^i \in \mathbb{C}((t))$, we denote $\partial(f(t))$ by $f'(t)$ and $a_{-1}$ by $\text{Res } f$. If $\text{Res } f = 0$, we can define the formal integral of $f(t)$ as

\[ \sum_{i \neq -1} \frac{a_i}{i+1} t^{i+1}, \]

denoted by $\int f$. Then $W = \text{Span}_{\mathbb{C}}\{ f(t)\partial \mid f(t) \in \mathbb{C}((t)) \}$ is a Lie algebra under the bracket operation:

\[ [f(t)\partial, g(t)\partial] = f(t)g'(t)\partial - g(t)f'(t)\partial, \] (1.1)

for any $f(t), g(t) \in \mathbb{C}((t))$.

More generally, denote $\mathcal{D} = \text{Diff } \mathbb{C}((t))$ the algebra of differential operators over $\mathbb{C}((t))$. As a vector space over $\mathbb{C}$,

\[ \mathcal{D} = \text{Span}_{\mathbb{C}}\{ f(t)\partial^l \mid l \in \mathbb{N}, f(t) \in \mathbb{C}((t)) \}, \]

its Lie bracket is given by

\[ [f(t)\partial, g(t)\partial^k] = f(t)\partial^k(g(t)\partial) - g(t)\partial^k(f(t)\partial) \]

\[ = \sum_{i=0}^{l} \binom{l}{i} f(t)\partial^i(g(t))\partial^{k-l+i} - \sum_{j=0}^{k} \binom{k}{j} g(t)\partial^j(f(t))\partial^{k-l-j}, \]

for any $f(t), g(t) \in \mathbb{C}((t))$.

Let $M_n(\mathbb{C})$ be the $n \times n$ matrix algebra, $\text{gl}_n(\mathbb{C}) = M_n(\mathbb{C})^-$ the general linear Lie algebra.

The algebra of differential operators on $\mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ (resp. $\mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}((t))$) is the Lie algebra $\tilde{D} := \text{gl}_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{D}$ (resp. $\tilde{D} := \text{gl}_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{D}$) with Lie bracket

\[ [A \otimes t^m\partial^n, B \otimes t^{m_1}\partial^{n_1}] = AB \otimes t^m\partial^n(t^{m_1}\partial^{n_1}) - BA \otimes t^{m_1}\partial^{n_1}(t^m\partial^n) \]

\[ = AB \otimes \sum_{i=0}^{n} \binom{n}{i} t^m \partial^i(t^{m_1})\partial^{n-n_1+i} \]

\[ - BA \otimes \sum_{j=0}^{n_1} \binom{n_1}{j} t^{m_1} \partial^j(t^m)\partial^{n+n_1-j}, \]
for any \( m, m_1 \in \mathbb{Z}, n, n_1 \in \mathbb{N}, A, B \in \mathfrak{gl}_n(\mathbb{C}) \) (resp.
\[
[A \otimes f(t)\partial^j, B \otimes g(t)\partial^k] = AB \otimes f(t)\partial^j(g(t)\partial^k) - BA \otimes g(t)\partial^k(f(t)\partial^j)
= AB \otimes \sum_{i=0}^{l} \binom{l}{i} f(t)\partial^i(g(t))\partial^{k+l-i} - BA \otimes \sum_{j=0}^{k} \binom{k}{j} g(t)\partial^j(f(t))\partial^{k+l-j},
\]
for any \( f(t), g(t) \in \mathbb{C}(t) \), \( A, B \in \mathfrak{gl}_n(\mathbb{C}) \).

Note that one-dimensional central extensions on the above Lie algebras were studied by Li [7], Li and Wilson [9]. Their second cohomology groups with coefficients in their trivial module \( \mathbb{C} \) are all one-dimensional.

Next, we proceed to introduce the quantum 2-torus, the Virasoro-like algebra and its \( q \)-analog in the following.

By definition, the Virasoro-like algebra is a \( \mathbb{Z} \times \mathbb{Z} \)-graded vector space
\[
\tilde{\mathcal{V}} = \bigoplus_{(m,n) \in \mathbb{Z} \times \mathbb{Z}} \mathcal{C}L_{m,n},
\]
with Lie bracket
\[
[L_{m,n}, L_{m_1,n_1}] = (nm_1 - mn_1) L_{m+m_1,n+n_1}.
\]

**Lemma 1.1.** Any nontrivial 2-cocycle on \( \tilde{\mathcal{V}} \) is equivalent to \( \varphi \):
\[
\varphi(L_{m,n}, L_{m_1,n_1}) = \delta_{m+m_1,0} \delta_{n+n_1,0} (m \varphi(L_{1,0}, L_{-1,0}) + n \varphi(L_{0,1}, L_{0,-1})),
\]
for any \( m, n, m_1, n_1 \in \mathbb{Z} \).

Let \( \mathcal{C}_q[x, x^{-1}, y, y^{-1}] \) be a quantum 2-torus, which is an associative algebra generated by \( x, x^{-1}, y, y^{-1} \) with relations
\[
xx^{-1} = yy^{-1} = 1, \quad xy = qyx.
\]
As a vector space over \( \mathbb{C} \), it has a basis \( \{x^iy^j \mid i, j \in \mathbb{Z}\} \). For convenience, denote \( \mathcal{C}_q \) the Lie algebra of \( \mathcal{C}_q[x, x^{-1}, y, y^{-1}] \). Its Lie bracket is given by:
\[
[x^my^n, x^{m_1}y^{n_1}] = (q^{mn_1} - q^{mn}) x^{m+m_1}y^{n+n_1}, \quad \forall \ m, n \in \mathbb{Z}.
\]

Let \( \mathcal{V}_q \) be the Lie algebra of all inner derivations of \( \mathcal{C}_q \), which is called the \( q \)-analog of Virasoro-like algebra. Let \( E_{m,n} := \text{ad} x^m y^n \) for any \( m, n \in \mathbb{Z} \).

All nontrivial 2-cocycles on \( \mathcal{V}_q \) were determined respectively by Kirkman, et al [6], and Meng, et al [14].

**Lemma 1.2.** The second cohomology group of the Lie algebra \( \mathcal{V}_q \) with coefficients in the trivial module \( \mathbb{C} \) is a two-dimensional vector space when \( q \) is not a root of unity. Each nontrivial 2-cocycle on the Lie algebra \( \mathcal{V}_q \) is equivalent to \( \psi \), where
\[
\psi(E_{m,n}, E_{m_1,n_1}) = \delta_{m,-m_1} \delta_{n,-n_1} q^{-mn} (m \psi(E_{1,0}, E_{-1,0}) + n \psi(E_{0,1}, E_{0,-1})),
\]
for any \( m, n, m_1, n_1 \in \mathbb{Z} \).
2. BASICS: LEIBNIZ ALGEBRA AND LEIBNIZ 2-COCYCLE

We recall some notions of Leibniz algebra and its (co)homology (see [10], [11] and [13]).

A Leibniz algebra $L$ is a vector space over a field $F$ equipped with a $F$-bilinear map

$$[-,-]: L \times L \to L$$

satisfying the Leibniz identity

$$[[a,b],c] + [[a,c],b] = 0, \quad \forall a, b, c \in L.$$  \hspace{1cm} (2.1)

Obviously, a Lie algebra is a Leibniz algebra. A Leibniz algebra is a Lie algebra if and only if

$$[x,x] = 0, \quad \forall x \in L.$$

Let $L$ be a Leibniz algebra, then $M$ is said a representation of $L$ if $M$ is an $F$-vector space equipped with two actions (left and right) of $L$:

$$[-,-]: L \times M \to M \quad \text{and} \quad [-,-]: M \times L \to M$$

satisfying the following three axioms

(MLL) \hspace{1cm} $[[m, x], y] - [[m, y], x] = 0$

(LML) \hspace{1cm} $[x, [m, y]] - [[x, y], m] = 0$

(LLM) \hspace{1cm} $[x, [y, m]] - [[x, m], y] = 0$

for any $m \in M$ and $x, y \in L$.

Denote $C^n(L, M) := \text{Hom}_F(L^\otimes n, M), \quad n \geq 0$.

Let $d^n : C^n(L, M) \to C^{n+1}(L, M)$ be an $F$-homomorphism defined by

$$(d^n f)(x_1, \ldots, x_{n+1}) := [x_1, f(x_2, \ldots, x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}), x_i]$$

$$+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_{n+1}).$$

where $f \in C^n(L, M)$. Clearly,

$$d^{n+1} d^n = 0, \quad n \geq 0.$$

Therefore, $(C^*(L, M), d)$ is a cochain complex, whose cohomology is called the cohomology of the Leibniz algebra $L$ with coefficients in the representation $M$:

A Leibniz 2-cocycle on $L$ is a bilinear $F$-valued form $\psi$ satisfying the following condition:

$$\psi(a, [b, c]) = \psi([a, b], c) - \psi([a, c], b), \quad \forall a, b, c \in L. \quad (2.2)$$

For convenience, a 2-cocycle in the Lie algebras case is called a Lie 2-cocycle. If a Leibniz 2-cocycle $\psi$ is in addition anti-symmetric, by definition $\psi$ is then a Lie 2-cocycle.

As in the Lie algebras case, one-dimensional Leibniz central extensions of a Leibniz algebra $L$ are uniquely determined by Leibniz 2-cocycles on $L$. If a Leibniz 2-cocycle $\psi$ is induced by a linear function $f$ on $L$, that is, $\psi = \alpha f$, where

$$\alpha_f(x, y) = f([x, y]) \quad (2.3)$$

for any $x, y \in L$, then such a $\psi$ is called trivial while the corresponding one-dimensional Leibniz central extension is also trivial, i.e., it is isomorphic to $L \oplus Fc$ as a direct sum of Leibniz ideals. A Leibniz 2-cocycle $\varphi$ is equivalent to a Leibniz 2-cocycle $\psi$, if $\varphi - \psi$ is trivial.

Given a Leibniz 2-cocycle $\alpha$ on $L$, one can construct a Leibniz central extension of $L$ in a canonical way as follows

$$[x + \lambda c, y + \mu c]_0 = [x, y] + \alpha(x, y)c, \quad \forall x, y \in L, \lambda, \mu \in F,$$

where $[\cdot, \cdot]$ is the Leibniz bracket on $L$ and $[\cdot, \cdot]_0$ is the Leibniz bracket on $L \oplus Fc$. Every one-dimensional Leibniz central extension of $L$ can be obtained in this way.

The following two Lemmas will be used in the proofs of our main results (the corresponding two Lemmas in the Lie algebras case were given in [9]).

**Lemma 2.1.** Let $L$ be a Leibniz algebra and $S$ a subset of $L$ such that $S$ spans $L$ and for each $x \in S$, $x = [y_x, z_x]$ for some $y_x, z_x \in L$. If a Leibniz 2-cocycle $\varphi$ satisfies $\varphi(y_x, z_x) = 0$ for any $x \in S$, then either $\varphi = 0$ or $\varphi$ is nontrivial.

**Proof:** Suppose that $\varphi$ is trivial, so that $\varphi = \alpha_f$ for some linear function $f$. Then for each $x \in S$,

$$f(x) = f([y_x, z_x]) = \varphi(y_x, z_x) = 0.$$

Thus $f = 0$ since $S$ spans $L$. This implies that $\varphi = \alpha_f = 0$. \[\square\]

**Lemma 2.2.** Let $L$ be a Leibniz algebra and $\varphi$ a Leibniz 2-cocycle on $L$. Suppose there are linear endomorphisms $E$ and $F$ of $L$ such that

$$\varphi(Ex, y) = \varphi(x, Fy)$$

for any $x, y \in L$, $E$ is surjective and $F$ is local nilpotent (i.e., for any $y \in L$, there is a positive integer $n$ such that $F^n y = 0$). Then the Leibniz 2-cocycle $\varphi$ is 0.

**Proof:** For any $y \in L$, let $n$ be a positive integer $n$ such that $F^n y = 0$. Since $E$ is surjective, we have $x' \in L$ such that $x = E^n x'$. Thus,

$$\varphi(x, y) = \varphi(E^n x', y) = \varphi(x', F^n y) = 0. \quad \square$$

In what follows, we mainly discuss one-dimensional Leibniz central extensions on the above-mentioned Lie algebras. All nontrivial Leibniz 2-cocycles on the algebras of differential operators over $C[t, t^{-1}]$ and over $C((t))$ respectively are determined in Section 3. They are consistent with
their corresponding Lie 2-cocycles. Meanwhile, all nontrivial Leibniz 2-cocycles on the quantum 2-torus, the Virasoro-like algebra and its $q$-analog are described in Section 4 when $q$ is not a root of unity, which distinguish from the corresponding Lie 2-cocycles only on a specific central element (see Theorems 4.1 and 4.2).

3. LEIBNIZ 2-COCYCLES ON ALGEBRAS OF DIFFERENTIAL OPERATORS

In [13], all nontrivial Leibniz 2-cocycles on the Witt algebra were determined by Loday and Pirashvili.

Lemma 3.1. Each Leibniz 2-cocycle on the Witt algebra $W$ is equivalent to $\alpha$:

$$\alpha(e_i, e_j) = \delta_{i,-j}(i^3 - i),$$

where $\{e_i \mid i \in \mathbb{Z}\}$ is the basis of $W$ given in Section 1. Equivalently,

$$HL^2(W, \mathbb{C}) = \mathbb{C} \alpha = H^2(W, \mathbb{C}).$$

Now we shall prove the following

Theorem 3.1. $\dim HL^2(W, \mathbb{C}) = 1$.

Proof: (1) Noticing that $H^2(W, \mathbb{C}) \subseteq HL^2(W, \mathbb{C})$ (due to the Witt algebra $W$ being a Leibniz one) as well as $\dim H^2(W, \mathbb{C}) = 1$ (by Theorem 1 in [9]), we need only to show

$$HL^2(W, \mathbb{C}) \subseteq H^2(W, \mathbb{C}).$$

To this end, for any nontrivial $\varphi \in HL^2(W, \mathbb{C})$, one can define a linear function $f_\varphi : W \to \mathbb{C}$ by

$$f_\varphi(g(t)\partial) = \varphi(\partial, \int g(t)\partial),$$

for all $g(t) \in \mathbb{C}((t))$ with $\text{Res} \, g(t) = 0$, and

$$f_\varphi(t^{-1}\partial) = \frac{1}{2} \varphi(t^{-1}\partial, t\partial).$$

Then $\varphi_1 := \varphi - \alpha f_\varphi$ is a nontrivial Leibniz 2-cocycle on $W$ which is equivalent to $\varphi$.

For $f(t) = \sum_i a_i t^i \in \mathbb{C}((t))$ with $a_0 = f(0) = 0$, one has $\text{Res} \, f'(t) = 0$ and $f(t) = \int f'(t)$ (due to $a_0 = 0$), and

$$\varphi_1(\partial, f(t)\partial) = \varphi(\partial, f(t)\partial) - f_\varphi(f'(t)\partial) = \varphi(\partial, f(t)\partial) - \varphi(\partial, \int f'(t)\partial) = 0, \quad (3.1)$$

$$\varphi_1(t^{-1}\partial, t\partial) = \varphi(t^{-1}\partial, t\partial) - f_\varphi([t^{-1}\partial, t\partial]) = \varphi(t^{-1}\partial, t\partial) - 2f_\varphi(t^{-1}\partial) = 0. \quad (3.2)$$

Applying the Leibniz rule (2.2), one has $\varphi_1(\partial, \partial) = \varphi_1(\partial, [\partial, t\partial]) = \varphi_1([\partial, \partial], t\partial) - \varphi_1([\partial, t\partial], \partial) = -\varphi_1(\partial, \partial)$, which implies

$$\varphi_1(\partial, \partial) = 0. \quad (3.3)$$

Similarly, one gets

$$\varphi_1(t\partial, t^{-1}\partial) = -\frac{1}{2} \varphi_1(t\partial, [t\partial, t^{-1}\partial]) = \frac{1}{2} \varphi_1([t\partial, t^{-1}\partial], t\partial) = -\varphi_1(t^{-1}\partial, t\partial) = 0. \quad (3.4)$$
Lemma 3.2. For $\varphi \in HL^2(\mathcal{W}, \mathbb{C})$ and for $f(t) \in \mathbb{C}((t))$ with $\text{Res} f(t) = 0$, there hold

(i) $\varphi_1(t^s \partial, \mathcal{W}) = 0$ for $s = 0, 1, 2$; and $\varphi_1(t^3 \partial, f(t) \partial) = 0$.

(ii) $\varphi_1(\mathcal{W}, t^s \partial) = 0$ for $s = 0, 1, 2$; and $\varphi_1(f(t) \partial, t^3 \partial) = 0$.

(iii) $\varphi_1(t^{-1} \partial, t^3 \partial) = -\varphi_1(t^3 \partial, t^{-1} \partial)$.

Proof: (i) follows from adopting the same argument (only involved the Leibniz rule) as those of Lemmas 3 to 5 (see p. 2572–2573) in [9] to the Leibniz 2-cocycle $\varphi_1$.

(ii) For $f(t) \in \mathbb{C}((t))$ with $\text{Res} f(t) = 0$ and $s = 0, 1, 2, 3$, using (i), we have

$$\varphi_1(f(t) \partial, t^s \partial) = \varphi_1(\partial, \int f \partial), t^s \partial) = \varphi_1(\partial, \int f \partial, t^s \partial] + \varphi_1(\partial, t^s \partial], \int f \partial) = 0,$$

which, together with (3.2), implies $\varphi_1(t^{-1} \partial, t^s \partial) = 0$ for $s = 0, 1, 2$, owing to

$$\varphi_1(t^{-1} \partial, t^s \partial) = \frac{1}{2} \varphi_1([t^{-1} \partial, t \partial], t^s \partial) \quad = \frac{1}{2} \varphi_1(t^{-1} \partial, [t \partial, t^s \partial]) + \frac{1}{2} \varphi_1([t^{-1} \partial, t^s \partial], t \partial) \quad = \frac{s-1}{2} \varphi_1(t^{-1} \partial, t^s \partial).$$

(iii) Using (i), we get

$$\varphi_1(t^{-1} \partial, t^3 \partial) = \frac{1}{2} \varphi_1([t^{-1} \partial, t \partial], t^3 \partial) = \frac{1}{2} \varphi_1([t \partial, t^{-1} \partial], t^3 \partial) \quad = -\frac{1}{2} \varphi_1(t \partial, [t^{-1} \partial, t^3 \partial]) - \frac{1}{2} \varphi_1([t \partial, t^3 \partial], t^{-1} \partial) \quad = -\varphi_1(t^3 \partial, t^{-1} \partial).$$

(II) According to [9], $H^2(\mathcal{W}, \mathbb{C}) = 1$, we take a Lie 2-cocycle $\alpha : \mathcal{W} \times \mathcal{W} \rightarrow \mathbb{C}$ as follows

$$\alpha \left( \sum_i a_i t^{i+1} \partial, \sum_j b_j t^{j+1} \partial \right) = \sum_i a_i b_{i-1} (i^3 - i).$$

Then $\alpha \in HL^2(\mathcal{W}, \mathbb{C})$. By definition, we have

$$\alpha(t^s \partial, \mathcal{W}) = 0 = \alpha(\mathcal{W}, t^s \partial), \quad (s = 0, 1, 2),$$

$$\alpha(t^3 \partial, t^{-1} \partial) = 6 = -\alpha(t^{-1} \partial, t^3 \partial),$$

and for $f(t) \in \mathbb{C}((t))$ with $\text{Res} f(t) = 0$,

$$\alpha(t^3 \partial, f(t) \partial) = 0 = \alpha(f(t) \partial, t^3 \partial).$$

Let $S$ be the subset of $\mathcal{W}$ given by

$$S = \{ t^{-1} \partial \} \cup \{ f(t) \partial \in \mathcal{W} \mid \text{Res} f(t) = 0 \}.$$
Lemma 2.1 shows that $\alpha$ is nontrivial, that is, $\alpha$ is not a Leibniz 2-coboundary.

(III) Suppose that $\varphi_1(t^3 \partial, t^{-1} \partial) = 6r$ for some $r \in \mathbb{C}$. Define $\varphi_2 := \varphi_1 - r\alpha$, then combining with Lemma 3.2 (iii), we have

$$\varphi_2(t^s \partial, W) = 0 = \varphi_2(W, t^s \partial), \quad s = 0, 1, 2, 3.$$ 

In what follows we show that $\varphi_2 \equiv 0$, completing the proof of Theorem 1.

Let $\text{ad} t^s \partial : W \to W$ be the adjoint operator, then for $s = 0, 1, 2, 3$, and $f(t), g(t) \in \mathbb{C}((t))$, we have

$$\varphi_2(\text{ad} t^s \partial (f(t) \partial), g(t) \partial) = \varphi_2([[t^s \partial, f(t) \partial], g(t) \partial]) = -\varphi_2([f(t) \partial, t^s \partial], g(t) \partial)$$

$$= -\varphi_2(f(t) \partial, [t^s \partial, g(t) \partial]) - \varphi_2([f(t) \partial, g(t) \partial], t^s \partial)$$

$$= -\varphi_2(f(t) \partial, \text{ad} t^s \partial (g(t) \partial)).$$

Noting that for $f(t) = \sum a_i t^{i+1}, g(t) = \sum b_j t^{j+1} \in \mathbb{C}((t))$,

$$\text{(ad} \partial)^2\text{ad} (t^3 \partial) (f(t) \partial) = \sum_i (i^3 + 3i^2 - 4i - 12)a_i t^{i+1} \partial,$$

$$\text{(ad} t \partial)^s (f(t) \partial) = \sum_i i^s a_i t^{i+1} \partial,$$

we consider the following linear endomorphisms

$$E = (\text{ad} \partial)^2\text{ad} (t^3 \partial) - 3(\text{ad} t \partial)^2 + 4 \text{ad} t \partial,$$

$$F = -\text{ad} (t^3 \partial) (\text{ad} \partial)^2 - 3(\text{ad} t \partial)^2 - 4 \text{ad} t \partial,$$

which satisfy $E(f(t) \partial) = -12f(t) \partial, F(g(t) \partial) = 0$ and

$$\varphi_2(E(f(t) \partial), g(t) \partial) = \varphi_2(f(t) \partial, F(g(t) \partial)).$$

Finally, it deduces from Lemma 2.2 that $\varphi_2 \equiv 0$, or equivalently, $\varphi_1 = r\alpha$ (notice that here the $r \neq 0$, since otherwise, $\varphi = \alpha f_\varphi$ is trivial, which is contrary with our choice for $\varphi$ in (I)), which shows $HL^2(W, \mathbb{C}) \subseteq H^2(W, \mathbb{C})$.

Theorem 3.2. $\dim HL^2(D, \mathbb{C}) = 1$.

Proof: (I) Let $\varphi$ be a nontrivial Leibniz 2-cocycle on $D$. Define a linear function $f_\varphi : D \to \mathbb{C}$ by

$$f_\varphi(t^m \partial^n) = \frac{1}{n+1} \varphi(t^m \partial^{n+1}, t),$$

for $m \in \mathbb{Z}, n \in \mathbb{N}$. Then $\psi := \varphi - \alpha f_\varphi$ is a Leibniz 2-cocycle on $D$ and equivalent to $\varphi$. We have

$$\psi(t^m \partial^{n+1}, t) = 0,$$

for $m \in \mathbb{Z}, n \in \mathbb{N}$.

For $m \in \mathbb{Z}, n \in \mathbb{N}$, setting $a = b = t, c = t^m \partial^{n+1}$ in (2.2), we obtain

$$\psi(t^m \partial^n, t) = -\psi(t, t^m \partial^n).$$

(3.5)

So

$$\psi(t, t^m \partial^{n+1}) = -\psi(t^m \partial^{n+1}, t) = 0.$$  

(3.6)
Lemma 3.3. For any \( m \in \mathbb{Z} \), \( m \neq -1 \), \( n \in \mathbb{N} \),
\[
\psi(t^m, t) = -\psi(t, t^m) = 0. \tag{3.7}
\]

Proof of Lemma 3.3: Since there holds a relation \( \partial t - t \partial = 1 \) in \( D \), which implies \([ t\partial, t ] = t\), we have
\[
\psi(t^m, t) = \psi(t^m, [t\partial, t]) = \psi([t^m, t\partial], t) = -m \psi(t^m, t).
\]
Since \( m \neq -1 \), \( \psi(t^m, t) = 0 \). So \( \psi(t, t^m) = 0 \).

(II) According to \([7]\), we take a Lie 2-cocycle \( \beta : D \times D \to \mathbb{C} \) as follows
\[
\beta(t^{m+k}\partial^k, t^{m'+l}\partial^l) = \delta_{m,m'}(1-k!l!)(m+k \choose k+l+1).
\]

Then \( \beta \in HL^2(D, \mathbb{C}) \). Let \( S \) be the subset of \( D \) given by
\[
S = \{ t^m \partial^n | m \in \mathbb{Z}, n \in \mathbb{N} \}.
\]

By definition, we have
\[
\beta(t, t^{-1}) = 1 = -\beta(t^{-1}, t), \quad \beta(t^m, t) = 0 = \beta(t, t^m), \quad \text{for } m \neq -1,
\]
\[
\beta(t^m \partial^{n+1}, t) = 0 = \beta(t, t^m \partial^{n+1}), \quad \text{for } m \in \mathbb{Z}, n \in \mathbb{N},
\]
which imply by the formula \( t^m \partial^n = \frac{1}{n+1} [ t^m \partial^{n+1}, t ] \) and Lemma 2.1 that \( \beta \neq 0 \) in \( HL^2(D, \mathbb{C}) \).

(III) Now let \( \psi(t, t^{-1}) = s \), noting \( \psi(t^{-1}, t) = -s \) (by (3.5)), we define \( \psi_1 := \psi - s \beta \) so that
\[
\psi_1(t, t^{-1}) = 0 = \psi_1(t^{-1}, t). \quad \text{Then by (3.6), (3.7) and (II) we obtain}
\]
\[
\psi_1(t, D) = \psi_1(D, t) = 0,
\]
which, together with \( \psi_1 \big( t^m \partial^k, [t, t^n \partial^l] \big) = \psi_1 \big( [t^m \partial^k, t], t^n \partial^l \big) - \psi_1 \big( [t^m \partial^k, t^n \partial^l], t \big) \), gives
\[
\psi_1(\text{ad } t(t^m \partial^k), t^n \partial^l) = -\psi_1(t^m \partial^k, \text{ad } t(t^n \partial^l)) \tag{3.8}
\]
for any \( m, n \in \mathbb{Z} \), \( k, l \in \mathbb{N} \).

The formula \( t^m \partial^n = \frac{1}{n+1} [ t^m \partial^{n+1}, t ] \) implies that \( E = \text{ad } t \) is surjective and \( F = -\text{ad } t \) is locally nilpotent. By (3.8) and Lemma 2.2, we have \( \psi_1 \equiv 0 \). This gives \( \psi = s \beta \) with \( s \in \mathbb{C}^* \) (since \( \varphi \) is assumed to be nontrivial), which shows \( HL^2(D, \mathbb{C}) \subseteq H^2(D, \mathbb{C}) \).

We then complete the proof of Theorem 3.2.

By Theorem 2 and Corollary in \([9]\) and similar to the proof of Theorem 3.2, we have

Corollary 3.2. \( \dim HL^2(D, \mathbb{C}) = \dim HL^2(D, \mathbb{C}) = \dim HL^2(D, \mathbb{C}) = 1. \)

Proof: Here we give the proof of the first claim only, as for the other cases, the key points are the definitions of nontrivial 2-cocycles \( \beta \) (see the Remark in the end of \([9]\)).

(I) For any nontrivial \( \varphi \in HL^2(D, \mathbb{C}) \), we define a linear function \( f_\varphi : D \to \mathbb{C} \) by
\[
f_\varphi(g(t)\partial^n) = \frac{1}{n+1} \varphi(g(t)\partial^{n+1}, t),
\]
for \( n \in \mathbb{N} \) and \( g(t) \in \mathbb{C}((t)) \). Then \( \psi = \varphi - \alpha f_\varphi \in HL^2(D, \mathbb{C}) \) and equivalent to \( \varphi \). We have
\[
\psi(g(t)\partial^{n+1}, t) = 0,
\]
for \( n \in \mathbb{N} \) and \( g(t) \in \mathcal{C}(t) \).

Similarly, setting \( a = b = t, \ c = g(t)\partial^{n+1} \) in (2.2), we obtain
\[
\psi(g(t)\partial^n, t) = -\psi(t, g(t)\partial^n).
\]
(3.9)

Thus
\[
\psi(t, g(t)\partial^{n+1}) = -\psi(g(t)\partial^{n+1}, t) = 0.
\]
(3.10)

**Lemma 3.4.** For any \( f(t) \in \mathcal{C}(t) \) and \( \operatorname{Res} f(t) = 0 \), then
\[
\psi(f(t), t) = -\psi(t, f(t)) = 0.
\]
(3.11)

**Proof of Lemma 3.4:** Since \([t\partial, t] = t \in \mathcal{D}\), we have
\[
\psi(g(t), t) = \psi(g(t), [t\partial, t]) = \psi([g(t), t\partial], t) = -\psi(tg'(t), t),
\]
\[
\psi(t, g(t)) = \psi([t\partial, t], g(t)) = -\psi([t, t\partial], g(t)) = -\psi(t, tg'(t)),
\]
for any \( g((t)) \in \mathcal{C}(t) \), that is, \( \psi(g(t) + tg'(t), t) = 0 \) and \( \psi(t, g(t) + tg'(t)) = 0 \). Now (3.11) follows from the fact that every \( f(t) \in \mathcal{C}(t) \) with \( \operatorname{Res} f(t) = 0 \) can be written in the form \( g(t) + tg'(t) \) for some \( g(t) \in \mathcal{C}(t) \) with \( \operatorname{Res} g(t) = 0 \). [1]

(II) According to [9], \( \dim H^2(D, \mathcal{C}) = 1 \), we take a Lie 2-cocycle \( \beta : \mathcal{D} \times \mathcal{D} \to \mathcal{C} \) as follows
\[
\beta(\sum_m a_m t^{m+k}\partial^k, \sum_n b_n t^{n+l}\partial^l) = \sum_m a_m b_{-m}(-1)^k k! (\binom{m+k}{k+l+1}).
\]

Then \( \beta \in HL^2(D, \mathcal{C}) \). Let \( S \) be the subset of \( \mathcal{D} \) given by
\[
\mathcal{S} = \{ g(t)\partial^n \mid g(t) \in \mathcal{C}(t), \ n \in \mathbb{N} \}.
\]

By definition, for \( n \in \mathbb{N} \), \( f(t) \in \mathcal{C}(t) \) with \( \operatorname{Res} f(t) = 0 \), and for all \( g(t) \in \mathcal{C}(t) \), we have
\[
\beta(t, t^{-1}) = 1 = -\beta(t^{-1}, t), \quad \beta(f(t), t) = 0 = \beta(t, f(t)),
\]
\[
\beta(g(t)\partial^{n+1}, t) = 0 = \beta(t, g(t)\partial^{n+1}),
\]
which imply by the formula \( g(t)\partial^n = \frac{1}{n+1} [g(t)\partial^{n+1}, t] \) and Lemma 2.1 that \( \beta \neq \bar{0} \) in \( HL^2(D, \mathcal{C}) \).

(III) Owing to (3.9), we can assume that \( \psi(t, t^{-1}) = s = -\psi(t^{-1}, t) \). Set \( \psi_1 := \psi - s \beta \). Then by (3.10), (3.11) and (II), we have
\[
\psi_1(t, \mathcal{D}) = \psi_1(\mathcal{D}, t) = 0,
\]
which, together with \( \psi_1([f(t)\partial^k, t, g(t)\partial^l]) = \psi_1([f(t)\partial^k, t], g(t)\partial^l) - \psi_1([f(t)\partial^k, g(t)\partial^l], t) \), gives
\[
\psi_1(\operatorname{ad} t(f(t)\partial^k), g(t)\partial^l) = -\psi_1(f(t)\partial^k, \operatorname{ad} t(g(t)\partial^l))
\]
(3.12)
for all \( f(t), g(t) \in \mathcal{C}(t) \) and \( k, l \in \mathbb{N} \).

The formula \( g(t)\partial^n = \frac{1}{n+1} [g(t)\partial^{n+1}, t] \) implies that \( E = \operatorname{ad} t \) is surjective and \( F = -\operatorname{ad} t \) is locally nilpotent. (3.12) and Lemma 2.2 show \( \psi_1 \equiv 0 \), that is, \( \psi = s \beta \ (s \in \mathcal{C}^*) \). So \( \dim HL^2(D, \mathcal{C}) = 1 \).

We complete the proof of Corollary 3.2. [1]
4. LEIBNIZ 2-COCYCLES ON VIRASORO-LIKE ALGEBRA AND q-ANALOG

Now we discuss the Leibniz 2-cocycles on the quantum 2-torus $C_q$, the $q$-analog of Virasoro-like algebra $V_q$, the Virasoro-like algebra $V$ (see Section 1), respectively. Throughout this section, $q$ is assumed to be not any root of unity.

By (1.4), we have
\[
[y, x^m y^{n-1}] = (q^{m} - 1) x^m y^n, \tag{4.1}
\]
\[
[x^{-1}, x y^n] = (1 - q^{-n}) y^n. \tag{4.2}
\]

Let $\psi$ be a Leibniz 2-cocycle on $C_q$. We define a linear function $f_\psi$ with $f_\psi(1) = 0$ on $C_q$ by
\[
f_\psi(x^m y^n) = \frac{1}{q^m - 1} \psi(y, x^m y^{n-1}), \quad m \neq 0,
\]
\[
f_\psi(y^n) = \frac{1}{1 - q^{-n}} \psi(x^{-1}, x y^n), \quad n \neq 0.
\]

Then $\beta = \psi - \alpha f_\psi$ is a Leibniz 2-cocycle on $C_q$ and equivalent to $\psi$, where $\alpha f_\psi$ is the trivial Leibniz 2-cocycle induced by $f_\psi$ as (2.3). From (4.1) and (4.2) we have
\[
\beta(y, x^m y^n) = 0, \quad m \neq 0, \tag{4.3}
\]
\[
\beta(x^{-1}, x y^n) = 0, \quad n \neq 0. \tag{4.4}
\]

By (2.2), we have
\[
\beta(a, [a, c]) = -\beta([a, c], a), \quad \forall a, c \in C_q.
\]

In case $nm_1 \neq mn_1$, we take
\[
a = x^m y^n, \quad c = \frac{1}{q^{n(m_1 - m)} - q^{m(n_1 - n)}} x^{m_1-m} y^{n_1-n},
\]
and obtain
\[
\beta(x^m y^n, x^{m_1} y^{n_1}) = -\beta(x^{m_1} y^{n_1}, x^m y^n), \quad \text{if} \quad nm_1 \neq mn_1. \tag{4.5}
\]

Hence, it follows from (4.3)—(4.5) that
\[
\beta(x^m y^n, y) = 0, \quad m \neq 0, \tag{4.6}
\]
\[
\beta(x y^n, x^{-1}) = 0, \quad n \neq 0. \tag{4.7}
\]

**Lemma 4.1.**
\[
\beta(y^{n_1}, x^m y^n) = \beta(x^m y^n, y^{n_1}) = 0, \quad m \neq 0. \tag{4.8}
\]

**Proof:**
\[
\beta(y^{n_1}, x^m y^n) = \frac{1}{1 - q^n} \beta(y^{n_1}, [x^m y^{n-1}, y])
\]
\[
= \frac{1}{1 - q^n} \beta([y^{n_1}, x^m y^{n-1}], y)
\]
\[
= \frac{q^{n_1m} - 1}{1 - q^n} \beta(x^m y^{n_1} y^{n-1}, y)
\]
\[
= 0. \quad \text{(by (4.6))} \quad \blacksquare
\]
Lemma 4.2. $\beta(1, x^m y^n) = 0 = \beta(x^m y^n, 1)$ if $m, n$ are not all 0.

Proof: If $m \neq 0$, this is the case of Lemma 4.1. If $n \neq 0$, it follows from (2.2) that

$$\beta(1, x^m y^n) = \frac{1}{1 - q^n} \beta(1, [x, x^{m-1} y^n]) = 0.$$  

Lemma 4.3.

$$\beta(x^m, x^{-m} y^n) = 0, \ \forall \ m \in \mathbb{Z}, n \neq 0. \tag{4.9}$$

Proof:

$$\begin{align*}
\beta(x^m, x^{-m} y^n) &= \frac{1}{1 - q^{-n}} \beta(x^m, [x^{-1}, x^{-m+1} y^n]) \\
&= - \frac{1}{1 - q^{-n}} \beta([x^m, x^{-m+1} y^n], x^{-1}) \\
&= - \frac{1 - q^{-mn}}{1 - q^{-n}} \beta(xy^n, x^{-1}) \\
&= 0. \ (by \ (4.7))
\end{align*}$$

Lemma 4.4.

$$\beta(x^m, x^{m_1} y^n) = 0, \ \forall \ m + m_1 \neq 0, n \neq 0. \tag{4.10}$$

Proof:

$$\begin{align*}
\beta(x^m, x^{m_1} y^n) &= \frac{1}{1 - q^{(m+m_1)n}} \beta(x^m, [x^{m+m_1}, x^{-m} y^n]) \\
&= - \frac{1}{1 - q^{(m+m_1)n}} \beta([x^m, x^{-m} y^n], x^{m+m_1}) \\
&= - \frac{1 - q^{-nm}}{1 - q^{n(m+m_1)}} \beta(y^n, x^{m+m_1}) \\
&= 0. \ (by \ (4.8))
\end{align*}$$

By Lemmas 4.2, 4.3, 4.4 and (4.5), we have

$$\beta(x^m, x^{m_1} y^n) = \beta(x^{m_1} y^n, x^m) = 0 \ \text{if} \ n \neq 0. \tag{4.11}$$

Lemma 4.5.

$$\beta(y^n, y^{n_1}) = 0, \ n + n_1 \neq 0. \tag{4.12}$$

Proof: We may set $n \neq 0$,

$$\begin{align*}
\beta(y^n, y^{n_1}) &= \frac{1}{q^{n(n+n_1)} - q^{n_2}} \beta(y^n, [x^{-n} y^{n+n_1}, x^n y^{-n}]) \\
&= \frac{q^{-n^2} - 1}{q^{n(n+n_1)} - q^{n_2}} (\beta(x^{-n} y^{2n+n_1}, x^n y^{-n}) \\
&\quad - \frac{q^{n^2} - 1}{q^{n(n+n_1)} - q^{n_2}} \beta(x^n, x^{-n} y^{n+n_1}) \\
&= \frac{q^{-n^2} - 1}{q^{n(n+n_1)} - q^{n_2}} \beta(x^{-n} y^{2n+n_1}, x^n y^{-n}). \ (by \ (4.9))
\end{align*}$$
But
\[ \beta(x^n y^n, x^{-n} y^{2n+n_1}) = \frac{1}{q^{n_1(n+n_1)} - q^{-n(n+n_1)}} \beta(x^n y^n, [x^{-n-n_1} y^{n+n_1}, x^{n_1} y^n]) \]
\[ = -\frac{q^{-n_1} - q^n}{q^{n_1(n+n_1)} - q^{-n(n+n_1)}} \beta(x^{n+n_1}, x^{-n-n_1} y^{n+n_1}) \]
\[ = 0. \quad \text{(by (4.9))} \]

(4.5) implies (4.12) is true.

With the above Lemmas, we obtain

**Lemma 4.6.**
\[ \beta(x^m y^n, x^{m_1} y^{n_1}) = 0 \quad \text{if} \quad n + n_1 \neq 0. \] (4.13)

**Proof:** (I) If \( m = m_1 = 0 \), then \( \beta(y^n, y^{n_1}) = 0 \) by (4.12).

(II) If \( n_1 = 0 \), then \( n \neq 0 \), and (4.11) shows \( \beta(x^m y^n, x^{m_1}) = 0 \); If \( n = 0 \), then \( n_1 \neq 0 \), and (4.11) gives \( \beta(x^m, x^{m_1} y^{n_1}) = 0 \).

(III) If \( n n_1 \neq 0 \) and

(i) when \( m = 0 \) and \( m_1 \neq 0 \), we have \( \beta(y^n, x^{m_1} y^{n_1}) = 0 \) by (4.8);

(ii) when \( m_1 = 0 \) and \( m \neq 0 \), we have \( \beta(x^m y^n, y^{n_1}) = 0 \) by (4.8);

(iii) when \( m m_1 \neq 0 \), we have

\[ \beta(x^m y^n, x^{m_1} y^{n_1}) = \frac{1}{1 - q^{mn}} \beta([x^m, y^n], x^{m_1} y^{n_1}) \]
\[ = \frac{1}{1 - q^{mn}} (\beta([x^m, x^{m_1} y^{n_1}], y^n) + \beta(x^m, [y^n, x^{m_1} y^{n_1}])) \]
\[ = \frac{1 - q^{mn_1}}{1 - q^{mn}} \beta(x^{m+m_1}, y^{n_1}, y^n) \]
\[ + \frac{q^{mn_1} - 1}{1 - q^{mn}} \beta(x^m, x^{m_1} y^{n+n_1}) \]
\[ = 0. \quad \text{(by (4.8) or (4.12) and by (II))} \]

The proof is completed.

**Lemma 4.7.**
\[ \beta(x^m y^n, x^{m_1} y^{-n}) = 0 \quad \text{if} \quad m + m_1 \neq 0. \]

**Proof:** (I) If \( m = 0 \), then \( m_1 \neq 0 \), and in this case we have

\[ \beta(y^n, x^{m_1} y^{-n}) = \frac{1}{1 - q^{m_1}} \beta(y^n, [x^{m_1} y^{-n-1}, y]) \]
\[ = \frac{q^{m_1 n} - 1}{1 - q^{m_1}} \beta(x^{m_1}, y^{-1}, y) \]
\[ = 0. \quad \text{(by (4.6))} \]

(II) If \( m_1 = 0 \), then \( m \neq 0 \). In case \( n \neq 0 \), \( \beta(x^m y^n, y^{-n}) = -\beta(y^{-n}, x^m y^n) = 0 \) owing to (4.5) and (I). In case \( n = 0 \), \( \beta(x^m, 1) = 0 \) by Lemma 4.2.
(III) If $m m_1 \neq 0$:

(i) First, we have

$$
\beta(x^{-m} y^m, x^{m+m_1} y^{-m}) = \frac{1}{q^{m(m+m_1)} - q^{m_1(m+m_1)}} \beta(x^{-m} y^m, [x^{m+m_1} y^{-m-m_1}, x^m y^{m_1}])
$$

$$
= \frac{q^{m^2} - q^{-m_1}}{q^{m(m+m_1)} - q^{m_1(m+m_1)}} \beta(y^{m_1+m}, x^{m+m_1} y^{-m-m_1})
$$

$$
= 0. \quad \text{(by (I))}
$$

By (4.5), we have

$$
\beta(x^{m+m_1} y^{-m_1}, x^{-m_1} y^{m}) = 0. \quad (4.14)
$$

(ii) If $n = 0$,

$$
\beta(x^m, x^{m_1}) = \frac{1}{q^{m(m+m_1)} - q^{m_1}} \beta(x^m, [x^{-m} y^m, x^{m+m_1} y^{-m}])
$$

$$
= \frac{1 - q^{m^2}}{q^{m(m+m_1)} - q^{m_1}} \beta(y^m, x^{m+m_1} y^{-m})
$$

$$
- \frac{1 - q^{-m_2}}{q^{m(m+m_1)} - q^{m_1}} \beta(x^{m+m_1} y^{-m}, x^{-m_1} y^{m})
$$

$$
= 0. \quad \text{(by (4.14))}
$$

(iii) If $n \neq 0$, noting under the assumption of $m + m_1 \neq -m$ (otherwise, $2m + m_1 = 0$, and in this case, $\beta(x^{m+m_1}, x^{-m}) = \beta(1, x^{-m}) = 0$.)

$$
\beta(x^{m+m_1}, x^{-m}) = \frac{1}{q^{m(m+m_1)} - q^{m_1}} \beta([x^m y, x^{m+m_1} y^{-1}], x^{-m})
$$

$$
= \frac{q^{-m} - 1}{q^{m+m_1} - q^{-m}} \beta(y, x^{m+m_1} y^{-1})
$$

$$
+ \frac{q^{m} - 1}{q^{m+m_1} - q^{m}} \beta(x^m y, x^{m_1} y^{-1})
$$

$$
= \ast \beta([x^m y, x^{m_1} y^{-1}], x^{-m}) \quad \text{(by (4.3))}
$$

$$
= \ast' \beta(x^{m+m_1} y^{-1}, y) + \ast'' \beta(x^m, x^{m_1})
$$

$$
= 0, \quad \text{(by (4.6) and (ii))}
$$

we obtain

$$
\beta(x^m y^n, x^{m_1} y^{-n}) = \frac{1}{q^{m_1 n} - 1} \beta(x^m y^n, [x^{m_1+m} y^{-n}, x^{-m}])
$$

$$
= \frac{q^{n(m+m_1)} - q^{-mn}}{q^{m_1 n} - 1} \beta(x^{m+m_1}, x^{-m})
$$

$$
- \frac{q^{-mn} - 1}{q^{m_1 n} - 1} \beta(y^n, x^{m_1+m} y^{-n})
$$

$$
= 0. \quad \text{(by (I))}
$$

We complete the proof. \quad \blacksquare

It follows from Lemmas 4.6 and 4.7 that

$$
\beta(x^m y^n, x^{m_1} y^{n_1}) = 0 \quad \text{if} \quad m + m_1 \neq 0 \quad \text{or} \quad n + n_1 \neq 0. \quad (4.15)
$$
In what follows, we proceed to calculate $\beta(x^{-m}y^{-n}, x^m y^n)$.

When $mn \neq 0$, 
\[
\beta(x^{-m}y^{-n}, x^m y^n) = \frac{1}{1 - q^{mn}} \beta(x^{-m}y^{-n}, [x^m, y^n]) \\
= \frac{q^{-mn} - 1}{1 - q^{mn}} \beta(y^{-n}, y^n) - \frac{1 - q^{-mn}}{1 - q^{mn}} \beta(x^{-m}, x^m) \\
= q^{-mn}(\beta(x^{-m}, x^m) + \beta(y^{-n}, y^n)),
\]
that is, for $mn \neq 0$, we have
\[
\beta(x^{-m}y^{-n}, x^m y^n) = q^{-mn}(\beta(x^{-m}, x^m) + \beta(y^{-n}, y^n)). \tag{4.16}
\]

For $m > 0$, 
\[
\beta(x^{-m}, x^m) = \frac{1}{q^{1-m} - q} \beta(x^{-m}, [xy^{-1}, x^{m-1}y]) \\
= \frac{1 - q^m}{q^{1-m} - q} \beta(x^{-m+1}y^{-1}, x^{m-1}y) - \frac{1 - q^{-m}}{q^{1-m} - q} \beta(x^{-1}y, xy^{-1}) \\
= q^{m-1}\beta(x^{-m+1}y^{-1}, x^{m-1}y) + q^{-1}\beta(x^{-1}y, xy^{-1}) \\
= \beta(x^{-m+1}, x^{m-1}) + \beta(x^{-1}, x) + \beta(y^{-1}, x) + \beta(y^{-1}, y) \\
= \ldots. \\
= m \beta(x^{-1}, x) + (m - 1)(\beta(y, y^{-1}) + \beta(y^{-1}, y)),
\]
that is, if $m > 0$, we have
\[
\beta(x^{-m}, x^m) = m \beta(x^{-1}, x) + (m - 1)(\beta(y, y^{-1}) + \beta(y^{-1}, y)). \tag{4.17}
\]

For $n > 0$, a similar consideration gives rise to
\[
\beta(y^{-n}, y^n) = n \beta(y^{-1}, y) + (n - 1)(\beta(x, x^{-1}) + \beta(x^{-1}, x)). \tag{4.18}
\]

Claim:
\[
\beta(y, y^{-1}) + \beta(y^{-1}, y) = 0 = \beta(x, x^{-1}) + \beta(x^{-1}, x). \tag{4.19}
\]

In fact, from (4.16) and (4.17) we get
\[
\beta(x^{-2}y^{-1}, x^2y) = q^{-2}(\beta(x^{-2}, x^2) + \beta(y^{-1}, y)) \\
= q^{-2}(2\beta(x^{-1}, x) + 2\beta(y^{-1}, y) + \beta(y, y^{-1})).
\]

But with (2.2), we deduce
\[
\beta(x^{-2}y^{-1}, x^2y) = \frac{1}{1 - q} \beta(x^{-2}y^{-1}, [x, xy]) \\
= \frac{q^{-1} - 1}{1 - q} \beta(x^{-1}y^{-1}, xy) - \frac{q^{-1} - q^{-2}}{1 - q} \beta(x^{-1}, x) \\
= q^{-1}\beta(x^{-1}y^{-1}, xy) + q^{-2}\beta(x^{-1}, x) \\
= q^{-2}\beta(x^{-1}, x) + q^{-2}\beta(y^{-1}, y) + q^{-2}\beta(x^{-1}, x).
\]
So \( \beta(y^{-1}, y) + \beta(y, y^{-1}) = 0 \).

An analogous calculation for \( \beta(x^{-1}y^{-2}, xy^2) \) leads to \( \beta(x^{-1}, x) + \beta(x, x^{-1}) = 0 \). \(\blacksquare\)

On the other hand, for \( m > 0 \),

\[
\beta(x^m, x^{-m}) = \frac{1}{q^{1-m}-q} \beta([xy^{-1}, x^{m-1}], x^{-m})
\]

\[
= \frac{1}{q^{1-m}-q} \left[ \beta([xy^{-1}, x^{m-1}], x^{-m}) + \beta(xy^{-1}, [x^{m-1}, x^{-m}]) \right]
\]

\[
= \frac{1}{q^{1-m}-q} \left[ (q^m - 1) \beta(x^{1-m}y^{-1}, x^{m-1}y) + (q^m - 1) \beta(xy^{-1}, x^{-1}) \right]
\]

\[
= -(q^m - 1) \beta(x^{1-m}, x^{m-1}) + \beta(x, x^{-1}) \quad \text{(by (4.16))}
\]

that is, if \( m > 0 \), we have

\[
\beta(x^m, x^{-m}) = -m \beta(x^{-1}, x). \quad (4.20)
\]

Similarly, for \( n > 0 \), we get

\[
\beta(y^n, y^{-n}) = -n \beta(y^{-1}, y). \quad (4.21)
\]

Finally, from (4.17)–(4.21), we obtain

\[
\beta(x^{-m}, x^m) = m \beta(x^{-1}, x), \quad \beta(y^{-n}, y^n) = n \beta(y^{-1}, y), \quad \forall m n \neq 0. \quad (4.22)
\]

Now we obtain the following results.

**Theorem 4.1.** Any nontrivial Leibniz 2-cocycle on the quantum 2-torus \( C_q \) is equivalent to \( \varphi \):

(i) If \( m, n \in \mathbb{Z} \) and are not all 0, then

\[
\varphi(x^my^n, x^mn) = \delta_{m+1,0} \delta_{n+1,0} q^{-mn} (m \varphi(x, x^{-1}) + n \varphi(y, y^{-1}));
\]

(ii) \( \varphi(1, 1) = c \), for some \( c \in \mathbb{C} \).

As the \( q \)-analogue of Virasoro-like algebra \( V_q \) is the inner derivation algebra of \( C_q \), we have

**Corollary 4.1.** Any nontrivial Leibniz 2-cocycle on the \( q \)-analogue of Virasoro-like algebra \( V_q \) is equivalent to \( \varphi \):

\[
\varphi(E_{m,n}, E_{m_1,n_1}) = \delta_{m+m_1,0} \delta_{n+n_1,0} q^{-mn} (m \varphi(E_{1,0}, E_{-1,0}) + n \varphi(E_{0,1}, E_{0,-1})).
\]

With the similar methods, we obtain all nontrivial Leibniz 2-cocycles on the Virasoro-like algebra \( \mathcal{V} \) (see Section 1).

**Theorem 4.2.** Any nontrivial Leibniz 2-cocycle on the Virasoro-like algebra \( \mathcal{V} \) is equivalent to \( \varphi \):

(i) If \( m, n \in \mathbb{Z} \) and are not all 0, then

\[
\varphi(L_{m,n}, L_{m_1,n_1}) = \delta_{m+m_1,0} \delta_{n+n_1,0} (m \varphi(L_{1,0}, L_{-1,0}) + n \varphi(L_{0,1}, L_{0,-1})).
\]

(ii) \( \varphi(L_{0,0}, L_{0,0}) = c \), for some \( c \in \mathbb{C} \).
With the methods in [7] and [8], we can easily obtain the Leibniz 2-cocycles on the centerless extended affine Lie algebra $\mathfrak{gl}_n(C) \otimes_C V_q$ (see Gao [3]).

**Corollary 4.2.** Any nontrivial Leibniz 2-cocycle on $\mathfrak{gl}_n(C) \otimes_C V_q$ is equivalent to $\psi$:

$$\psi(A \otimes E_{m,n}, B \otimes E_{m_1,n_1}) = \delta_{m+m_1,0}\delta_{n+n_1,0}q^{-mn}(m \varphi(E_{1,0}, E_{-1,0}) + n \varphi(E_{0,1}, E_{0,-1})) \text{tr}(AB),$$

where $\varphi$ is defined in Corollary 4.1.

**Question:** From the above discussions, we see that the Leibniz 2-cocycles on the above Lie algebras are almost consistent with the corresponding Lie 2-cocycles. It seems interesting to ask if any infinite-dimensional (simple) Lie algebra over the field of characteristic 0 has this property.

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