CONVEX SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS IN CLASSICAL DIFFERENTIAL GEOMETRY

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1. Introduction

The issue of convexity is fundamental in the theory of partial differential equations. We discuss some recent progress of convexity estimates for solutions of nonlinear elliptic equations arising from some classical problems in differential geometry.

We first review some works in the literature on the convexity of solutions of quasilinear elliptic equations in $\mathbb{R}^n$. The study of geometric properties of the harmonic function and solutions of general elliptic partial differential equations was initiated long time ago, such as the location of the critical points and the star-shapeness of the level set, etc. Gabriel [16] obtained the strict convexity of level set for the Green function in thee dimension convex domain in $\mathbb{R}^3$. Makar-Limanov [34] studied equation

\[ \Delta u = -1 \quad \text{in} \quad \Omega, \]
\[ u = 0 \quad \text{on} \quad \partial \Omega, \]

in bounded plane convex domain $\Omega$. He considered the auxiliary function

\[ 2u(u_{11}u_{22} - u_{12}^2) + 2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11}, \]

where $u_i = \frac{\partial u}{\partial x_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. He proved that $\sqrt{u}$ is concave using the maximum principle. His method has great impact on the later development related to the topic of convexity.

Brascamp-Lieb [9] used the heat equation technique to prove that $\log u$ is a concave, where $u$ is the first eigenfunction of homogeneous Dirichlet problem:

\[ \Delta u + \lambda_1 u = 0 \quad \text{in} \quad \Omega, \]
\[ u = 0 \quad \text{on} \quad \partial \Omega, \]

where $\Omega$ is bounded and convex in $\mathbb{R}^n$. In the case of dimension two, another proof of Brascamp-Lieb’s result was given in Acker-Payne-Philippin [1]. It was observed that the...
function \( v = \log u \) satisfies the following equation,

\[
\Delta v = -(\lambda_1 + |Dv|^2) \quad \text{in } \Omega,
\]
\[
v \rightarrow -\infty \quad \text{on } \Omega.
\]

If we instead let \( w = \sqrt{u} \), then \( w \) satisfies equation

\[
w \Delta w = -(1 + |Dw|^2) \quad \text{in } \Omega,
\]
\[w = 0 \quad \text{on } \partial \Omega.
\]

The second author [33] gave a new proof that the function \( w \) is concave if \( \Omega \) is a bounded convex plane domain, moreover a sharp estimate on the lower bound of the Gauss curvature of the graph of \( w \) is obtained in term of the curvature of \( \partial \Omega \). The methods in [1] and [33] are restricted to two dimensions. In [29, 30], Korevaar studied the convexity of the capillary surface. He introduced a very useful maximum principle in convex domain, under certain boundary value conditions, he established some convexity results for the mean curvature type equations. New proofs of the concavity of \( \log u \) for solution \( u \) in (1.2) were also given by Korevaar [30] and Caffarelli-Spruck [12]. Their methods were developed further by Kawohl [27] (for the intermediate case) and Kennington [28] to establish an improved maximum principle, which enables them to give a higher dimensional generalization of the theorem of Makar-Limanov [34]. Recently, a new approach to the convexity problem was found by Alvarez-Lasry-Lions [5] and they treated a large class fully nonlinear elliptic equations.

In an important development in 1985, Caffarelli-Friedman [10] devised a new deformation technique to deal with the convexity issue via strong minimum principle. They establish the strictly convexity of \( \sqrt{u} \) of the solution in (1.1) in two dimensions. Their method was generalized by Korevaar-Lewis [32] to higher dimensions. Korevaar-Lewis gave another proof of the theorem Makar-Limanov [34] in the higher dimensions case. As pointed in [31], Yau also had similar ideas on the deformation technique. This deformation idea is the main motivation for our discussion on the convexity problem of some nonlinear elliptic equations associated to some classical problems in differential geometry in the next sections.

2. The Christoffel-Minkowski problem

The Minkowski problem is a problem of finding a convex hypersurface with the prescribed Gauss curvature on its outer normals. The general problem of finding a convex hypersurface with \( k \)th symmetric function of principal radii prescribed on its outer normals is often called Christoffel-Minkowski problem. It corresponds to finding convex solutions
of the nonlinear elliptic Hessian equation:

\[(2.1) \quad S_k(\{u_{ij} + u\delta_{ij}\}) = \varphi \quad \text{on} \quad S^n,\]

where we have let \(\{e_1, e_2, ..., e_n\}\) be an orthonormal frame on \(S^n\), \(S_k\) is the \(k\)-th elementary symmetric function (see Definition 1).

It is known that for (2.1) to be solvable, the function \(\varphi(x)\) has to satisfy

\[(2.2) \quad \int_{S^n} x_i \varphi(x) \, dx = 0, \quad i = 1, ..., n + 1.\]

At one end \(k = n\), this is the Minkowski problem. (2.2) is also sufficient in this case. But it is not sufficient for the cases \(1 \leq k < n\). The natural solution class for this of type equations is in general (for \(k < n\)) consisting of functions not necessary convex. Hence the major issue is to find conditions for the existence of convex solution of (2.1). At the other end \(k = 1\), equation (2.1) is linear and it corresponds to the Christoffel problem. The necessary and sufficient conditions for the existence of a convex solution can be read off from the Green function [15]. For the intermediate cases \((2 \leq k \leq n - 1)\), (2.1) is a fully nonlinear equation. The first existence theorem was obtained by Pogorelov in [38] under certain restrictive condition on \(\varphi\) (see Remark 5.5 in [24]). In [24], we deal with the problem using continuity method as a deformation process together with strong minimum principle to force the convexity. We recall some definitions.

**Definition 1.** For \(\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n\), \(S_k(\lambda)\) is defined as

\[S_k(\lambda) = \sum \lambda_{i_1} \cdots \lambda_{i_k},\]

where the sum is taking over for all increasing sequences \(i_1, \ldots, i_k\) of the indices chosen from the set \(\{1, \ldots, n\}\). The definition can be extended to symmetric matrices \(A\) by taking \(S_k(A) = S_k(\lambda(A))\), where \(\lambda(A)\) the eigenvalues of \(A\). For \(1 \leq k \leq n\), define

\[\Gamma_k = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, ..., S_k(\lambda) > 0\}.\]

A function \(u \in C^2(S^n)\) is called \(k\)-convex if the eigenvalues of \(W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\}\) is in \(\Gamma_k\) for each \(x \in S^n\). \(u\) is called an admissible solution of (2.1) if it is \(k\)-convex. \(u\) is simply called convex if \(u\) is \(n\)-convex.

**Definition 2.** Let \(f\) be a positive \(C^{1,1}\) function on \(S^n\) satisfies (2.2), \(\forall s \in \mathbb{R}\), we say \(f\) is in \(C_s\) if \((f^s + \delta_{ij} f^s)\) is semi-positive definite almost everywhere in \(S^n\).

The following full rank theorem was proved in [24].

**Theorem 1.** Suppose \(u\) is an admissible solution of equation (2.1) with semi-positive definite spherical hessian \(W = \{u_{ij} + u\delta_{ij}\}\) on \(S^n\). If \(\varphi \in C_{-\frac{1}{k}}\), then \(W\) is positive definite on \(S^n\).
As a consequence, an existence result can be established for the Christoffel-Minkowski problem.

**Theorem 2.** Let \( \varphi(x) \in C^{1,1}(\mathbb{S}^n) \) be a positive function and suppose \( \varphi \in C_{-\frac{1}{k}} \), then Christoffel-Minkowski problem (2.1) has a unique convex solution up to translations. More precisely, there exists a \( C^{3,\alpha} (\forall 0 < \alpha < 1) \) closed strictly convex hypersurface \( M \) in \( \mathbb{R}^{n+1} \) whose \( k \)th elementary symmetric function of principal radii on the outer normals is \( \varphi(x) \). \( M \) is unique up to translations. Furthermore, if \( \varphi(x) \in C^{1,\gamma} (\mathbb{S}^n) \) (\( l \geq 2, \gamma > 0 \)), then \( M \) is \( C^{2+l,\gamma} \).

Theorem 2 was first proved in [24] under further assumption that \( \varphi \) is connected to 1 in \( C_{-\frac{1}{k}} \). It turns out this extra condition is redundant as \( C_{-\frac{1}{k}} \) is indeed connected. This fact was first proved in the joint work of Andrews and the second author [7] via curvature flow approach. More recently, this fact again was also verified directly by Trudinger-Wang [41].

The proof of Theorem 1 relies on a deformation lemma for Hessian equation (2.1). This approach was motivated by works of Caffarelli-Friedman [10] and Korevaar-Lewis [32]. This type of deformation lemma enables us to apply the strong maximum principle to enforce the constant rank of \((u_{ij} + u_\delta_{ij})\) on \( \mathbb{S}^n \). The proof of such deformation lemma in [24] is delicate, since equation (2.1) is fully nonlinear. Theorem 1 was generalized to Hessian quotient equation in [26]. The same result is in fact valid for a general class of fully nonlinear elliptic equations in our recent work [25]. This general phenomenon follows from the ellipticity and concavity of the fully nonlinear operators \( F(W) = -S^{-\frac{1}{k}}(W) \) and \( G(W) = -F(W^{-1}) \). Here we state a sample of this type of results for the equation in the domains of \( \mathbb{R}^n \). There is also a corresponding version for equations of spherical hessian on \( \mathbb{S}^n \), we refer [25] for the proof of Theorem 3 and other related results.

**Theorem 3.** Let \( f \) be a \( C^2 \) symmetric function defined on a symmetric domain \( \Psi \subset \Gamma_1 \) in \( \mathbb{R}^n \). Let \( \hat{\Psi} = \{ A \in \text{Sym}(n) : \lambda(A) \in \Psi \} \), and define \( F : \hat{\Psi} \to \mathbb{R} \) by \( F(A) = f(\lambda(A)) \). If \( \hat{F}(A) = -F(A^{-1}) \) and \( F(A) \) are concave functions on the positive definite matrices, \( f_\lambda = \frac{\partial f}{\partial \lambda} > 0 \). Then if \( u \) is a \( C^4 \) convex solution of the following equation in a domain \( \Omega \) in \( \mathbb{R}^n \)

\[
F(\{u_{ij}\}) = g(x)
\]

and \( g(x) \) is concave function in \( \Omega \). Then the Hessian \( u_{ij} \) is constant rank in \( \Omega \).

**Remark 1.** To our knowledge, the conditions in Theorem 3 was introduced by Alvarez-Lasry-Lions in [5], see also [6]. Theorem 1 and Theorem 3 provide a positive lower bound on the eigenvalues of the corresponding Hessians. In particular, Theorem 1 implies that there
is a priori upper bound of principal curvatures of the convex hypersurface $M$ satisfying (2.1). The existence of such estimate has been known for sometime if a stronger condition \(((\varphi^{-\frac{1}{k}})_{ij} + \varphi^{-\frac{1}{k}}\delta_{ij}) > 0\) is imposed. Under this condition, the upper bound of the principal curvatures can be deduced from the equation (2.1) combining the ellipticity and concavity of the fully nonlinear operators $F(W) = -S^{-\frac{1}{k}}(W)$ and $G(W) = -F(W^{-1})$ or directly from the curvature equation on the hypersurface easily. This has been recently written out in [41]. Nevertheless, this kind of direct estimates on the upper bound of the principal curvatures will blow up when some of the eigenvalues of \((\varphi^{-\frac{1}{k}} + \varphi^{-\frac{1}{k}}\delta_{ij})\) tends to zero. In this respect, Theorem 1 and Theorem 3 are better suited, they even allow \((\varphi^{-\frac{1}{k}} + \varphi^{-\frac{1}{k}}\delta_{ij})\) to be negative. For an illustration, we state the following result as a consequence of Theorem 1.

**Theorem 4.** For any constant $1 > \beta > 0$, there is a positive constant $\gamma > 0$ such that if $\varphi(x) \in C^{1,1}(S^n)$ is a positive function with $\inf_{S^n} \varphi \geq \beta$, $\sup_{S^n} \varphi \geq \beta$, and $\varphi$ satisfies the necessary condition (2.2) and

\[
(\varphi^{-\frac{1}{k}} + \varphi^{-\frac{1}{k}}\delta_{ij}) \geq -\gamma \varphi^{-\frac{1}{k}}\delta_{ij} \quad \text{on} \quad S^n,
\]

then Christoffel-Minkowski problem (2.1) has a unique $C^{3,\alpha}$ \((0 < \alpha < 1)\) convex solution.

Proof. We argue by contradiction. If the result is not true, for some $0 < \beta < 1$, there is a sequence of positive functions $\varphi_l \in C^{1,1}(S^n)$ such that $\sup_{S^n} \varphi_l = 1$, $\inf_{S^n} \varphi_l \geq \beta$, $\|\varphi_l\|_{C^{1,1}(S^n)} \leq \frac{1}{l}$, $((\varphi_l^{-\frac{1}{k}})_{ij} + \varphi_l^{-\frac{1}{k}}\delta_{ij}) \geq -\frac{1}{l}\varphi_l^{-\frac{1}{k}}\delta_{ij}$, $\varphi_l$ satisfies (2.2), and equation (2.1) has no convex solution. By [26], equation (2.2) has an admissible solution $u_l$ with

\[
\|u_l\|_{C^{3,\alpha}(S^n)} \leq C,
\]

independent of $l$. Therefore, there exist subsequences, we still denote $\varphi_l$ and $u_l$, $\varphi_l \to \varphi$ in $C^{1,\alpha}(S^n)$, $u_l \to u$ in $C^{3,\alpha}(S^n)$, for some positive $\varphi \in C^{1,1}(S^n)$ with $(\varphi^{-\frac{1}{k}} + \varphi^{-\frac{1}{k}}\delta_{ij}) \geq 0$, and $u$ satisfies equation (2.1) and $(u_{ij}(x) + u(x)\delta_{ij}) \leq 0$ at some point $x$. On the other hand, Theorem 2 yields a convex solution $\tilde{u} \in C^{3,\alpha}(S^n)$ for such $\varphi$. By the uniqueness theorem in [26], $u - \tilde{u} = \sum_{i=1}^{n+1} a_i x_i$. In turn, $(u_{ij} + u\delta_{ij}) = (u_{ij} + \tilde{u}\delta_{ij}) > 0$ everywhere. This is a contradiction. \(\square\)

**Remark 2.** For differential equation $F(u_{ij} + \delta_{ij} u) = \varphi$ on $S^n$ with $F$ satisfying the structural conditions in Theorem 3, one may easily obtained a priori estimates for solutions $u$ following the same lines of proof as in [19]. On the other hand, the constant rank theorems like Theorem 1 and Theorem 3 provide a positive lower bound on the Hessian.
\((u_{ij} + \delta_{ij}u)\). In turn, the hypersurface \(M\) with \(u\) as its support function is uniformly convex and smooth. The delicate part is the existence of such convex surface \(M\) with prescribed \(\varphi\). As pointed out in [19], the condition (2.2) is neither necessary, nor sufficient for \(F = \frac{S_{n-k}}{S_{n-k}}\) \((0 < k < n)\), which corresponds to the problem of prescribing \(k^{th}\) Weingarten curvature on outer normals. The existence was proved in [19] for \(F = \frac{S_{n-k}}{S_{n-k}}\) under some group invariant assumption on \(\varphi\). We note that under the same assumption, and the structure conditions on \(F\) as in Theorem 3, we may state a similar existence result as in Theorem 4 for the problem of prescribing general Weingarten curvature function \(F(\kappa_1, \cdots, \kappa_n) = f(\nu)\), where \(\kappa_1, \cdots, \kappa_n\) are the principal curvatures of \(M\) and \(\nu\) is the outer normal of \(M\). In particular, Theorem 4 is valid for

\[
\frac{S_{k+l}}{S_l}(\kappa_1, \cdots, \kappa_n) = \frac{1}{\varphi}(\nu),
\]

\((0 \leq l < k + l \leq n)\) if we assume in addition that \(\varphi\) is invariant under an automorphic group \(\mathcal{G}\) which has no fixed point on \(\mathbb{S}^n\) (see [26]). We also refer [25] for the explicit statements of general existence results of this type.

3. Weingarten curvature equations

The Christoffel-Minkowski problem was deduced to a convexity problem of a spherical hessian equation on \(\mathbb{S}^n\) in the last section. It can also be considered as a curvature equation on the hypersurface via inverse Gauss map. In this section, we discuss some curvature equations related to problems in the classical differential geometry. We will indicate how the techniques in convexity estimates for fully nonlinear equations may help us for this type of problems.

For a compact hypersurface \(M\) in \(\mathbb{R}^{n+1}\), the \(k^{th}\) Weingarten curvature at \(x \in M\) is defined as

\[
W_k(x) = S_k(\kappa_1(x), \kappa_2(x), \cdots, \kappa_n(x))
\]

where \(\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n)\) the principal curvatures of \(M\). In particular, \(W_1\) is the mean curvature, \(W_2\) is the scalar curvature, and \(W_n\) is the Gauss-Kronecker curvature. If the surface is starshaped about the origin, it follows that the surface can be parameterized as a graph over \(\mathbb{S}^n\):

\[
X = \rho(x)x, \quad x \in \mathbb{S}^n,
\]

where \(\rho\) is the radial function. In this correspondence, the Weingarten curvature can be considered as a function on \(\mathbb{S}^n\) or in \(\mathbb{R}^{n+1}\). The problem of prescribing curvature functions has attracted much attention. For example, given a positive function \(F\) in \(\mathbb{R}^{n+1} \setminus \{0\}\),
one would like to find a starshaped hypersurface $M$ about the origin such that its $k$th Weingarten curvature is $F$. The problem is equivalent to solve the following equation

$$S_k(\kappa_1, \kappa_2, \ldots, \kappa_n)(X) = F(X) \quad \text{for any } X \in M.$$  

(3.2)

The uniqueness question of starshaped hypersurfaces with prescribed curvature was studied by Alexandrov [4] and Aeppli [2]. The prescribing Weingarten curvature problem and similar problems have been studied by various authors, we refer to [8, 42, 40, 36, 11, 13, 43, 17, 18, 19] and references there for related works.

We will use notions of admissible solutions as in last section

**Definition 3.** A $C^2$ surface $M$ is called $k$-admissible if at every point $X \in M$, $\kappa \in \Gamma_k$.

Under some barrier conditions, an existence result of equation (3.2) was obtained by Bakelman-Kantor [8], Treibergs-Wei [40] for $k = 1$, and by Caffarelli-Nirenberg-Spruck in [11] for general $1 \leq k \leq n$. The solution of the problem [11] in general is not convex if $k < n$. The question of convexity of solution in [11] was treated by Chou [13] (see also [43]) for the mean curvature case under concavity assumption on $F$, and by Gerhardt [17] for general Weingarten curvature case under concavity assumption on $\log F$, see also [18] for the work on general Riemannian manifolds.

The following is a general principle for the convexity proved in [22].

**Theorem 5.** Suppose $M$ is a $k$-admissible surface of equation (3.2) in $\mathbb{R}^{n+1}$ with semi-positive definite second fundamental form $W = \{h_{ij}\}$ and $F(X) : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^+$ is a given smooth positive function. If $F(X)^{-\frac{1}{k}}$ is a convex function in a neighborhood of $M$, then $\{h_{ij}\}$ is constant rank, so $M$ is strictly convex.

As a consequence, we deduce the existence of convex hypersurface with prescribed Weingarten curvature in (3.2) in [22]: if in addition to the barrier condition in [11], $F(X)^{-\frac{1}{k}}$ is a convex function in the region $r_1 < |X| < r_2$, then the $k$-admissible solution in Theorem [11] is strictly convex.

In the literature, the homogeneous Weingarten curvature problem

$$S_k(k_1, k_2, \ldots, k_n)(X) = \gamma f\left(\frac{X}{|X|}\right)|X|^{-k}, \quad \forall X \in M,$$

also draws some attention. If $M$ is a starshaped hypersurface about the origin in $\mathbb{R}^{n+1}$, by dilation property of the curvature function, the $k$th Weingarten curvature can be considered as a function of homogeneous degree $-k$ in $\mathbb{R}^{n+1} \setminus \{0\}$. If $F$ is of homogeneous degree $-k$, then the barrier condition in [11] can not be valid unless the function is constant. Therefore equation (3.3) needs a different treatment. In fact, this problem is a nonlinear
eigenvalue problem for the curvature equation. When \( k = n \), then equation (3.3) can be expressed as a Monge-Ampère equation of radial function \( \rho \) on \( S^n \), the problem was studied by Delanoë [14]. The other special case \( k = 1 \) was considered by Treibergs in [39]. The difficulty for equation (3.3) is the lack of gradient estimate, such kind of estimate does not hold in general (see [39, 22]). Therefore, some conditions have to be in place for \( f \) in (3.3). In [22], a uniform treatment for \( 1 \leq k \leq n \) was given, and together with some discussion on the existence of convex solutions.

**Theorem 6.** Suppose \( n \geq 2 \), \( 1 \leq k \leq n \) and \( f \) is a positive smooth function on \( S^n \). If \( k < n \), assume further that \( f \) satisfies
\[
\sup_{S^n} \frac{\vert \nabla f \vert}{f} < 2k,
\]
Then there exist a unique constant \( \gamma > 0 \) with
\[
\frac{C^k_n}{\max_{S^n} f} \leq \gamma \leq \frac{C^k_n}{\min_{S^n} f}
\]
and a smooth \( k \)-admissible hypersurface \( M \) satisfying (3.3) and solution is unique up to homothetic dilations. Furthermore, for \( 1 \leq k < n \), if in addition \( |X| f (\frac{X}{|X|})^{-\frac{k}{2}} \) is convex in \( \mathbb{R}^{n+1} \setminus \{0\} \), then \( M \) is strictly convex.

**Remark 3.** Condition (3.4) in Theorem 6 can be weakened, we refer to [22] for the precise statement. When \( k = n \), the above result was proved by Delanoë [14]. In this case, the solution is convex automatically. The treatment in [22] is different from [14]. When \( k = 1 \), the existence part of Theorem 6 was proved in [39], along with a sufficient condition (which is quite complicated) for convexity.

We now switch to a similar curvature equation arising from the problem of prescribing curvature measures in the theory of convex bodies. For a bounded convex body \( \Omega \) in \( \mathbb{R}^{n+1} \) with \( C^2 \) boundary \( M \), the corresponding curvature measures of \( \Omega \) can be defined according to some geometric quantities of \( M \). The \( k \)-th curvature measure of \( \Omega \) is defined as
\[
C^k_k(\Omega, \beta) := \int_{\beta \cap M} W_{n-k} dF_n,
\]
for every Borel measurable set \( \beta \) in \( \mathbb{R}^{n+1} \), where \( dF_n \) is the volume element of the induced metric of \( \mathbb{R}^{n+1} \) on \( M \). Since \( M \) is convex, \( M \) is star-shaped about some point. We may assume that the origin is inside of \( \Omega \). Since \( M \) and \( S^n \) is diffeomorphism through radial correspondence \( R_M \). Then the \( k \)-th curvature measure can also be defined as a measure on each Borel set \( \beta \) in \( S^n \):
\[
C^k_k(M, \beta) = \int_{R_M(\beta)} W_{n-k} dF_n.
\]
Note that $C_k(M, S^n)$ is the $k$-th quermassintegral of $\Omega$.

The problem of prescribing curvature measures is dual to the Christoffel-Minkowski problem in the previous section. The case $k = 0$ is named as the Alexandrov problem, which can be considered as a counterpart to Minkowski problem. The existence and uniqueness were obtained by Alexandrov [3]. The regularity of the Alexandrov problem in elliptic case was proved by Pogorelov [37] for $n = 2$ and by Oliker [35] for higher dimension case. The general regularity results (degenerate case) of the problem were obtained in [20]. Yet, very little is known for the existence problem of prescribing curvature measures $C_{n-k}$ for $k < n$.

The problem is equivalent to solve the following curvature equation

$$S_k(\kappa_1, \kappa_2, ..., \kappa_n) = \frac{f(x)}{g(x)}, \quad 1 \leq k \leq n \quad \text{on} \quad S^n$$

where $f$ is the given function on $S^n$ and $g(x)$ is a function involves the gradient of solution.

The major difficulty around equation (3.6) is the lack of $C^2$ a priori estimates for admissible solutions. Though equation (3.6) is similar to the equation of prescribing Weingarten curvature equation (3.2), the function $g$ (depending on the gradient of solution) makes the matter very delicate. Equation (3.6) was studied in an unpublished notes [21] by Yanyan Li and the first author. The uniqueness and $C^1$ estimates were established for admissible solutions there. In [23], we make use of some ideas in the convexity estimate for curvature equations to overcome the difficulty on $C^2$ estimate.

**Theorem 7.** Suppose $f(x) \in C^2(S^n)$, $f > 0$, $n \geq 2$, $1 \leq k \leq n - 1$. If $f$ satisfies the condition

$$|X|^\frac{n+1}{n} f\left(\frac{X}{|X|}\right)^{-\frac{1}{k}} \quad \text{is a strictly convex function in} \quad \mathbb{R}^{n+1} \setminus \{0\},$$

then there exists a unique strictly convex hypersurface $M \in C^{3,\alpha}, \alpha \in (0,1)$ such that it satisfies (3.6).

When $k = 1$ or $2$, the strict convex condition (3.7) can be weakened.

**Theorem 8.** Suppose $k = 1$, or $2$ and $k < n$, and suppose $f(x) \in C^2(S^n)$ is a positive function. If $f$ satisfies

$$|X|^\frac{n+1}{n} f\left(\frac{X}{|X|}\right)^{-\frac{1}{k}} \quad \text{is a convex function in} \quad \mathbb{R}^{n+1} \setminus \{0\},$$

then there exists unique strictly convex hypersurface $M \in C^{3,\alpha}, \alpha \in (0,1)$ such that it satisfies equation (3.6).

Theorem 8 yields solutions to two other important measures, the mean curvature measure and scalar curvature measure under convex condition (3.8). For the existence of
convex solutions, some condition on $f$ is necessary. In the proof of Theorem 7, the novel feature is the $C^2$ estimates. Instead of obtaining an upper bound of the principal curvatures, we look for a lower bound of the principal curvatures (the upper bound of principal radii) by transforming (3.6) to a new equation of support function on $S^n$ through Gauss map. For the proof of Theorem 8, the key part is the $C^2$ estimates for the case $k = 2$, which we make use of some special structure of $S_2$. We also establish a deformation lemma as in Theorem 1 and Theorem 5 to ensure the convexity of solutions in the process of applying the method of continuity.

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