BOSONIC REPRESENTATIONS OF THE LIE ALGEBRAS

\( \mathcal{W}_{1+\infty} \) AND \( \mathcal{W}_{1+\infty}(gl_N) \)

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ABSTRACT

In the paper, we construct bosonic negative-energy representations of the central extension of the Lie algebra of differential operators on the circle, the Lie algebra \( \mathcal{W}_{1+\infty} \), as well as of the Lie algebra \( \mathcal{W}_{1+\infty}(gl_N) \). In particular, when restricted to the Virasoro subalgebra of the Lie algebra \( \mathcal{W}_{1+\infty} \), we obtain a bosonic realization of the Virasoro algebra with central charge \( c = 2 \) and negative-energies, which is completely reducible.

Keywords: Bosonic representation, Virasoro algebra, differential operator, negative-energy

1. INTRODUCTION

Spinor representations for the affine Kac-Moody Lie algebras were first developed by Frenkel [F] and Kac-Peterson [KP] independently. The idea is to use a Clifford algebra with infinitely many generators to construct certain quadratic elements, which, together with the identity element, span an orthogonal affine Kac-Moody Lie algebra. Thereafter, Feingold-Frenkel [FF] constructed the so-called fermionic or bosonic representations for all classical affine Kac-Moody Lie algebras by using Clifford or Weyl algebras with infinitely many generators. Recently, Gao [G2] constructed fermionic and bosonic representations for the extended affine Lie algebra \( \widehat{gl_N}(C_q) \).

As we know, the Lie algebra \( \widehat{D}^- \), as the universal central extension of the Lie algebra of differential operators on the circle (see [KP]), has appeared in various models of two-dimensional quantum field theory and integrable systems (see the references in [FKRW], [KR]). A systematic study of the quasifinite highest weight representation theory of the Lie algebra \( \widehat{D}^- \), which is often referred to as \( \mathcal{W}_{1+\infty} \) algebra by physicists, has been investigated by Kac et al (see [KR], [FKRW], etc.). Motivated by [G2], this paper is devoted to constructing bosonic representations of the Lie algebra \( \mathcal{W}_{1+\infty} \) and of the Lie algebra \( \mathcal{W}_{1+\infty}(gl_N) \). In particular, we obtain a bosonic realization of the Virasoro algebra with central charge \( c = 2 \) and negative-energies when restricted to the Virasoro subalgebra of the Lie algebra \( \mathcal{W}_{1+\infty} \), which differs from the typical one given in the literature (for instance, see [F2], or [KRa]).

Throughout this paper, \( \mathbb{Z}, \mathbb{N} \) and \( \mathbb{C} \) denote the set of integers, non-negative integers and complex numbers, respectively.

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Lemma 1.1 ([KR]). Any non-trivial 2-cocycle on $\mathcal{D}^-$ is equivalent to $\phi$:

$$
\phi(t^{m_1}D^{n_1}, t^{m_2}D^{n_2}) = \begin{cases} 
0, & \text{if } m_1 = 0, \\
(-1)^{n_1} \delta_{m_1+m_2,0} \frac{1}{2} \sum_{i=1}^{m_1} (m_1 - i)^{n_1} i^{n_2}, & \text{if } m_1 > 0, \\
(-1)^{n_1} \delta_{m_1+m_2,0} \frac{1}{2} \sum_{i=1}^{m_1} (m_1 - i)^{n_1} i^{n_2}, & \text{if } m_1 < 0.
\end{cases}
$$

Let $\mathcal{W}_{1,+\infty}$ denote the universal (one-dimensional) central extension $\widehat{\mathcal{D}^-}$ of the Lie algebra $\mathcal{D}^-$ by the above 2-cocycle $\phi$. In particular, $\text{Vir} = \text{Span}_\mathbb{C}\{L_m = t^m D, c \mid m \in \mathbb{Z}\}$ is the Virasoro subalgebra of $\mathcal{W}_{1,+\infty}$, where its Lie bracket is given as follows (since $\sum_{i=1}^{m} (m - i)i = \frac{1}{6} (m - 1) m (m + 1)$ for $m > 0$)

$$
[L_m, L_n] = (n - m) L_{m+n} + \frac{1}{12} (m - 1) m (m + 1) \delta_{m+n,0} c, \\
c, L_m] = 0.
$$

2. BOSONIC REPRESENTATION OF $\mathcal{W}_{1,+\infty}$

Define $S$ to be the unital associative algebra with infinitely many generators: $a(n), a^*(n) \ (n \in \mathbb{Z})$ with relations

$$
[a(n), a(m)] = [a^*(n), a^*(m)] = 0, 
$$

$$
[a(n), a^*(m)] = -\delta_{n+m,0}.
$$

We define the normal ordering as follows.

$$
a(n)a^*(m) := \begin{cases} 
a(n)a^*(m), & n \leq m, \\
a^*(m)a(n), & n > m,
\end{cases}
$$

for $n, m \in \mathbb{Z}$. Set

$$
\theta(n) = \begin{cases} 
1, & n > 0, \\
0, & n \leq 0.
\end{cases}
$$
Then
\[ a(n) a^*(m) =: a(n) a^*(m) = -\delta_{n+m,0} \theta(n-m), \] (2.5)
and
\[ [a(m) a^*(n), a(p)] = \delta_{n+p,0} a(m), \]
\[ [a(m) a^*(n), a^*(p)] = -\delta_{m+p,0} a^*(n), \] (2.6)
for \( m, n, p \in \mathbb{Z}. \)

Let \( \mathcal{S}^+ \) be the subalgebra generated by \( a(n), a^*(0), a^*(m) \) for \( n, m > 0 \). Let \( \mathcal{S}^- \) be the subalgebra generated by \( a(0), a(n), a^*(m) \) for \( n, m < 0 \). Those generators in \( \mathcal{S}^+ \) are called annihilation operators while those in \( \mathcal{S}^- \) are called creation operators. Let \( V \) be a simple \( \mathcal{S} \)-module containing an element \( v_0 \), called a “vacuum vector”, and satisfying
\[ \mathcal{S}^+ v_0 = 0. \] (2.7)

So all annihilation operators kill \( v_0 \) and
\[ V = \mathcal{S}^- v_0. \] (2.8)

Now we may construct a class of bosons on \( V \). For any \( m \in \mathbb{Z}, n \in \mathbb{N} \), set
\[ f(m, n) = \sum_{i \in \mathbb{Z}} (-i)^n : a(m - i) a^*(i) :. \] (2.9)

Although \( f(m, n) \) are infinite sums, they are well-defined as operators on \( V \). Indeed, for any vector \( v \in V = \mathcal{S}^- v_0 \), only finitely many terms in (2.9) can make a non-zero contribution to \( f(m, n)v \).

**Lemma 2.1.** For \( m, p, s \in \mathbb{Z}, n \in \mathbb{N} \),
\[ [f(m, n), a(p)] = p^n a(m+p), \] (2.10)
\[ [f(m, n), a^*(p)] = -(-m-p)^n a^*(m+p), \] (2.11)
\[ [f(m, n), a(p)a^*(s)] = p^n a(m+p)a^*(s) - (-m-s)^n a(p)a^*(m+s). \] (2.12)

**Proof.** Since
\[ [f(m, n), a(p)] = \sum_{i \in \mathbb{Z}} (-i)^n [a(m-i)a^*(i), a(p)] \]
\[ = \sum_{i \in \mathbb{Z}} (-i)^n [a(m-i)a^*(i), a(p)] \]
\[ = p^n a(m+p), \]
(2.10) is true. The proof of (2.11) is similar, and (2.12) follows from (2.10) and (2.11). \( \Box \)

**Proposition 2.2.** For \( m_1, m_2 \in \mathbb{Z}, n_1, n_2 \in \mathbb{N} \), we have
\[ [f(m_1, n_1), f(m_2, n_2)] = \sum_{i=0}^{n_1} \binom{n_1}{i} m_2^i f(m_1 + m_2, n_1 + n_2 - i) \]
\[ - \sum_{j=0}^{n_2} \binom{n_2}{j} m_1^j f(m_1 + m_2, n_1 + n_2 - j) \]
\[ + \varphi(f(m_1, n_1), f(m_2, n_2)), \]
where \( \varphi \) is given by

\[
\varphi(f(m_1, n_1), f(m_2, n_2)) = \begin{cases} 
0, & \text{if } m_1 = 0, \\
(-1)^{n_1+1}\delta_{m_1+m_2,0}\sum_{i=1}^{m_1}(m_1-i)^{n_1}i^{n_2}, & \text{if } m_1 > 0, \\
(-1)^{n_1}\delta_{m_1+m_2,0}\sum_{i=m_1}^{m_1}(m_1-i)^{n_1}i^{n_2}, & \text{if } m_1 < 0.
\end{cases}
\]

**Proof.** By Lemma 2.1, we have

\[
[f(m_1, n_1), f(m_2, n_2)] = [f(m_1, n_1), \sum_{i \in \mathbb{Z}} (-t)^{n_2} : a(m_2-t)a^*(t)] \\
= [f(m_1, n_1), \sum_{i \in \mathbb{Z}} (-t)^{n_2}a(m_2-t)a^*(t)] \\
= \sum_{i \in \mathbb{Z}} (-t)^{n_2}(m_2-t)^{n_1}a(m_1+m_2-t)a^*(t) - \sum_{i \in \mathbb{Z}} (-m_1-t)^{n_1}(-t)^{n_2}a(m_2-t)a^*(m_1+t) \\
= \sum_{i \in \mathbb{Z}} (-t)^{n_2}(m_2-t)^{n_1}a(m_1+m_2-t)a^*(t) - \sum_{i \in \mathbb{Z}} (-m_1-t)^{n_1}(-t)^{n_2}a(m_2-t)a^*(m_1+t) \\
- \delta_{m_1+m_2,0}\left(\sum_{i \in \mathbb{Z}} (-m_2-t)^{n_1}(-t)^{n_2}\theta(m_1+m_2-t) - \sum_{i \in \mathbb{Z}} (-m_1-t)^{n_1}(-t)^{n_2}\theta(m_2-m_1-2t)\right) \\
= \sum_{i=0}^{n_1} \binom{n_1}{i} m_1^i \sum_{i=0}^{n_2} \binom{n_2}{j} m_2^j f(m_1+m_2, n_1+n_2-i) - \sum_{j=0}^{n_2} \binom{n_2}{j} m_2^j f(m_1+m_2, n_1+n_2-j) \\
- \delta_{m_1+m_2,0}\left(\sum_{i \in \mathbb{Z}} (-m_1-t)^{n_1}(-t)^{n_2}\theta(-2t) - \sum_{i \in \mathbb{Z}} (-m_1-t)^{n_1}(-t)^{n_2}\theta(-2m_1-2t)\right) \\
= \sum_{i=0}^{n_1} \binom{n_1}{i} m_1^i \sum_{i=0}^{n_2} \binom{n_2}{j} m_2^j f(m_1+m_2, n_1+n_2-i) - \sum_{j=0}^{n_2} \binom{n_2}{j} m_2^j f(m_1+m_2, n_1+n_2-j) \\
+ \delta_{m_1+m_2,0}\sum_{i \in \mathbb{Z}} (-\theta(-2t) + \theta(-2m_1-2t)) (-m_1-t)^{n_1}(-t)^{n_2} \\
= \sum_{i=0}^{n_1} \binom{n_1}{i} m_1^i f(m_1+m_2, n_1+n_2-i) - \sum_{j=0}^{n_2} \binom{n_2}{j} m_2^j f(m_1+m_2, n_1+n_2-j) \\
+ \varphi(f(m_1, n_1), f(m_2, n_2)),
\]

where the last equality is given by

\[
\sum_{i \in \mathbb{Z}} (-\theta(-2t) + \theta(-2m_1-2t)) (-m_1-t)^{n_1}(-t)^{n_2} = \begin{cases} 
0, & m_1 = 0, \\
-\sum_{i=1}^{m_1}(t-m_1)^{n_1}t^{n_2}, & m_1 > 0, \\
\sum_{i=m_1}^{m_1}(t-m_1)^{n_1}t^{n_2}, & m_1 < 0.
\end{cases}
\]

The proof is completed. 

Let \( T = f(0, 0) \), then Lemma 2.1 gives

\[
[T, a(n)] = a(n), \quad [T, a^*(n)] = -a^*(n),
\]

for all \( n \in \mathbb{Z} \). For any \( v = a(n_1) \cdots a(n_s)a^*(m_1) \cdots a^*(m_l)v_0 \in V \), noting that \( Tv_0 = 0 \), one has

\[
Tv = (s-l)v.
\]
According to Proposition 2.2 and Lemma 2.1, we obtain

**Theorem 2.3.** \( V \) is a module for the Lie algebra \( W_{1+\infty} \) with central charge \( c = 2 \) under the action given by

\[
\pi(t^m D^n) = f(m, n), \\
\pi(c) = 2 \text{id.}
\]

for all \( m \in \mathbb{Z}, n \in \mathbb{N} \). Moreover,

\[
V = \bigoplus_{k \in \mathbb{Z}} V_k,
\]

is completely reducible, where \( V_k \) is the eigenspace with eigenvalue \( k \) of the operator \( T \), and each component \( V_k \) is irreducible as a \( W_{1+\infty} \)-module.

**Proof.** Notice that \( \varphi(f(m_1, n_1), f(m_2, n_2)) = 2 \phi(t^{m_1} D^{n_1}, t^{m_2} D^{n_2}) \). Therefore, Proposition 2.2 shows that \( V \) is a \( W_{1+\infty} \)-module with central charge \( c = 2 \). On the other hand, Lemma 2.1 indicates that each eigenspace \( V_k \) of the operator \( T \) is \( W_{1+\infty} \)-stable. In what follows, we shall prove that \( V_k \) is also irreducible under the actions of all \( f(m, n) \)'s.

To this end, we need introduce some notation. Fix a \( k \in \mathbb{Z} \), for any \( s \in \mathbb{N} \) such that \( s + k \geq 0 \), set \( v_k^{(s)} := a(0)^s a^* (-1)^{s+k} v_0 \) and

\[
V_k^{(s)} := \text{Span}_{\mathbb{C}} \{ a(n_1) \cdots a(n_s) a^*(m_1) \cdots a^*(m_{s+k}) v_0 \mid n_i \leq 0, m_j < 0 \}.
\]

It is clear that \( V_k = \bigoplus_{s \in \mathbb{N}, s+k \geq 0} V_k^{(s)} \). On the other hand, if we define the weight by

\[
\text{wt}(t^m D^n) = m,
\]

which induces a principle \( \mathbb{Z} \)-gradation of \( W_{1+\infty} \):

\[
W_{1+\infty} = \bigoplus_{j \in \mathbb{Z}} W_{1+\infty}^{(j)},
\]

we then have a triangular decomposition of \( W_{1+\infty} \) as follows

\[
W_{1+\infty} = W_{1+\infty}^{(-)} \bigoplus W_{1+\infty}^{(0)} \bigoplus W_{1+\infty}^{(+)}
\]

where \( W_{1+\infty}^{(-)} = \{ t^m D^n \mid m < 0, n \in \mathbb{N} \}, W_{1+\infty}^{(0)} = \{ D^n \mid n \in \mathbb{N} \}, W_{1+\infty}^{(+)} = \{ t^m D^n \mid m > 0, n \in \mathbb{N} \} \).

Lemma 2.1 shows that \( V_k \) is a weight module with respect to the abelian subalgebra \( W_{1+\infty}^{(0)} \). Since

\[
[f(0,n), f(m,0)] = \sum_{j=1}^{n} \binom{n}{j} (n-j)^j f(m,n-j),
\]

the actions of \( f(m,0), f(m,1), \ldots, f(m,n-1) \) on \( V \) can be expressed as some combinations of \( F(k,m) \)'s for \( k = 1, \ldots, n \), where \( F(k,m) := [f(0,k), f(m,0)] \). Therefore, it suffices to consider the actions of \( f(m,0) \)'s in the analysis of irreducibility of \( V_k \). By Lemma 2.1, it is easily seen that \( V_k^{(s)} \) is \( W_{1+\infty}^{(+)} \)-stable, and \( \{ v_k^{(s)} \mid s \in \mathbb{N}, s+k \geq 0 \} \) is the complete set of singular vectors of \( W_{1+\infty} \)-module \( V_k \) (here \( v \in V \) is called singular if \( W_{1+\infty}^{(+)} v = 0 \) (since \( v_k^{(s)} \) is a unique (\( \mathbb{C} \)-linear independent) singular vector in \( V_k^{(s)} \) according to the acting rule of \( f(m,0) \) for \( m > 0 \)).

Finally, noticing that \( f(-1,0) v_k^{(s)} \equiv v_k^{(s+1)} \) (mod \( V_k^{(s)} \)), \( f(m,0) V_k^{(s)} \subseteq V_k^{(s)} + V_k^{(s+1)} \) for \( m < 0 \), we see that \( V_k \) is irreducible owing to \( f(-m,0) V_k^{(s)} \subseteq V_k^{(s)} + V_k^{(s+1)} \) for \( m > 0 \).
Corollary 2.4. $V = \bigoplus_{k \in \mathbb{Z}} V_k$ is a completely reducible module for the Virasoro algebra Vir with central charge $c = 2$ under the action given by
\[
\pi(L_m) = f(m, 1),
\]
\[
\pi(c) = 2 \text{id}.
\]
for all $m \in \mathbb{Z}$. Each component $V_k$ is also irreducible for Vir.

Proof. Lemma 2.1 indicates $L_0 = f(0, 1)$ acts diagonalizably on the weight $W_{1+\infty}$-module $V$. The proof of irreducibility of the weight $W_{1+\infty}$-module $V_k$ (see the proof of Theorem 2.3) is reduced to consider the actions of operators $f(m, 0)$'s for $m \in \mathbb{Z}$. Now the same observation applies to the proof of irreducibility of the weight Vir-module $V_k$ provided that we note the formula:
\[
f(m, 1) = \frac{1}{2m} [f(0, 2), f(m, 0)] - \frac{m}{2} f(m, 0), \quad \text{for } m \neq 0
\]
derived from Proposition 2.2.

Remark. In the Virasoro algebra Vir, the operator $L_0$ is usually called the energy operator by physicists (see [KRa]). In [KRa], only positive-energy representations (that is, $L_0$ is diagonal and all its eigenvalues are nonnegative) were discussed there and all irreducible positive-energy representations are proven to be of the form $V(c, h)$ with $h \geq 0$ (see Remark 3.5 in [KRa], here $h$ is the eigenvalue of $L_0$, the highest weight module $V(c, h)$ is the irreducible quotient of the Vema Vir-module $M(c, h)$). The negative-energy representations of Vir, which are related to the Dirac positron theory, was pointed out to be interesting but lack of investigation there (see Section 4.2 [KRa]). In our case, Corollary 2.4 affords such negative-energy representations for Vir.

On the other hand, [KR] classified positive-energy representations with finite degeneracies of the Lie algebra $W_{1+\infty}$, while our bosonic construction in Theorem 2.3 then gives some negative-energy representations for $W_{1+\infty}$.

3. BOSONIC REPRESENTATION OF $W_{1+\infty}(gl_N)$

Let $M_N(C)$ be the $N \times N$ matrix algebra, $gl_N(C) = M_N(C)^{-}$ the general linear Lie algebra over $C$, then $gl_N(D) := gl_N(C) \otimes_C D$ is the general linear Lie algebra with coefficients in $D$. Let $e_{ij}$ be the $N \times N$ matrix unit with 1 in the $(i, j)$-entry and 0 elsewhere, then $gl_N(D)$ has a basis
\[
\{e_{ij} \otimes t^m D^l \mid m \in \mathbb{Z}, l \in \mathbb{N}, 1 \leq i, j \leq n\}.
\]

Consider the subsequent central extension $\widehat{gl}_N(D)$ by $C$ of the Lie algebra $gl_N(D)$, also denoted by $W_{1+\infty}(gl_N)$ (since $W_{1+\infty}(gl_N) = W_{1+\infty}$ when $N = 1$).

\[
\left[ e_{ij} \otimes t^{m_1} D^{n_1}, e_{kl} \otimes t^{m_2} D^{n_2} \right] = \delta_{jk} e_{il} \otimes \sum_{i=0}^{n_1} \left( \begin{array}{c} n_1 \\ i \end{array} \right) m_2^i t^{m_1 + m_2} D^{n_1 + n_2 - i} \]
\[
- \delta_{il} e_{kj} \otimes \sum_{j=0}^{n_2} \left( \begin{array}{c} n_2 \\ j \end{array} \right) m_1^j t^{m_1 + m_2} D^{n_1 + n_2 - j} \]
\[
+ \Phi(e_{ij} \otimes t^{m_1} D^{n_1}, e_{kl} \otimes t^{m_2} D^{n_2}) c,
\]
for all \( m_1, m_2 \in \mathbb{Z}, n_1, n_2 \in \mathbb{N} \) and \( 1 \leq i, j, k, l \leq N \), where \( \Phi \) is given by

\[
\Phi(e_{ij} \otimes t^{m_1}D^{n_1}, e_{kl} \otimes t^{m_2}D^{n_2}) = \begin{cases} 
0, & \text{if } m_1 = 0, \\
(-1)^{n_1+1}j_{k,i} \delta_{m_1+m_2,0} \frac{1}{2} \sum_{i=1}^{m_1} (m_1 - i)^{n_1}i^{n_2}, & \text{if } m_1 > 0, \\
(-1)^{n_2}j_{i,k} \delta_{m_1+m_2,0} \frac{1}{2} \sum_{i=1}^{m_1} (m_1 - i)^{n_1}i^{n_2}, & \text{if } m_1 < 0.
\end{cases}
\]

Now we give a representation of the Lie algebra \( \mathcal{W}_{1+\infty}(gl_N) \).

Define \( S(N) \) to be the unital associative algebra with infinite many generators: \( a_i(n), a_j^*(n) \) \( (n \in \mathbb{Z}, 1 \leq i, j \leq N) \) with the relations

\[
[a_i(n), a_j^*(m)] = [a_i^*(n), a_j^*(m)] = 0, \tag{3.1}
\]

\[
[a_i(n), a_j^*(m)] = -\delta_{i,j}\delta_{n+m,0}. \tag{3.2}
\]

We define the normal ordering as follows.

\[
:a_i(n)a_j^*(m): = \begin{cases} 
    a_i(n)a_j^*(m), & n \leq m, \\
    a_j^*(m)a_i(n), & n > m,
\end{cases} \tag{3.3}
\]

for \( n, m \in \mathbb{Z}, 1 \leq i, j \leq N \).

Similar to (2.5)–(2.6), we have

\[
a_i(n)a_j^*(m) =: a_i(n)a_j^*(m) = -\delta_{i,j}\delta_{n+m,0}\theta(n-m), \tag{3.4}
\]

and

\[
[a_i(m)a_j^*(n), a_k(p)] = \delta_{j,k}\delta_{n+p,0}a_i(m),
\]

\[
[a_i(m)a_j^*(n), a_k^*(p)] = -\delta_{i,k}\delta_{m+p,0}a_j^*(n), \tag{3.5}
\]

for \( m, n, p \in \mathbb{Z}, 1 \leq i, j, k \leq N \).

Let \( S(N)^+ \) be the subalgebra generated by \( a_i(n), a_i^*(n), a_i^*(0) \) for \( n > 0 \) and \( 1 \leq i \leq N \). Let \( S(N)^- \) be the subalgebra generated by \( a_i(0), a_i(n), a_i^*(n) \) for \( n < 0 \) and \( 1 \leq i \leq N \). Those generators in \( S(N)^+ \) are called annihilation operators while those in \( S(N)^- \) are called creation operators. Let \( V(N) \) be a simple \( S(N) \)-module containing an element \( v_0 \), called a “vacuum vector”, and satisfying

\[
S(N)^+v_0 = 0. \tag{3.6}
\]

So all annihilation operators kill \( v_0 \) and

\[
V(N) = S(N)^-v_0. \tag{3.7}
\]

Now we may construct a class of bosons on \( V(N) \). For any \( m, n \in \mathbb{Z}, n \in \mathbb{N} \) and \( 1 \leq i, j \leq N \), set

\[
f_{i,j}(m, n) = \sum_{k \in \mathbb{Z}} (-k)^n : a_i(m-k)a_j^*(k) : \tag{3.8}
\]

Although \( f_{i,j}(m, n) \) are infinite sums, they are well-defined as operators on \( V(N) \). Since, for any vector \( v \in V(N) = S(N)^-v_0 \), only finitely many terms in (3.8) can make a non-zero contribution to \( f_{i,j}(m, n)v \).
Lemma 3.1. For \( m, p, s \in \mathbb{Z} \), \( n \in \mathbb{N} \) and \( 1 \leq i, j, k \leq N \),
\[
[f_{i,j}(m, n), a_k(p)] = \delta_{j,k} p^a a_i(m + p), \tag{3.9}
\]
\[
[f_{i,j}(m, n), a_k^*(p)] = -\delta_{i,k} p^a (-m - p)^n a_i^*(m + p), \tag{3.10}
\]
\[
[f_{i,j}(m, n), a_k(p)a_k^*(s)] = \delta_{j,k} p^a a(m + p)a^*(s) - \delta_{i,l} (-m - s)^n a(p)a^*(m + s). \tag{3.11}
\]

**Proof.** The proof is similar to that of Lemma 2.1. 

Proposition 3.2. For \( m_1, m_2 \in \mathbb{Z} \), \( n_1, n_2 \in \mathbb{N} \) and \( 1 \leq i, j, k, l \leq N \), we have
\[
[f_{i,j}(m_1, n_1), f_{k,l}(m_2, n_2)] = \sum_{i=0}^{n_1} \binom{n_1}{i} m_2^i f_{i,l}(m_1 + m_2, n_1 + n_2 - i) \]
\[
- \sum_{j=0}^{n_2} \binom{n_2}{j} m_1^j f_{k,j}(m_1 + m_2, n_1 + n_2 - j) \]
\[
+ \Psi(f_{i,j}(m_1, n_1), f_{k,l}(m_2, n_2)),
\]
where \( \Psi \) is given by
\[
\Psi(f_{i,j}(m_1, n_1), f_{k,l}(m_2, n_2)) = \begin{cases} 
0, & \text{if } m_1 = 0, \\
(-1)^{n_1+1} \delta_{j,k} \delta_{i,l} \delta_{m_1+m_2,0} \sum_{i,m_1 = 1}^{m_1 + i} (m_1 - i)^{m_1} m_2^i, & \text{if } m_1 > 0, \\
(-1)^{n_1} \delta_{j,k} \delta_{i,l} \delta_{m_1+m_2,0} \sum_{i,m_1 = 1}^{m_1 + i} (m_1 - i)^{m_1} m_2^i, & \text{if } m_1 < 0.
\end{cases}
\]

**Proof.** The proof is similar to that of Proposition 2.2. 

Let \( T = \sum_{i=1}^{N} f_{i,i}(0, 0) \), then one can easily show that
\[
[T, a_j(n)] = a_j(n), \quad [T, a_j^*(n)] = -a_j^*(n), \tag{3.13}
\]
for all \( n \in \mathbb{Z} \), \( 1 \leq j \leq N \). For any
\[
v = a_{i_1}(n_1) \cdots a_{i_s}(n_s) a_{j_1}^*(m_1) \cdots a_{j_t}^*(m_t)v_0
\]
from \( V(N) \), noting that \( Tv_0 = 0 \), one has
\[
Tv = (s-t)v. \tag{3.14}
\]

Noting that
\[
\Psi(f_{i,j}(m_1, n_1), f_{k,l}(m_2, n_2)) = 2 \Phi(c_{ij} \otimes t^{m_1} D^{n_1}, e_{kl} \otimes t^{m_2} D^{n_2}),
\]
we may prove similarly

**Theorem 3.3.** \( V(N) \) is a level 2 module for the Lie algebra \( W_{1+\infty}(gl_N) \) under the action given by
\[
\pi(c) = 2 \text{id},
\]
for all \( m \in \mathbb{Z} \), \( n \in \mathbb{N} \) and \( 1 \leq i, j \leq N \). Moreover,
\[
V(N) = \bigoplus_{k \in \mathbb{Z}} V_k,
\]
is completely reducible, where \( V_k \) is the eigenspace with eigenvalue \( k \) of the operator \( T \), and each component \( V_k \) is irreducible as a \( W_{1+\infty}(gl_N) \)-module. 

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