BOSONIC REPRESENTATIONS OF THE LIE ALGEBRAS

$\mathcal{W}_{1+\infty}$ AND $\mathcal{W}_{1+\infty}(gl_N)$

Dong Liu and Naihong Hu*

Department of Mathematics, East China Normal University

Shanghai, 200062, P.R. China

ABSTRACT

In the paper, we construct bosonic negative-energy representations of the central extension of the Lie algebra of differential operators on the circle, the Lie algebra $\mathcal{W}_{1+\infty}$, as well as of the Lie algebra $\mathcal{W}_{1+\infty}(gl_N)$. In particular, when restricted to the Virasoro subalgebra of the Lie algebra $\mathcal{W}_{1+\infty}$, we obtain a bosonic realization of the Virasoro algebra with central charge $c = 2$ and negative-energies, which is completely reducible.

Keywords: Bosonic representation, Virasoro algebra, differential operator, negative-energy

1. INTRODUCTION

Spinor representations for the affine Kac-Moody Lie algebras were first developed by Frenkel [F] and Kac-Peterson [KP] independently. The idea is to use a Clifford algebra with infinitely many generators to construct certain quadratic elements, which, together with the identity element, span an orthogonal affine Kac-Moody Lie algebra. Thereafter, Feingold-Frenkel [FF] constructed the so-called fermionic or bosonic representations for all classical affine Kac-Moody Lie algebras by using Clifford or Weyl algebras with infinitely many generators. Recently, Gao [G2] constructed fermionic and bosonic representations for the extended affine Lie algebra $\widetilde{gl_N(\mathbf{C}_q)}$.

As we know, the Lie algebra $\widehat{\mathcal{D}^-}$, as the universal central extension of the Lie algebra of differential operators on the circle (see [KP]), has appeared in various models of two-dimentional quantum field theory and integrable systems (see the references in [FKRW], [KR]). A systematic study of the quasifinite highest weight representation theory of the Lie algebra $\widehat{\mathcal{D}^-}$, which is often referred to as $\mathcal{W}_{1+\infty}$ algebra by physicists, has been investigated by Kac et al (see [KR], [FKRW], etc.). Motivated by [G2], this paper is devoted to constructing bosonic representations of the Lie algebra $\mathcal{W}_{1+\infty}$ and of the Lie algebra $\mathcal{W}_{1+\infty}(gl_N)$. In particular, we obtain a bosonic realization of the Virasoro algebra with central charge $c = 2$ and negative-energies when restricted to the Virasoro subalgebra of the Lie algebra $\mathcal{W}_{1+\infty}$, which differs from the typical one given in the literature (for instance, see [F2], or [KRa]).

Throughout this paper, $\mathbf{Z}$, $\mathbf{N}$ and $\mathbf{C}$ denote the set of integers, non-negative integers and complex numbers, respectively.

* Corresponding author. Email: nhhu@math.ecnu.edu.cn

Let $\mathbf{C}[\,t,t^{-1}\,]$ be the algebra of Laurent polynomials over $\mathbf{C}$, and $\mathcal{D} = \mathrm{Diff}\,\mathbf{C}[\,t,t^{-1}\,]$ the associative algebra of all differential operators over $\mathbf{C}[\,t,t^{-1}\,]$, whose basis over $\mathbf{C}$ is $\{t^m D^n \mid m \in \mathbf{Z},\ n \in \mathbf{N}\}$ with multiplication:

$$(t^a D^b)\cdot(t^c D^d) = \sum_{i=0}^{b} \binom{b}{i} c^i t^{a+c} D^{b+d-i},$$

where $D = t\partial,\ \partial = d/dt$.

Let $\mathcal{D}^-$ be the Lie algebra of $\mathcal{D}$ under Lie bracket given by

$$\left[\, t^{m_1} D^{n_1}, t^{m_2} D^{n_2} \,\right] = \sum_{i=0}^{n_1} \binom{n_1}{i} m_2^i\, t^{m_1+m_2} D^{n_1+n_2-i} - \sum_{j=0}^{n_2} \binom{n_2}{j} m_1^j\, t^{m_1+m_2} D^{n_1+n_2-j}$$

for all $m_1,\ m_2 \in \mathbf{Z},\ n_1,\ n_2 \in \mathbf{N}$.

Li [L] proved $H^2(\mathcal{D}^-, \mathbf{C}) = 1$ (also see [KP], [Kas], [KR] and [LW]). In this paper, we will adopt a convenient form of a specific 2-cocycle on $\mathcal{D}^-$, which is due to Kac and Radul (see formula (1.5.5) in [KR]) up to a sign, more precisely, we take $f(D) = D^{n_2}$, $g(D) = D^{n_1}$ and $\phi(t^{m_1}D^{n_1}, t^{m_2}D^{n_2}) := \frac{1}{2}\,\psi(t^{m_2}D^{n_2}, t^{m_1}D^{n_1})$ in the notation of [KR]. So we have the following

Lemma 1.1 ([KR]). *Any non-trivial 2-cocycle on $\mathcal{D}^-$ is equivalent to ϕ:*

$$\phi(t^{m_1}D^{n_1}, t^{m_2}D^{n_2}) = \begin{cases} 0, & \text{if} \quad m_1 = 0, \\ (-1)^{n_1+1}\,\delta_{m_1+m_2,0}\,\frac{1}{2}\sum_{i=1}^{m_1}(m_1-i)^{n_1}\,i^{n_2}, & \text{if} \quad m_1 > 0, \\ (-1)^{n_1}\,\delta_{m_1+m_2,0}\,\frac{1}{2}\sum_{i=m_1}^{-1}(m_1-i)^{n_1}\,i^{n_2}, & \text{if} \quad m_1 < 0. \end{cases}$$

Let $\mathcal{W}_{1+\infty}$ denote the universal (one-dimensional) central extension $\widehat{\mathcal{D}^-}$ of the Lie algebra $\mathcal{D}^-$ by the above 2-cocycle ϕ. In particular, $\mathrm{Vir} = \mathrm{Span}_{\mathbf{C}}\{\,L_m = t^m D, c \mid m \in \mathbf{Z}\,\}$ is the Virasoro subalgebra of $\mathcal{W}_{1+\infty}$, where its Lie bracket is given as follows (since $\sum_{i=1}^{m}(m-i)i = \frac{1}{6}\,(m-1)\,m\,(m+1)$ for $m > 0$)

$$[\,L_m, L_n\,] = (n-m)\,L_{m+n} + \frac{1}{12}\,(m-1)\,m\,(m+1)\,\delta_{m+n,0}\,c,$$
$$[\,c, L_m\,] = 0.$$

2. BOSONIC REPRESENTATION OF $\mathcal{W}_{1+\infty}$

Define $\mathcal{S}$ to be the unital associative algebra with infinitely many generators: $a(n),\ a^*(n)$ $(n \in \mathbf{Z})$ with relations

$$[\,a(n), a(m)\,] = [\,a^*(n), a^*(m)\,] = 0, \tag{2.1}$$

$$[\,a(n), a^*(m)\,] = -\delta_{n+m,0}. \tag{2.2}$$

We define the normal ordering as follows.

$$:a(n)a^*(m): = \begin{cases} a(n)a^*(m), & n \le m, \\ a^*(m)a(n), & n > m, \end{cases} \tag{2.3}$$

for $n,\ m \in \mathbf{Z}$. Set

$$\theta(n) = \begin{cases} 1, & n > 0, \\ 0, & n \le 0. \end{cases} \tag{2.4}$$

Then

$$a(n)a^*(m) =: a(n)a^*(m) : -\, \delta_{n+m,0}\, \theta(n-m), \tag{2.5}$$

and

$$\begin{aligned} [\, a(m)a^*(n), a(p)\,] &= \delta_{n+p,0}\, a(m), \\ [\, a(m)a^*(n), a^*(p)\,] &= -\delta_{m+p,0}\, a^*(n), \end{aligned} \tag{2.6}$$

for $m,\, n,\, p \in \mathbf{Z}$.

Let $\mathcal{S}^+$ be the subalgebra generated by $a(n)$, $a^*(0)$, $a^*(m)$ for $n,\, m > 0$. Let $\mathcal{S}^-$ be the subalgebra generated by $a(0)$, $a(n)$, $a^*(m)$ for $n,\, m < 0$. Those generators in $\mathcal{S}^+$ are called annihilation operators while those in $\mathcal{S}^-$ are called creation operators. Let V be a simple $\mathcal{S}$-module containing an element v_0, called a "vacuum vector", and satisfying

$$\mathcal{S}^+ v_0 = 0. \tag{2.7}$$

So all annihilation operators kill v_0 and

$$V = \mathcal{S}^- v_0. \tag{2.8}$$

Now we may construct a class of bosons on V. For any $m \in \mathbf{Z}$, $n \in \mathbf{N}$, set

$$f(m,n) = \sum_{i\in\mathbf{Z}} (-i)^n : a(m-i)a^*(i) : . \tag{2.9}$$

Although $f(m,n)$ are infinite sums, they are well-defined as operators on V. Indeed, for any vector $v \in V = \mathcal{S}^- v_0$, only finitely many terms in (2.9) can make a non-zero contribution to $f(m,n)v$.

Lemma 2.1. *For* $m,\, p,\, s \in \mathbf{Z}$, $n \in \mathbf{N}$,

$$[\, f(m,n), a(p)\,] = p^n a(m+p), \tag{2.10}$$

$$[\, f(m,n), a^*(p)\,] = -(-m-p)^n a^*(m+p), \tag{2.11}$$

$$[\, f(m,n), a(p)a^*(s)\,] = p^n a(m+p)a^*(s) - (-m-s)^n a(p)a^*(m+s). \tag{2.12}$$

Proof. Since

$$\begin{aligned} [\, f(m,n), a(p)\,] &= \sum_{i\in\mathbf{Z}} (-i)^n [\, : a(m-i)a^*(i) :, a(p)\,] \\ &= \sum_{i\in\mathbf{Z}} (-i)^n [\, a(m-i)a^*(i), a(p)\,] \\ &= p^n a(m+p), \end{aligned}$$

(2.10) is true. The proof of (2.11) is similar, and (2.12) follows from (2.10) and (2.11). ∎

Proposition 2.2. *For* $m_1,\, m_2 \in \mathbf{Z}$, $n_1,\, n_2 \in \mathbf{N}$, *we have*

$$\begin{aligned} [\, f(m_1,n_1), f(m_2,n_2)\,] &= \sum_{i=0}^{n_1} \binom{n_1}{i} m_2^i\, f(m_1+m_2, n_1+n_2-i) \\ &\quad - \sum_{j=0}^{n_2} \binom{n_2}{j} m_1^j\, f(m_1+m_2, n_1+n_2-j) \\ &\quad + \varphi\big(f(m_1,n_1), f(m_2,n_2)\big), \end{aligned}$$

where φ is given by

$$\varphi(f(m_1,n_1),f(m_2,n_2))=\begin{cases}0, \quad \text{if} \quad m_1=0,\\ (-1)^{n_1+1}\delta_{m_1+m_2,0}\sum_{i=1}^{m_1}(m_1-i)^{n_1}i^{n_2}, \quad \text{if} \quad m_1>0,\\ (-1)^{n_1}\delta_{m_1+m_2,0}\sum_{i=m_1}^{-1}(m_1-i)^{n_1}i^{n_2}, \quad \text{if} \quad m_1<0.\end{cases}$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}
&[\,f(m_1,n_1),f(m_2,n_2)\,]\\
&=[\,f(m_1,n_1),\sum_{t\in\mathbf{Z}}(-t)^{n_2}:a(m_2-t)a^*(t):]\\
&=[\,f(m_1,n_1),\sum_{t\in\mathbf{Z}}(-t)^{n_2}a(m_2-t)a^*(t)\,]\\
&=\sum_{t\in\mathbf{Z}}(-t)^{n_2}(m_2-t)^{n_1}a(m_1+m_2-t)a^*(t)-\sum_{t\in\mathbf{Z}}(-m_1-t)^{n_1}(-t)^{n_2}a(m_2-t)a^*(m_1+t)\\
&=\sum_{t\in\mathbf{Z}}(-t)^{n_2}(m_2-t)^{n_1}:a(m_1+m_2-t)a^*(t):-\sum_{t\in\mathbf{Z}}(-m_1-t)^{n_1}(-t)^{n_2}:a(m_2-t)a^*(m_1+t):\\
&-\delta_{m_1+m_2,0}\left(\sum_{t\in\mathbf{Z}}(m_2-t)^{n_1}(-t)^{n_2}\theta(m_1+m_2-2t)-\sum_{t\in\mathbf{Z}}(-m_1-t)^{n_1}(-t)^{n_2}\theta(m_2-m_1-2t)\right)\\
&=\sum_{i=0}^{n_1}\binom{n_1}{i}m_2^i\,f(m_1+m_2,n_1+n_2-i)-\sum_{j=0}^{n_2}\binom{n_2}{j}m_1^j\,f(m_1+m_2,n_1+n_2-j)\\
&-\delta_{m_1+m_2,0}\left(\sum_{t\in\mathbf{Z}}(-m_1-t)^{n_1}(-t)^{n_2}\theta(-2t)-\sum_{t\in\mathbf{Z}}(-m_1-t)^{n_1}(-t)^{n_2}\theta(-2m_1-2t)\right)\\
&=\sum_{i=0}^{n_1}\binom{n_1}{i}m_2^i\,f(m_1+m_2,n_1+n_2-i)-\sum_{j=0}^{n_2}\binom{n_2}{j}m_1^j\,f(m_1+m_2,n_1+n_2-j)\\
&+\delta_{m_1+m_2,0}\sum_{t\in\mathbf{Z}}\left(-\theta(-2t)+\theta(-2m_1-2t)\right)(-m_1-t)^{n_1}(-t)^{n_2}\\
&=\sum_{i=0}^{n_1}\binom{n_1}{i}\,m_2^i\,f(m_1+m_2,n_1+n_2-i)\\
&-\sum_{j=0}^{n_2}\binom{n_2}{j}\,m_1^j\,f(m_1+m_2,n_1+n_2-j)\\
&+\varphi\big(f(m_1,n_1),f(m_2,n_2)\big),
\end{aligned}$$

where the last equality is given by

$$\sum_{t\in\mathbf{Z}}\left(-\theta(-2t)+\theta(-2m_1-2t)\right)(-m_1-t)^{n_1}(-t)^{n_2}=\begin{cases}0, \quad m_1=0,\\ -\sum_{t=1}^{m_1}(t-m_1)^{n_1}t^{n_2}, \quad m_1>0,\\ \sum_{t=m_1}^{-1}(t-m_1)^{n_1}t^{n_2}, \quad m_1<0.\end{cases}$$

The proof is completed. ∎

Let $T=f(0,0)$, then Lemma 2.1 gives

$$[\,T,a(n)\,]=a(n),\quad [\,T,a^*(n)\,]=-a^*(n), \tag{2.13}$$

for all $n\in\mathbf{Z}$. For any $v=a(n_1)\cdots a(n_s)a^*(m_1)\cdots a^*(m_l)v_0\in V$, noting that $Tv_0=0$, one has

$$Tv=(s-l)v. \tag{2.14}$$

According to Proposition 2.2 and Lemma 2.1, we obtain

Theorem 2.3. *V is a module for the Lie algebra $\mathcal{W}_{1+\infty}$ with central charge $c = 2$ under the action given by*

$$\pi(t^m D^n) = f(m,n),$$

$$\pi(c) = 2\,\mathrm{id}.$$

for all $m \in \mathbf{Z}$, $n \in \mathbf{N}$. Moreover,

$$V = \bigoplus_{k\in\mathbf{Z}} V_k,$$

is completely reducible, where V_k is the eigenspace with eigenvalue k of the operator T, and each component V_k is irreducible as a $\mathcal{W}_{1+\infty}$-module.

Proof. Notice that $\varphi(f(m_1,n_1), f(m_2,n_2)) = 2\,\phi(t^{m_1}D^{n_1}, t^{m_2}D^{n_2})$. Therefore, Proposition 2.2 shows that V is a $\mathcal{W}_{1+\infty}$-module with central charge $c = 2$. On the other hand, Lemma 2.1 indicates that each eigenspace V_k of the operator T is $\mathcal{W}_{1+\infty}$-stable. In what follows, we shall prove that V_k is also irreducible under the actions of all $f(m,n)$'s.

To this end, we need introduce some notation. Fix a $k \in \mathbf{Z}$, for any $s \in \mathbf{N}$ such that $s+k \geq 0$, set $v_k^{(s)} := a(0)^s\, a^*(-1)^{s+k}.v_0$ and

$$V_k^{(s)} := \mathrm{Span}_{\mathbf{C}}\{\, a(n_1)\cdots a(n_s)a^*(m_1)\cdots a^*(m_{s+k}).v_0 \mid n_i \leq 0, m_j < 0\,\}.$$

It is clear that $V_k = \bigoplus_{s\in\mathbf{N}, s+k\geq 0} V_k^{(s)}$. On the other hand, if we define the *weight* by

$$\mathrm{wt}(t^m D^n) = m,$$

which induces a principle $\mathbf{Z}$-gradation of $\mathcal{W}_{1+\infty}$:

$$\mathcal{W}_{1+\infty} = \bigoplus_{j\in\mathbf{Z}} \mathcal{W}_{1+\infty}{}^{(j)},$$

we then have a triangular decomposition of $\mathcal{W}_{1+\infty}$ as follows

$$\mathcal{W}_{1+\infty} = \mathcal{W}_{1+\infty}^{(-)} \bigoplus \mathcal{W}_{1+\infty}^{(0)} \bigoplus \mathcal{W}_{1+\infty}^{(+)}\,,$$

where $\mathcal{W}_{1+\infty}^{(-)}=\{t^m D^n \mid m < 0, n \in \mathbf{N}\}$, $\mathcal{W}_{1+\infty}^{(0)}=\{D^n \mid n \in \mathbf{N}\}$, $\mathcal{W}_{1+\infty}^{(+)}=\{t^m D^n \mid m > 0, n \in \mathbf{N}\}$.

Lemma 2.1 shows that V_k is a weight module with respect to the abelian subalgebra $\mathcal{W}_{1+\infty}^{(0)}$. Since

$$[\, f(0,n), f(m,0)\,] = \sum_{j=1}^{n} \binom{n}{j} m^j\, f(m, n-j),$$

the actions of $f(m,0), f(m,1), \cdots, f(m,n-1)$ on V can be expressed as some combinations of $F(k,m)$'s for $k = 1, \cdots, n$, where $F(k,m) := [\, f(0,k), f(m,0)\,]$. Therefore, it suffices to consider the actions of $f(m,0)$'s in the analysis of irreducibility of V_k. By Lemma 2.1, it is easily seen that $V_k^{(s)}$ is $\mathcal{W}_{1+\infty}^{(+)}$-stable, and $\{\, v_k^{(s)} \mid s \in \mathbf{N}, s+k \geq 0\,\}$ is the complete set of *singular vectors* of $\mathcal{W}_{1+\infty}$-module V_k (here $v \in V$ is called singular if $\mathcal{W}_{1+\infty}^{(+)}.v = 0$) (since $v_k^{(s)}$ is a unique ($\mathbf{C}$-linear independent) singular vector in $V_k^{(s)}$ according to the acting rule of $f(m,0)$ for $m > 0$).

Finally, noticing that $f(-1,0)(v_k^{(s)}) \equiv v_k^{(s+1)} \pmod{V_k^{(s)}}$, $f(m,0)(V_k^{(s)}) \subseteq V_k^{(s)} + V_k^{(s+1)}$ for $m < 0$, we see that V_k is irreducible owing to $f(-m,0).\, f(m,0).\, v_k^{(s)} = [\, f(-m,0), f(m,0)\,].\, v_k^{(s)} = m\, v_k^{(s)} \neq 0$ (by Proposition 2.2). ∎

Corollary 2.4. *$V=\bigoplus_{k\in\mathbf{Z}}V_k$ is a completely reducible module for the Virasoro algebra Vir with central charge $c=2$ under the action given by*

$$\pi(L_m)=f(m,1),$$

$$\pi(c)=2\,\mathrm{id}.$$

for all $m\in\mathbf{Z}$. Each component V_k is also irreducible for Vir.

Proof. Lemma 2.1 indicates $L_0=f(0,1)$ acts diagonalizably on the weight $\mathcal{W}_{1+\infty}$-module V. The proof of irreducibility of the weight $\mathcal{W}_{1+\infty}$-module V_k (see the proof of Theorem 2.3) is reduced to consider the actions of operators $f(m,0)$'s for $m\in\mathbf{Z}$. Now the same observation applies to the proof of irreducibility of the weight Vir-module V_k provided that we note the formula:

$$f(m,1)=\frac{1}{2\,m}\,[\,f(0,2),f(m,0)\,]-\frac{m}{2}\,f(m,0),\quad \textit{for}\ \ m\neq 0$$

derived from Proposition 2.2. ∎

Remark. In the Virasoro algebra Vir, the operator L_0 is usually called the *energy operator* by physicists (see [KRa]). In [KRa], only *positive-energy* representations (that is, L_0 is diagonal and all its eigenvalues are nonnegative) were discussed there and all irreducible positive-energy representations are proven to be of the form $V(c,h)$ with $h\geq 0$ (see *Remark 3.5* in [KRa], here h is the eigenvalue of L_0, the highest weight module $V(c,h)$ is the irreducible quotient of the Vema Vir-module $M(c,h)$). The negative-energy representations of Vir, which are related to the Dirac positron theory, was pointed out to be interesting but lack of investigation there (see Section 4.2 [KRa]). In our case, Corollary 2.4 affords such negative-energy representations for Vir.

On the other hand, [KR] classified positive-energy representations with finite degeneracies of the Lie algebra $\mathcal{W}_{1+\infty}$, while our bosonic construction in Theorem 2.3 then gives some negative-energy representations for $\mathcal{W}_{1+\infty}$.

3. BOSONIC REPRESENTATION OF $\mathcal{W}_{1+\infty}(gl_N)$

Let $M_N(\mathbf{C})$ be the $N\times N$ matrix algebra, $\mathrm{gl}_N(\mathbf{C})=M_N(\mathbf{C})^-$ the general linear Lie algebra over $\mathbf{C}$, then $\mathrm{gl}_N(\mathcal{D}):=\mathrm{gl}_N(\mathbf{C})\otimes_{\mathbf{C}}\mathcal{D}$ is the general linear Lie algebra with coefficients in $\mathcal{D}$. Let e_{ij} be the $N\times N$ matrix unit with 1 in the (i,j)-entry and 0 elsewhere, then $\mathrm{gl}_N(\mathcal{D})$ has a basis

$$\{e_{ij}\otimes t^mD^l\mid m\in\mathbf{Z},\ l\in\mathbf{N},\ 1\leq i,j\leq n\}.$$

Consider the subsequent central extension $\widehat{\mathrm{gl}_N}(\mathcal{D})$ by $\mathbf{C}$ of the Lie algebra $\mathrm{gl}_N(\mathcal{D})$, also denoted by $\mathcal{W}_{1+\infty}(gl_N)$ (since $\mathcal{W}_{1+\infty}(gl_N)=\mathcal{W}_{1+\infty}$ when $N=1$).

$$\begin{aligned}[\,e_{ij}\otimes t^{m_1}D^{n_1},e_{kl}\otimes t^{m_2}D^{n_2}\,] \;&=\; \delta_{jk}e_{il}\otimes\sum_{i=0}^{n_1}\binom{n_1}{i}m_2^i\,t^{m_1+m_2}D^{n_1+n_2-i}\\ &\quad-\delta_{il}e_{kj}\otimes\sum_{j=0}^{n_2}\binom{n_2}{j}m_1^j\,t^{m_1+m_2}D^{n_1+n_2-j}\\ &\quad+\Phi\big(e_{ij}\otimes t^{m_1}D^{n_1},e_{kl}\otimes t^{m_2}D^{n_2}\big)\,c,\end{aligned}$$

for all $m_1, m_2 \in \mathbf{Z},\ n_1, n_2 \in \mathbf{N}$ and $1 \leq i, j, k, l \leq N$, where Φ is given by

$$\Phi(e_{ij} \otimes t^{m_1} D^{n_1}, e_{kl} \otimes t^{m_2} D^{n_2}) = \begin{cases} 0, & \text{if} \quad m_1 = 0, \\ (-1)^{n_1+1} \delta_{j,k} \delta_{i,l} \delta_{m_1+m_2,0} \frac{1}{2} \sum_{i=1}^{m_1} (m_1 - i)^{n_1} i^{n_2}, & \text{if} \quad m_1 > 0, \\ (-1)^{n_1} \delta_{j,k} \delta_{i,l} \delta_{m_1+m_2,0} \frac{1}{2} \sum_{i=m_1}^{-1} (m_1 - i)^{n_1} i^{n_2}, & \text{if} \quad m_1 < 0. \end{cases}$$

.

Now we give a representation of the Lie algebra $\mathcal{W}_{1+\infty}(gl_N)$.

Define $\mathcal{S}(N)$ to be the unital associative algebra with infinite many generators: $a_i(n), a_j^*(n)$ $(n \in \mathbf{Z}, 1 \leq i,\ j \leq N)$ with the relations

$$[\, a_i(n), a_j(m)\,] = [\, a_i^*(n), a_j^*(m)\,] = 0, \tag{3.1}$$

$$[\, a_i(n), a_j^*(m)\,] = -\delta_{i,j}\delta_{n+m,0}. \tag{3.2}$$

We define the normal ordering as follows.

$$: a_i(n) a_j^*(m) := \begin{cases} a_i(n) a_j^*(m), & n \leq m, \\ a_j^*(m) a_i(n), & n > m, \end{cases} \tag{3.3}$$

for $n, m \in \mathbf{Z}, 1 \leq i, j \leq N$.

Similar to (2.5)–(2.6), we have

$$a_i(n) a_j^*(m) =: a_i(n) a_j^*(m) : - \delta_{i,j} \delta_{n+m,0} \theta(n-m), \tag{3.4}$$

and

$$\begin{aligned} [\, a_i(m) a_j^*(n), a_k(p)\,] &= \delta_{j,k} \delta_{n+p,0} a_i(m), \\ [\, a_i(m) a_j^*(n), a_k^*(p)\,] &= -\delta_{i,k} \delta_{m+p,0} a_j^*(n), \end{aligned} \tag{3.5}$$

for $m,\ n,\ p \in \mathbf{Z},\ 1 \leq i, j, k \leq N$.

Let $\mathcal{S}(N)^+$ be the subalgebra generated by $a_i(n),\ a_i^*(n),\ a_i^*(0)$ for $n > 0$ and $1 \leq i \leq N$. Let $\mathcal{S}(N)^-$ be the subalgebra generated by $a_i(0),\ a_i(n),\ a_i^*(n)$ for $n < 0$ and $1 \leq i \leq N$. Those generators in $\mathcal{S}(N)^+$ are called annihilation operators while those in $\mathcal{S}(N)^-$ are called creation operators. Let $V(N)$ be a simple $\mathcal{S}(N)$-module containing an element v_0, called a "vacuum vector", and satisfying

$$\mathcal{S}(N)^+ v_0 = 0. \tag{3.6}$$

So all annihilation operators kill v_0 and

$$V(N) = \mathcal{S}(N)^- v_0. \tag{3.7}$$

Now we may construct a class of bosons on $V(N)$. For any $m \in \mathbf{Z}, n \in \mathbf{N}$ and $1 \leq i, j \leq N$, set

$$f_{i,j}(m, n) = \sum_{k \in \mathbf{Z}} (-k)^n : a_i(m-k) a_j^*(k) : \tag{3.8}$$

Although $f_{i,j}(m,n)$ are infinite sums, they are well-defined as operators on $V(N)$. Since, for any vector $v \in V(N) = \mathcal{S}(N)^- v_0$, only finitely many terms in (3.8) can make a non-zero contribution to $f_{i,j}(m,n)v$.

Lemma 3.1. *For* $m,p,s\in\mathbf{Z},\ n\in\mathbf{N}$ *and* $1\le i,j,k\le N$,

$$[\,f_{i,j}(m,n),a_k(p)\,]=\delta_{j,k}p^n a_i(m+p), \tag{3.9}$$

$$[\,f_{i,j}(m,n),a^*_k(p)\,]=-\delta_{i,k}(-m-p)^n a^*_j(m+p), \tag{3.10}$$

$$[\,f_{i,j}(m,n),a_k(p)a^*_l(s)\,]=\delta_{j,k}p^n a(m+p)a^*(s)-\delta_{i,l}(-m-s)^n a(p)a^*(m+s). \tag{3.11}$$

Proof. The proof is similar to that of Lemma 2.1. ∎

Proposition 3.2. *For* $m_1,m_2\in\mathbf{Z},\ n_1,n_2\in\mathbf{N}$ *and* $1\le i,j,k,l\le N$, *we have*

$$\begin{aligned}[\,f_{i,j}(m_1,n_1),f_{k,l}(m_2,n_2)\,] \;&=\; \delta_{j,k}\sum_{i=0}^{n_1}\binom{n_1}{i}m_2^i f_{i,l}(m_1+m_2,n_1+n_2-i)\\ &\quad-\delta_{i,l}\sum_{j=0}^{n_2}\binom{n_2}{j}m_1^j f_{k,j}(m_1+m_2,n_1+n_2-j)\\ &\quad+\Psi\big(f_{i,j}(m_1,n_1),f_{k,l}(m_2,n_2)\big),\end{aligned}$$

where Ψ *is given by*

$$\Psi(f_{i,j}(m_1,n_1),f_{k,l}(m_2,n_2))=\begin{cases}0, & \text{if}\quad m_1=0,\\ (-1)^{n_1+1}\delta_{j,k}\delta_{i,l}\delta_{m_1+m_2,0}\sum_{i=1}^{m_1}(m_1-i)^{n_1}i^{n_2}, & \text{if}\quad m_1>0,\\ (-1)^{n_1}\delta_{j,k}\delta_{i,l}\delta_{m_1+m_2,0}\sum_{i=m_1}^{-1}(m_1-i)^{n_1}i^{n_2}, & \text{if}\quad m_1<0.\end{cases}$$

Proof. The proof is similar to that of Proposition 2.2. ∎

Let $T=\sum_{i=1}^N f_{i,i}(0,0)$, then one can easily show that

$$[\,T,a_j(n)\,]=a_j(n),\quad [\,T,a^*_j(n)\,]=-a^*_j(n), \tag{3.13}$$

for all $n\in\mathbf{Z},\ 1\le j\le N$. For any

$$v=a_{i_1}(n_1)\cdots a_{i_s}(n_s)a^*_{j_1}(m_1)\cdots a^*_{j_t}(m_t)v_0$$

from $V(N)$, noting that $Tv_0=0$, one has

$$Tv=(s-t)v. \tag{3.14}$$

Noting that

$$\Psi(f_{i,j}(m_1,n_1),f_{k,l}(m_2,n_2))=2\,\Phi(e_{ij}\otimes t^{m_1}D^{n_1},e_{kl}\otimes t^{m_2}D^{n_2}),$$

we may prove similarly

Theorem 3.3. $V(N)$ *is a level* 2 *module for the Lie algebra* $\mathcal{W}_{1+\infty}(gl_N)$ *under the action given by*

$$\pi(e_{ij}\otimes t^mD^n)=f_{i,j}(m,n),$$

$$\pi(c)=2\,\mathrm{id},$$

for all $m\in\mathbf{Z},\ n\in\mathbf{N}$ *and* $1\le i,\ j\le N$. *Moreover,*

$$V(N)=\bigoplus_{k\in\mathbf{Z}}V_k,$$

is completely reducible, where V_k *is the eigenspace with eigenvalue* k *of the operator* T, *and each component* V_k *is irreducible as a* $\mathcal{W}_{1+\infty}(gl_N)$*-module.* ∎

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REFERENCES

[FF] Feingold, A. J.; Frenkel, I.B., *Classical affine Lie algebras,* Adv. in Math. **56 (1985)**, 117–172.

[F1] Frenkel, I.B., *Spinor representations of affine Lie algebras,* Proc. Natl. Acad. Sci. USA **77 (1980)**, 6303–6306.

[F2] Frenkel, I.B., *Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory,* J. Funct. Anal. **44 (1981)**, 259–327.

[FKRW] Frenkel, E.; Kac, V.; Radul, A.; Wang, W., $\mathcal{W}_{1+\infty}$ *and* $\mathcal{W}(gl_N)$ *with central charge* N, Comm. Math. Phys. **170 (1995)**, 337–357.

[G1] Gao, Y., *Representations of the extended affine Lie algebra coordinatized by certain quantum tori,* Composito Mathematica **132 (2000)**, 1–25.

[G2] Gao, Y., *Fermionic and bosonic representations of the extended affine Lie algebra* $\widetilde{gl_N(\mathbf{C}_q)}$, Canad. Math. Bull. **45 (4) (2002)**, 623–633.

[KP] Kac, V.G. and Peterson, D.H., *Spin and wedge representations of infinfite dimensional Lie algebras and groups,* Natl. Acad. Sci. USA **78 (1981)**, 3308–3312.

[KR] Kac, V.G.; Radul, A., *Quasifinite highest weight modules over the Lie algebra of differential operators on the circle,* Comm. Math. Phys. **157 (1993)**, 429–457.

[KRa] Kac, V.G.; Raina, A., *Highest Weight Representations of Infinite Dimensional Lie Algebras,* Advanced Series in Math. Physics, Vol. **2 (1987)**.

[Kas] Kassel, C., *Cyclic homology of differential operators, the Virasoro algebra and a q-analogue,* Comm. Math. Phys. **164 (1992)**, 343–356.

[L] Li, Wanglai, 2-*Cocycles on the algebra of differential operators,* J. Algebra. **122 (1989)**, 64–80.

[LH] Liu, Dong; Hu, Naihong, *Derivation algebras and 2-cocycles of the algebras of q-differential operators,* preprint, **(2002)** (submitted).

[LW] Li, Wanglai; Wilson, R.L., *Central extensions of some Lie algebras,* Proc. AMS **126 (9) (1998)**, 2569–2577.