Quantum Group Structure Associated to the Quantum Affine Space

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Abstract. A braided monoidal category $G_{\Lambda, \theta}$ of $\Lambda$-graded associative algebras over a field $k$ is established. The structural feature (including its PBW-basis) of the braided universal enveloping algebra $U(L)$ of a $\theta$-Lie algebra $L$ is investigated as an object in $G_{\Lambda, \theta}$ and a class of quantum groups arising from $U(L)$ is presented. In addition, the quantized universal enveloping algebra of any abelian Lie algebra is given, which is a twisted quantum group associated to the quantum affine space $k[A^0_q]$.

1. Introduction

By definition, the quantum affine space over a field $k$ is the quadratic algebra: $k[A^0_q] = k\{x_1, \cdots, x_n\}/(x_jx_i - q x_ix_j, \ i < j)$ (i.e., the quantum symmetric algebra), which plays a crucial role in studying both quantum linear groups $GL_q(n)$, $SL_q(n)$ (cf. [8], etc.) and quantized universal enveloping algebras $U_q(\mathfrak{g}_n)$, $U_q(\mathfrak{sl}_n)$ (cf. [9], etc). As is known, the quantum affine space is usually viewed as a $q$-analogue of the usual polynomial (Hopf) algebra in $n$ variables. However, it hadn’t been clear whether $k[A^0_q]$ has a Hopf algebra structure until $k[A^0_q]$ turned out to be a free object ([3]) in a certain braided category $GB$ of $\Lambda$-graded $\theta$-commutative associative algebras over $k$, which is distinguished from the category of usual Hopf algebras over $k$ (see [1]). Actually, a negative answer for the above question is then known. A natural problem is to ask what is the quantum group structure (i.e., a noncommutative and noncocommutative Hopf algebra in the sense of Drinfeld [6]) associated to a polynomial (Hopf) algebra, or equivalently, what is the quantized universal enveloping algebra of any abelian Lie algebra (this problem is open for Lie algebras in general except for those of the semisimple Lie algebras, the Kac-Moody algebras, the toroidal Lie algebras). The answer was partially obtained in [3] under the assumption that $q \in k^*$ is generic or a primitive $l$-th root of unity with an odd $l$. In the present paper, we will remove this restriction and yield a new and more natural version in a broader sense, actually we have done much more.

The paper is organized as follows. In section 2, a braided monoidal category $G_{\Lambda, \theta}$ is introduced, which consists of $\Lambda$-graded associative algebras over a field $k$ relative to a
braiding $\psi$, where $\psi$ is provided by a skew bicharacter $\theta$ on the abelian group $\Lambda$. In section 3, the concept of $\theta$-Lie algebras is recalled, which is due to M. Scheunert ([10]). The braided universal enveloping algebra of a $\theta$-Lie algebra is showed to be a braided Hopf algebra as an object in the braided monoidal category $\mathcal{G}_{\Lambda, \theta}$ and its PBW-basis Theorem is proved by using Bergman’s approach [2]. Section 4 presents a class of quantum groups arising from $\theta$-Lie algebras, which can be viewed as the genuine universal enveloping algebras of $\theta$-Lie algebras in the category of the usual Hopf algebras over $k$. Particularly, the quantum affine space $k[A^0_q]$ can be naturally interpreted as the braided universal enveloping algebra of an abelian $\theta$-Lie algebra. In section 5, a twisted quantum group structure associated to the quantum affine space is constructed, which meets simultaneously the two basic requirements such as: (1) It is exactly the usual polynomial Hopf algebra when $q$ takes 1; (2) The restriction to its central (polynomial) Hopf subalgebra $(\text{char } q)_l = 1$ and $(\text{char } q)_{l-1} = 0$ for $1 < r < l$. So, $(\text{char } q)_{l-1} = 0$ for any $l \in \mathbb{N}$. Denote it by $\text{char}(q) = l$. Thus $\text{char}(q) \in \mathbb{Z}_+.

2. Braided Monoidal Category $\mathcal{G}_{\Lambda, \theta}$

Let $\Lambda$ be an abelian group and $\theta : \Lambda \times \Lambda \rightarrow k^*$ a skew bicharacter of the abelian group $\Lambda$ (that is, for any fixed $\alpha, \beta \in \Lambda$, both $\theta(\alpha, \cdot)$ and $\theta(\cdot, \beta) : \Lambda \rightarrow k^*$ are characters of the abelian group $\Lambda$), namely, the mapping $\theta$ obeys the relations below:

\[
\begin{align*}
\theta(\alpha + \beta, \gamma) &= \theta(\alpha, \gamma)\theta(\beta, \gamma), \\
\theta(\alpha, \beta + \gamma) &= \theta(\alpha, \beta)\theta(\alpha, \gamma), \\
\theta(\alpha, 0) &= 1 = \theta(0, \alpha), \\
\theta(\alpha, \beta)\theta(\beta, \alpha) &= 1,
\end{align*}
\]

which is a 2-cocycle on $\Lambda$.

For a fixed pair $(\Lambda, \theta)$, let $\mathcal{G}_{\Lambda, \theta}$ denote a category of $\Lambda$-graded associative (unitary) algebras over $k$. That is, for any $A \in Ob(\mathcal{G}_{\Lambda, \theta})$, $A = \bigoplus_{\alpha \in \Lambda} A_\alpha$ is a $\Lambda$-graded associative algebra with $k \subseteq A_0$ and $A_\alpha \cdot A_\beta \subseteq A_{\alpha + \beta}$. Moreover, $\phi : A \rightarrow B$ is a morphism from $A$ to $B$, if $\phi : A \rightarrow B$ is a graded algebra homomorphism (of degree 0) such that $\phi(A_\alpha) \subseteq B_\alpha$ and $\phi(1) = 1$. Denote by $\text{Hom}_{\mathcal{G}_{\Lambda, \theta}}(A, B)$ all the morphisms from object $A$ to object $B$. 

Notation. Let $k$ be a field, $q \in k^*$. For $n, r \in \mathbb{Z}_+ (n \geq r)$, set

\[
(n)_q = \frac{1 - q^n}{1 - q}, \quad (n)_q! = (1)_q(2)_q \cdots (n)_q, \quad \binom{n}{r}_q = \frac{(n)_q!}{(n-r)_q!(r)_q!}.
\]

Then $(-n)_q := -q^{-n}(n)_q$, where the $q$-binomial number $(\binom{n}{r})_q$ is a Gaussian polynomial in $q$. Convention: $(0)_q! = 1$, $(\binom{0}{r})_q = 1$ and $(\binom{n}{r})_q = 0$ for $n < r$. So, $(\binom{n}{r})_q$ is well defined for all $q \in k$. Particularly, when $q$ is a primitive $l$-th root of unity, we have $(\binom{1}{r})_q = 0$ and $(\binom{n}{r})_q = 0$ for $1 \leq r < l$. For any $q \in k^*$, we define the characteristic of $q$ as the minimal positive integer $l$ such that $q^l = 1$, or $0$ if $q^l \neq 1$ for any $l \in \mathbb{N}$. Denote it by $\text{char}(q) = l$. Thus $\text{char}(q) \in \mathbb{Z}_+$. 


For any object \( A \in \text{Ob}(\mathcal{G}_{\Lambda,\theta}) \), to the skew bicharacter \( \theta \), one can associate an opposite object \( A^{\text{op}} := (A,\circ) \), where \( A^{\text{op}} = \bigoplus_{a \in \Lambda} A_a \), \( a \circ b := \theta(a, \beta) ba \), for any \( a \in A_{\alpha} \), \( b \in A_{\beta} \). \( A^{\text{op}} \) is associative due to the 2-cocycle property of \( \theta \). That is, \( A^{\text{op}} \in \text{Ob}(\mathcal{G}_{\Lambda,\theta}) \).

For any \( U = \bigoplus_{a \in \Lambda} U_a, \quad V = \bigoplus_{\beta \in \Lambda} V_{\beta} \in \text{Ob}(\mathcal{G}_{\Lambda,\theta}) \), one has

\[
U \otimes V = \bigoplus_{\gamma} (U \otimes V)_{\gamma} = \bigoplus_{\gamma = \alpha + \beta} U_{\alpha} \otimes V_{\beta} \in \text{Ob}(\mathcal{G}_{\Lambda,\theta}),
\]

where its algebra structure on the tensor vector space \( U \otimes V \) is given by

\[
(a \otimes b)(c \otimes d) = \theta(a, \beta) ac \otimes bd, \quad \text{for a.e.} \quad a \in U, \quad d \in V, \quad b \in V_{\alpha}, \quad c \in U_{\beta},
\]

moreover, it is an associative algebra according to (1) & (2).

\( \mathcal{G}_{\Lambda,\theta} \) is a category equipped with a functor \( \otimes : \mathcal{G}_{\Lambda,\theta} \times \mathcal{G}_{\Lambda,\theta} \to \mathcal{G}_{\Lambda,\theta} \), which satisfies the associativity: \((U \otimes V) \otimes W = U \otimes (V \otimes W)\) in \( \mathcal{G}_{\Lambda,\theta} \), for any object \( U, \quad V, \quad W \in \text{Ob}(\mathcal{G}_{\Lambda,\theta}) \).

Again, we have an unit object \( \mathbf{1} = k = \bigoplus_{a \in \Lambda} k_a \in \text{Ob}(\mathcal{G}_{\Lambda,\theta}) \) with \( k_a = \delta_{0, a} k \), such that \( V \otimes \mathbf{1} = V = \mathbf{1} \otimes V \) as isomorphic objects in \( \mathcal{G}_{\Lambda,\theta} \), due to property (3) of the \( \theta \). In other words, \( \mathcal{G}_{\Lambda,\theta} \) is therefore a (strict) monoidal category in the sense of [5] (cf. p. 16).

Furthermore, we have the subsequent

**Proposition.** The (strict) monoidal category \( \mathcal{G}_{\Lambda,\theta} \) is a braided monoidal category.

**Proof.** For any couple \((U, V)\) of objects of \( \mathcal{G}_{\Lambda,\theta} \), we define the mapping \( \psi_{U,V} : U \otimes V \to V \otimes U \) as

\[
\psi_{U,V}(a \otimes b) = \theta(a, \beta) b \otimes a, \quad \forall a \in U_{\alpha}, \quad b \in V_{\beta}.
\]

Since \( \theta \) is a skew bicharacter, we easily see that

\[
\psi_{U,V}((a \otimes b)(a' \otimes b')) = \psi_{U,V}(a \otimes b) \psi_{U,V}(a' \otimes b'),
\]

for any \( a \in U_{\alpha}, \quad b \in V_{\beta}, \quad a' \in U_{\alpha}', \quad \text{and} \quad b' \in V_{\beta}' \). By (4), we get \( \psi_{U,V} \circ \psi_{V,U} = \text{id}_{V \otimes U} \) and \( \psi_{V,U} \circ \psi_{U,V} = \text{id}_{U \otimes V} \). So \( \psi_{U,V} \in \text{Hom}_{\mathcal{G}_{\Lambda,\theta}}(U \otimes V, V \otimes U) \) is an (algebra) isomorphism (as a morphism in \( \mathcal{G}_{\Lambda,\theta} \)) between the above objects. Furthermore, it is easy to check that the diagram is commutative

\[
\begin{array}{ccc}
U \otimes V & \xrightarrow{\psi_{U,V}} & V \otimes U \\
\downarrow f \otimes g & & \downarrow g \otimes f \\
U' \otimes V' & \xrightarrow{\psi_{U',V'}} & V' \otimes U'
\end{array}
\]

for all morphisms \( f, \quad g \).

We thus obtain a commutativity constraint \( \psi \) in the monoidal category \( \mathcal{G}_{\Lambda,\theta} \), which consists of a family of natural isomorphisms \( \{ \psi_{U,V} : U \otimes V \to V \otimes U \mid U, \quad V \in \text{Ob}(\mathcal{G}_{\Lambda,\theta}) \} \).

Obviously, \( \psi \) obeys the two hexagon conditions:

\[
\begin{align*}
\psi_{U \otimes V, W} &= (\psi_{U,W} \otimes \text{id}_V)(\text{id}_U \otimes \psi_{V,W}), \\
\psi_{U, V \otimes W} &= (\text{id}_V \otimes \psi_{U,W})(\psi_{U,V} \otimes \text{id}_W)
\end{align*}
\]
for all objects $U$, $V$, $W$. Hence, the commutativity constraint $\psi$ is a braiding in $\mathcal{G}_{\Lambda, \theta}$, that is, $\mathcal{G}_{\Lambda, \theta}$ is a braided monoidal category (for definition, see [5], p. 18).

For $H \in \text{Ob}(\mathcal{G}_{\Lambda, \theta})$, call $H$ a braided Hopf algebra over $k$, if $H$ is a Hopf algebra object in the braided monoidal category under consideration, and the algebra structure on $H \otimes H$ is provided by the braiding $\psi$ in $\mathcal{G}_{\Lambda, \theta}$ in a manner: $(m \otimes m) \circ (\text{id}_H \otimes \psi \otimes \text{id}_H)$, where $m : H \otimes H \rightarrow H$ is the multiplication mapping on algebra $H$.

3. $\theta$-Lie Algebra and its Braided Universal Enveloping Algebra

Let $\Lambda$ be an abelian group and $\theta$ is a (nontrivial) skew bicharacter on $\Lambda$. Let $L = \bigoplus_{\alpha \in \Lambda} L_\alpha$ be a $\Lambda$-graded vector space over a field $k$ with a bilinear product $[\cdot, \cdot] : L \times L \rightarrow L$ satisfying:

\[
[L_\alpha, L_\beta] \subseteq L_{\alpha + \beta},
\]

\[
[a, b] = -\theta(\alpha, \beta) [b, a],
\]

\[
\theta(\gamma, \alpha) [a, [b, c]] + \theta(\alpha, \beta) [b, [c, a]] + \theta(\beta, \gamma) [c, [a, b]] = 0,
\]

for any $a \in L_\alpha$, $b \in L_\beta$ and $c \in L_\gamma$. Then $L$ is called a color Lie algebra over $k$ (cf. [10]).

In particular, $L$ is said an abelian color Lie algebra, if $[a, b] \equiv 0$, for any $a \in L_\alpha$, $b \in L_\beta$ and for any $\alpha, \beta \in \Lambda$.

**Examples.** Any Lie algebra can be regarded as a color Lie algebra with a trivial $\Lambda$-grading and a trivial skew bicharacter $\theta$ (i.e., $L = \bigoplus_{\alpha \in \Lambda} L_\alpha$ with $L_\alpha = \delta_{\alpha0} L$, and $\theta(\alpha, \beta) \equiv 1$ for any $\alpha, \beta \in \Lambda = \Lambda_0$).

Again for any $A = \bigoplus_{\alpha \in \Lambda} A_\alpha \in \text{Ob}(\mathcal{G}_{\Lambda, \theta})$, a $\Lambda$-graded associative algebra over $k$, if we let $[a, b] := ab - \theta(\alpha, \beta) ba$ for any $a \in A_\alpha$ and $b \in A_\beta$, it is clear that $A^{-} := (A, [\cdot, \cdot])$ is a color Lie algebra over $k$.

**Remark.** When $\Lambda = \mathbb{Z}_2$, any $\mathbb{Z}_2$-graded color Lie algebra is a Lie superalgebra. It is not a Lie algebra provided that there is an $\alpha \in \Lambda$ such that $\theta(\alpha, \alpha) = -1$.

Throughout the paper, for the sake of simplicity, we always suppose that $\Lambda$ is a free abelian group of finite rank and $\theta(\alpha, \alpha) = 1$ for all $\alpha \in \Lambda$ in order to avoid involving Lie superalgebras. Particularly, in this case we refer such a color Lie algebra as to a $\theta$-Lie algebra.

Let $T(L)$ denote the tensor associative algebra of $L$ over $k$. Clearly, $T(L) \in \text{Ob}(\mathcal{G}_{\Lambda, \theta})$, which is derived from the $\Lambda$-grading of $L$ as well as (6). Assume that $J$ is the $\Lambda$-graded two-sided ideal in $T(L)$ generated by homogeneous elements $a \otimes b - \theta(\alpha, \beta) b \otimes a - [a, b]$, for any $a \in L_\alpha$, $b \in L_\beta$.

The associative algebra $U(L) := T(L)/J$ over $k$, which is $\Lambda$-graded (i.e., $U(L) \in \text{Ob}(\mathcal{G}_{\Lambda, \theta})$), is said the braided universal enveloping algebra of the $\theta$-Lie algebra $L$.

$U(L)$ is universal as a braided object in $\mathcal{G}_{\Lambda, \theta}$ in the following sense.
Lemma. Let \( \sigma \) be the canonical map of \( L \) into \( U(L) \), for any \( A \in \text{Ob}(\mathcal{G}_{\Lambda, \delta}) \), \( \tau \) a homogeneous linear map of \( L \) into \( A \) of degree 0 with respect to the \( \Lambda \)-gradation such that \( \tau(a) \tau(b) - \theta(\alpha, \beta) \tau(b) \tau(a) = \tau([a, b]) \), for all \( a \in \Lambda_\alpha, b \in \Lambda_\beta \). Then there exists a unique morphism \( \tau' \in \text{Hom}_{\mathcal{G}_{\Lambda, \delta}}(U(L), A) \) such that \( \tau' \cdot \sigma = \tau \). 

Similar to the case of Lie algebras, \( U(L) \) has the Poincaré-Birkhoff-Witt Theorem, which was stated in [10] and no detailed proof was given. For the reader’s convenience and self-contained, we give a proof by employing Bergman’s idea of Diamond Lemma — reduction system (see [2], [4]).

We need some notions and notation from [2]. Fix an ordered homogeneous basis \( \{a_{i_1}, a_{i_2}, \ldots \} \) \((i_1 < i_2 < \ldots)\) of \( L \) relative to its \( \Lambda \)-gradation such that \( a_{i_k} \in \Lambda_\alpha \) for some \( \alpha \). Let \( \mathcal{L} \) denote the set of monomials in \( T(L) \) relative to such a chosen ordered homogeneous basis. Define the disordering index of a monomial \( a_{j_1} \otimes \cdots \otimes a_{j_n} \) as the number of pairs \((j_i, j_k)\) with \( l < k \) but \( j_i > j_k \). Thus we can partially order monomials in \( \mathcal{L} \) by setting \( C \prec D \) if \( C \) is of a smaller length than \( D \), or if \( C \) is a permutation of the terms of \( D \) but of a smaller disordering index. Obviously, this defines a semigroup partial ordering on \( \mathcal{L} \) in the sense: \( B \prec B' \) implies \( A \otimes B \otimes C \prec A \otimes B' \otimes C \) (\( A, B, B', C \in \{\mathcal{L}\} \)).

Moreover, let \( S = \{\tau = (\omega_\tau, f_\tau) \mid \omega_\tau \in \mathcal{L}, f_\tau \in T(L)\} \) be a reduction system of \( T(L) \) relative to a fixed ordered homogeneous basis of \( L \). If for all \( \tau \in S \), \( f_\tau \) is a linear combination of monomials \( \prec \omega_\tau \), then the semigroup partial ordering \( \prec \) is called compatible with \( S \). For each \( \tau \in S \), let \( r_\tau \) denote the map on \( T(L) \) which maps each monomial \( A \otimes \omega_\tau \otimes B \) into \( A \otimes f_\tau \otimes B \) and fixes those monomials not containing the subword \( \omega_\tau \). Denote \( r_S := \{r_\tau \mid \tau \in S\} \) and each \( r_\tau \) is called a reduction of \( T(L) \). For any \( \mathcal{L} \), define \( J_\mathcal{L} := \{C = \sum_i C_i \in J \mid \mathcal{C}_i \prec A \text{ for each } i\} \). An overlap ambiguity of \( S \) means a 5-tuple \((\sigma, \tau, A, B, C)\) with \( \sigma, \tau \in S \) and \( A, B, C \in \mathcal{L} \) such that \( \omega_\sigma = A \otimes B, \omega_\tau = B \otimes C \). The ambiguity is said resolvable relative to \( \prec \) if \( f_\sigma \otimes C - A \otimes f_\tau \in J_{A \otimes B \otimes C} \).

Denote by \( \widetilde{a} \) the class in \( U(L) \) of any element \( a \) of \( T(L) \).

Theorem (Poincaré-Birkhoff-Witt). Assume that \( \Lambda \) is a free abelian group (of finite rank) and \( \theta \) is a (nontrivial) skew bicharacter on \( \Lambda \) with \( \theta(\alpha, \alpha) = 1 \) for all \( \alpha \in \Lambda \). Let \( L = \bigoplus_{\alpha \in \Lambda} \Lambda_\alpha \) be a \( \theta \)-Lie algebra over a field \( k \), \( \{a_{i_1}, a_{i_2}, \ldots \} \) \((i_1 < i_2 < \ldots)\) an ordered homogeneous basis of \( L \) relative to its \( \Lambda \)-gradation. Then the unit 1 and the ordered monomials \( \bar{a}_{j_1} \cdots \bar{a}_{j_n} \) \((j_1 \leq \cdots \leq j_n, n \geq 1)\) form a basis of \( U(L) \) over \( k \).

Proof. Take \( S = \{\tau = r_{\tau_{ab}} := (\omega_{\tau_{ab}}, f_{\tau_{ab}}) \mid a < b, a \in \Lambda_\alpha, b \in \Lambda_\beta\} \) as a reduction system of \( T(L) \) relative to the ordered homogeneous basis \( \{a_{i_1}, a_{i_2}, \ldots \} \) of \( L \), where \( \omega_{\tau_{ab}} = b \otimes a \in \mathcal{L}, f_{\tau_{ab}} = \theta(\beta, \alpha) a \otimes b + [b, a] \in T(L) \) and the semigroup partial ordering \( \prec \) is compatible with \( S \). Note that the ideal generated by the differences \( \omega_\tau - f_\tau \) \((\tau \in S)\) is precisely \( J \).

Since each reduction \( r_\tau \) on \( T(L) \) maps each monomial \( A \otimes \omega_\tau \otimes B \) into \( A \otimes f_\tau \otimes B \) but fixes those monomials without containing subword \( \omega_\tau \), the images in \( U(L) \) of the fixed monomials under \( S \) are precisely the alleged basis.

Since each monomial of degree \( n \) for a finite number of reductions will be stable under \( r_S \), the above semigroup partial ordering \( \prec \) on \( \mathcal{L} \) satisfies the descending chain condition.

Obviously, the ambiguities of \( S \) are precisely the 5-tuples \((\tau_{bc}, \tau_{ab}, c, b, a)\) with \( a < b < c \)
and $a \in L_\alpha$, $b \in L_\beta$, $c \in L_\gamma$. To see that the ambiguity is resolvable relative to $\prec$, we study
\[
\tau_{\epsilon}(c \otimes b \otimes a) - \tau_{\epsilon}(c \otimes b \otimes a)
= \theta(\gamma, \beta) b \otimes c \otimes a + [c, b] \otimes a
- \theta(\beta, \alpha) c \otimes a \otimes b - c \otimes [b, a].
\]
To further reduce the term $b \otimes c \otimes a$, we apply first $\tau_{\epsilon}$, and then to handle $b \otimes a \otimes c$ which results, $\tau_{\epsilon}$. Similarly, to deal with $c \otimes a \otimes b$ we apply $\tau_{\epsilon}$ and then $\tau_{\epsilon}$. Thus we get
\[
\left(\theta(\gamma, \beta) \theta(\gamma, \alpha) \theta(\beta, \alpha) a \otimes b \otimes c + \theta(\gamma, \alpha + \beta) [b, a] \otimes c
\right.
+ \theta(\gamma, \beta) b \otimes [c, a] + [c, b] \otimes a
- (\theta(\beta, \alpha) \theta(\gamma, \alpha) \theta(\beta, \beta) a \otimes b \otimes c + \theta(\beta + \gamma, \alpha) a \otimes [c, b]
+ \theta(\beta, \alpha) [c, a] \otimes b + c \otimes [b, a])
= [c, b], a] + \theta(\gamma, \beta) b, [c, a] - [c, [b, a]]
= 0 \in J_{c \otimes b \otimes a}.
\]
This means that our ambiguities are resolvable relative to $\prec$. Hence by Theorem 1.2 of Bergman in [2], $\mathcal{U}(\Lambda)$ has the basis indicated.

**Corollary.** The canonical map $\sigma : L \to \mathcal{U}(\Lambda)$ is injective and $\mathcal{U}(\Lambda)$ has no zero divisors, for any $\theta$-Lie algebra $L = \bigoplus_{\alpha \in \Lambda} L_\alpha$, where $\Lambda$ is a free abelian group (of finite rank) and $\theta$ is a (nontrivial) skew bicharacter on $\Lambda$ with $\theta(\alpha, \alpha) = 1$ for all $\alpha \in \Lambda$.

Now we can identify $\bar{a}$ in $\mathcal{U}(\Lambda)$ with $a$ for any $a \in L$.

**Theorem.** Let $L$ be a free abelian group and $\theta : \Lambda \times \Lambda \to k^*$ a skew bicharacter on $\Lambda$ with $\theta(\alpha, \alpha) = 1$ for all $\alpha \in \Lambda$. Let $G_{\Lambda, \theta}$ denote the braided monoidal category. Assume that $L = \bigoplus_{\alpha \in \Lambda} L_\alpha$ be a $\theta$-Lie algebra over $k$, then the universal enveloping algebra $\mathcal{U}(L)$ is a braided Hopf algebra over $k$ in $G_{\Lambda, \theta}$ with the following morphisms in $G_{\Lambda, \theta}$
\[
\Delta : \mathcal{U}(L) \to \mathcal{U}(L) \otimes \mathcal{U}(L), \quad \Delta(a) = a \otimes 1 + 1 \otimes a, \quad \forall a \in L_\alpha,
\]
\[
\epsilon : \mathcal{U}(L) \to k, \quad \epsilon(a) = \delta_{0, \alpha} a, \quad \forall a \in L_\alpha,
\]
\[
S : \mathcal{U}(L) \to \mathcal{U}(L)^{\text{op}}, \quad S(a) = -a, \quad \forall a \in L_\alpha
\]
as its comultiplication, counity and antipode, respectively.

4. A Class of Quantum Groups Arising from $\theta$-Lie Algebras

By a remarkable observation (cf. [3, Section 3.3]), we are able to convert a braided object in the braided monoidal category $G_{\Lambda, \theta}$ to an object in the non-braided monoidal category. In what follows, we shall give the quantum group object for any $\theta$-Lie algebra, which can be regarded as the genuine universal enveloping algebra object of the $\theta$-Lie algebra in the category $\mathcal{H}_\Lambda$ of usual Hopf algebras over $k$.

Let $\mathfrak{A}(L)$ be the associative algebra over $k$ generated by the braided universal enveloping algebra $\mathcal{U}(L)$ of a $\theta$-Lie algebra $L = \bigoplus_{\alpha \in \Lambda} L_\alpha$, together with the symbols $\Theta(\gamma)$ ($\gamma \in \Lambda$),
associated to the given (nontrivial) skew bicharacter \( \theta \) on the free abelian group \( \Lambda \) (of finite rank), subject to the relations

\[
\Theta(\alpha) = \Theta(-\alpha), \quad \Theta(0) = 1, \quad (i)
\]

\[
\Theta(\alpha) \Theta(\beta) = \Theta(\alpha + \beta) = \Theta(\beta) \Theta(\alpha), \quad (ii)
\]

\[
\Theta(\alpha) b \Theta(\alpha)^{-1} = \theta(\alpha, \beta) b, \quad \forall \ b \in U(L)_\beta, \quad (iii)
\]

\[
a b - \theta(\alpha, \beta) b a = [a, b], \quad \forall \ a \in L_\alpha, b \in L_\beta. \quad (iv)
\]

Furthermore, \( \mathcal{A}(L) \) can be equipped with a quantum group structure if define the mappings \( \Delta, \epsilon \), and \( S \) on the generators of \( \mathcal{A}(L) \) as

\[
\Delta : \mathcal{A}(L) \to \mathcal{A}(L) \otimes \mathcal{A}(L) \quad (v)
\]

\[
\Delta(\Theta(\alpha)) = \Theta(\alpha) \otimes \Theta(\alpha),
\]

\[
\Delta(a) = a \otimes 1 + \Theta(\alpha) \otimes a, \quad \forall \ a \in L_\alpha.
\]

\[
\epsilon : \mathcal{A}(L) \to k \quad (vi)
\]

\[
\epsilon(\Theta(\alpha)) = 1,
\]

\[
\epsilon(a) = 0, \quad \forall \ a \in L_\alpha.
\]

\[
S : \mathcal{A}(L) \to \mathcal{A}(L) \quad (vii)
\]

\[
S(\Theta(\alpha)) = \Theta(-\alpha),
\]

\[
S(a) = -\Theta(-\alpha) a, \quad \forall \ a \in L_\alpha.
\]

Again we extend the definitions of \( \Delta, \epsilon \) (resp. \( S \)) on \( \mathcal{A}(L) \) (anti-)algebraically. Thereby we obtain the following

**Theorem.** Assume that the skew bicharacter \( \theta \) of the free abelian group \( \Lambda \) is nontrivial and \( \theta(\alpha, \alpha) = 1 \) for all \( \alpha \in \Lambda \), then \( (\mathcal{A}(L), \Delta, \epsilon, S) \in \text{Ob}(\mathcal{HA}) \) is a noncommutative and noncocommutative Hopf algebra with respect to the above relations (i) – (vii). \( \square \)

**Example.** Now let \( \Lambda = \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i \) be a free abelian group of rank \( n \), where \( \epsilon_i = (\delta_{i1}, \cdots, \delta_{in}) \) and \( \{ \epsilon_i \}_{1 \leq i \leq n} \) is a canonical basis of \( \Lambda \). As in [3], we take a skew bicharacter \( \theta \) on \( \Lambda \) defined by

\[
\theta(\alpha, \beta) = q^{\alpha \ast_\beta - \beta \ast_\alpha}, \quad (11)
\]

for any \( \alpha, \beta \in \Lambda \) and \( q \in k^* \), where \( \alpha \ast_\beta = \sum_{j=1}^{n-1} \sum_{i>j} \alpha_i \beta_j \). In particular,

\[
\theta(\epsilon_i, \epsilon_j) = \begin{cases} q, & \text{if } i > j, \\ 1, & \text{if } i = j, \\ q^{-1}, & \text{if } i < j. \end{cases} \quad (12)
\]

Denote \( \Lambda_+ := \{ \alpha = \sum_{i=1}^n \alpha_i \epsilon_i \in \Lambda \mid \alpha_i \in \mathbb{Z}_+, \ \forall \ i \} \). Let \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) be any nonzero monomial in \( k[A_q^{n[0]}] \), then \( \{ x^\alpha \mid \alpha \in \Lambda_+ \} \) constitutes a canonical basis of \( k[A_q^{n[0]}] \). Moreover, by definition (see Section 1),

\[
x^\alpha x^\beta = q^{\alpha \ast_\beta} x^{\alpha + \beta} = \theta(\alpha, \beta) x^{\beta} x^\alpha. \quad (13)
\]
k[A_q^{n[0]}] = \bigoplus_{\alpha \in \Lambda} k x^\alpha is \Lambda\text{-graded (with } x^\alpha = 0 \text{ for } \alpha \notin \Lambda_+ \text{ and } x^{e_i} = x_i) , \text{ and } k[A_q^{n[0]}] \in \text{Ob}(G_{\Lambda, \theta}) .

By (13), \( k[A_q^{n[0]}]^- = ( k[A_q^{n[0]}], [\cdot, \cdot] ) \) is an abelian \( \theta \)-Lie algebra. Consider \( L = \bigoplus_{i=1}^n k x_i \) as an abelian \( \theta \)-Lie algebra. Obviously, \( \mathcal{U}(L) = k[A_q^{n[0]}] \).

5. Quantized Universal Enveloping Algebra of Abelian Lie Algebra

Let \( \Lambda = \bigoplus_{i=1}^n \mathbb{Z} \varepsilon_i \), and \( \theta \) be the skew bicharacter on \( \Lambda \) given by (11). For any \( \alpha \in \Lambda \), consider the algebra automorphisms \( \mathcal{K}(\alpha) \) of \( k[A_q^{n[0]}] \):

\[
\mathcal{K}(\alpha)(x^\gamma) = \theta(\alpha, \gamma) q^{\langle \alpha, \gamma \rangle} x^\gamma, \quad \text{for } x^\gamma \in k[A_q^{n[0]}],
\]

where \( \langle \cdot, \cdot \rangle : \Lambda \times \Lambda \to \mathbb{Z} \) is a symmetric bilinear form such that \( \langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij} \). Obviously, \( \mathcal{K}(\alpha) \mathcal{K}(\beta) = \mathcal{K}(\alpha + \beta) = \mathcal{K}(\beta) \mathcal{K}(\alpha) \). When \( q = 1 \), one has \( \mathcal{K}(\alpha) = \text{id} \).

Let \( \mathcal{A}_q(n) \) be the associative algebra over \( k \) generated by the symbols \( \mathcal{K}(\varepsilon_i) \pm 1, \ x_i \) \((1 \leq i \leq n)\), associated to the skew bicharacter \( \theta \) on \( \Lambda \), subject to the relations:

\[
\mathcal{K}(\varepsilon_i) \mathcal{K}(\varepsilon_j) = \mathcal{K}(\varepsilon_i + \varepsilon_j) = \mathcal{K}(\varepsilon_j) \mathcal{K}(\varepsilon_i), \quad \mathcal{K}(\varepsilon_i)^{\pm 1} \mathcal{K}(\varepsilon_i)^{\mp 1} = \mathcal{K}(0) = 1, \quad (15)
\]

\[
\mathcal{K}(\varepsilon_i)^l = \mathcal{K}(l \varepsilon_i) = 1, \quad (l = \text{char}(q)), \quad (16)
\]

\[
\mathcal{K}(\varepsilon_i) x_j \mathcal{K}(\varepsilon_i)^{-1} = \theta(\varepsilon_i, \varepsilon_j) q^{\delta_{ij}} x_j, \quad (17)
\]

\[
x_i x_j = \theta(\varepsilon_i, \varepsilon_j) x_j x_i. \quad (18)
\]

Obviously, \( \mathcal{A}_q(n) \) contains the quantum affine space \( k[A_q^{n[0]}] \) as its subalgebra. Note that the relation (16) is superfluous in the case when \( \text{char}(q) = 0 \).

Define the comultiplication \( \Delta \), the counity \( \epsilon \) and the antipode \( S \) for \( \mathcal{A}_q(n) \) as follows:

\[
\Delta : \mathcal{A}_q(n) \to \mathcal{A}_q(n) \otimes \mathcal{A}_q(n) \quad (19)
\]

\[
\Delta(\mathcal{K}(\varepsilon_i)^{\pm 1}) = \mathcal{K}(\varepsilon_i)^{\pm 1} \otimes \mathcal{K}(\varepsilon_i)^{\mp 1},
\]

\[
\Delta(x_i) = x_i \otimes 1 + \mathcal{K}(\varepsilon_i) \otimes x_i . \quad (20)
\]

\[
\epsilon : \mathcal{A}_q(n) \to k
\]

\[
\epsilon(\mathcal{K}(\varepsilon_i)^{\pm 1}) = 1,
\]

\[
\epsilon(x_i) = 0 .
\]

\[
S : \mathcal{A}_q(n) \to \mathcal{A}_q(n) \quad (21)
\]

\[
S(\mathcal{K}(\varepsilon_i)^{\pm 1}) = \mathcal{K}(\varepsilon_i)^{\mp 1},
\]

\[
S(x_i) = -\mathcal{K}(\varepsilon_i)^{-1} x_i .
\]

**Theorem.** For any \( q \in k^* \), the algebra \( \mathcal{A}_q(n) \) under the comultiplication \( \Delta \), the counity \( \epsilon \) and the antipode \( S \) defined by (19) — (21) forms a Hopf algebra structure, which is just the corresponding object of the usual polynomial algebra in \( n \) variables in the context of quantum groups, or equivalently, the quantized universal enveloping algebra of an abelian Lie algebra of dimension \( n \).
Proof. First we need to show that \( \Delta, \epsilon \) and \( S \) preserve the algebraic relations (15) — (18) of \( A_q(n) \). This is clear for \( \epsilon \) and \( S \), and is clear for \( \Delta \) preserving relations (15) (or (16)).

So it remains to check it for \( \Delta \) with respect to relations (17) — (18). Note that

\[
\mathcal{K}(\epsilon_i) \ x_j = \theta(\epsilon_i, \epsilon_j) \ q^{\delta_{ij}} \ x_j \ \mathcal{K}(\epsilon_i),
\]

\[
x_i \ \mathcal{K}(\epsilon_j) = \theta(\epsilon_i, \epsilon_j) \ q^{\delta_{ij}} \ \mathcal{K}(\epsilon_j) \ x_i.
\]

Hence, we have

\[
\Delta(\mathcal{K}(\epsilon_i))\Delta(\mathcal{K}(\epsilon_i)^{-1}) = \mathcal{K}(\epsilon_i) \ x_j \ \mathcal{K}(\epsilon_i)^{-1} \ \mathcal{K}(\epsilon_j) \ \mathcal{K}(\epsilon_i) \ x_j \ \mathcal{K}(\epsilon_i)^{-1}
\]

\[
= \theta(\epsilon_i, \epsilon_j) \ q^{\delta_{ij}} \ \Delta(x_j),
\]

\[
\Delta(x_i)\Delta(x_j) = (x_i \mathcal{K}(\epsilon_i)^{-1} \ + \ \mathcal{K}(\epsilon_i) \ x_j) \ x_j \ x_j
\]

\[
+ \ \mathcal{K}(\epsilon_i) \ x_j \ \mathcal{K}(\epsilon_i) \ x_j
\]

\[
= \theta(\epsilon_i, \epsilon_j)(x_j \mathcal{K}(\epsilon_j)^{-1} \ + \ \mathcal{K}(\epsilon_j) \ x_i) \ x_i
\]

\[
+ \ \mathcal{K}(\epsilon_j) \ x_i \ \mathcal{K}(\epsilon_j) \ x_i
\]

\[
= \theta(\epsilon_i, \epsilon_j)\Delta(x_j)\Delta(x_i).
\]

In view of the fact above, together with (19) & (20), we see that \( (1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta \) and \( (1 \otimes \epsilon)\Delta = 1 = (\epsilon \otimes 1)\Delta \) hold. Again by (19) & (21), we have \( m \circ (1 \otimes S) \circ \Delta(x_i) = x_i + \mathcal{K}(\epsilon_i) \ (-\mathcal{K}(\epsilon_i)^{-1} \ x_i) = x_i - x_i = 0 \) and \( m \circ (S \otimes 1) \circ \Delta(x_i) = -\mathcal{K}(\epsilon_i)^{-1} \ x_i + \mathcal{K}(\epsilon_i)^{-1} \ x_i = \mathcal{K}(\epsilon_i)^{-1} \ (-x_i + x_i) = 0 \), thus we get \( m \circ (1 \otimes S) \circ \Delta(x_i) = \epsilon(x_i) = m \circ (S \otimes 1) \circ \Delta(x_i).\)

On the other hand, owing to (19) and \( \mathcal{K}(\epsilon_i)^{\pm 1} \mathcal{K}(\epsilon_i)^{\mp 1} = 1 \), there holds \( m \circ (1 \otimes S) \circ \Delta = \eta \circ \epsilon = m \circ (S \otimes 1) \circ \Delta \), where \( (A_q(n), m, \eta) \) is the algebra structure of \( A_q(n) \).

Consequently, \( (A_q(n), m, \eta, \Delta, \epsilon, S) \) is a noncommutative and noncocommutative Hopf algebra, namely, a quantum group in the sense of Drinfeld.

Finally, when \( q \) takes 1, \( l = 1 \), thus by (16), the group \( \mathcal{K} \) generated by \( \mathcal{K}(\epsilon_i)^{\pm 1} \ (1 \leq i \leq n) \) degenerates into the unit group and then \( A_q(n) \) into a polynomial algebra in \( n \) variables with a known Hopf algebra structure. On the other hand, when \( q \) is a primitive \( l \)-th root of unity, the usual polynomial algebra \( k[x_1^n, \cdots, x_n^n] \) is a central subalgebra of \( k[A_q^{n[0]}] \).

Particularly, the Hopf algebra structure of \( A_q(n) \) obtained in the Theorem restricted to its central subalgebra \( k[x_1^n, \cdots, x_n^n] \) coincides with the usual polynomial Hopf algebra structure. This fact is clear from the Proposition below. On basis of these two kinds of reasons, the quantum group \( A_q(n) \) we obtained is the corresponding quantization object of the usual polynomial algebra in \( n \) variables, which can be also viewed as the quantized universal enveloping algebra of an \( n \)-dimensional abelian Lie algebra. \( \square \)

Proposition. For \( x_i \in k[A_q^{n[0]}] \subset A_q(n) \), \( m \in \mathbb{N} \), we have

\[
\Delta(x_i^m) = \sum_{r=0}^{m} \binom{m}{r} q^{m-r} \mathcal{K}(\epsilon_i)^r \otimes x_i^r.
\]  

(22)

In particular, if \( q \) is a primitive \( l \)-th root of unity whenever \( l \) is even or odd, we have

\[
\Delta(x_i^l) = x_i^l \otimes 1 + 1 \otimes x_i^l, \quad S(x_i^l) = -x_i^l.
\]
Proof. (22) is deduced by induction on $m$ and $K(\varepsilon_i) x_i = q x_i K(\varepsilon_i)$.

For $q$ being a primitive $l$-th root of unity, noting that \( \binom{l}{r}_q = 0 \) for $1 \leq r < l$, we get $\Delta(x_l^i) = x_l^i \otimes 1 + 1 \otimes x_l^i$, which is compatible with the natural requirement in the usual polynomial (Hopf) algebra $k[x_1^i, \ldots, x_n^i]$ since $x_l^i$ are central in $A_q(n)$ (noting that $K(\varepsilon_i) x_l^i = \theta(\varepsilon_i, \ell \varepsilon_i) q^l x_l^i K(\varepsilon_i)$ and $x_l^i x_j = \theta(l \varepsilon_i, \varepsilon_j) x_j x_l^i = x_j x_l^i$).

On the other hand, observing $q^l(l+1)2 = q^r = -1$ when $l = 2$, we have
\[
S(x_l^i) = (-1)^l \left( K(\varepsilon_i)^{-1} x_l^i \right)^l \\
= (-1)^l q^{l(l+1)} K(\varepsilon_i)^{-l} x_l^i \\
= \begin{cases} 
- x_l^i, & \text{if } l \text{ is odd;} \\
q^{l(l+1)} x_l^i = - x_l^i, & \text{if } l \text{ is even.}
\end{cases}
\]

That means the restriction of the Hopf algebra structure of $A_q(n)$ on its central (polynomial) subalgebra $k[x_1^i, \ldots, x_n^i]$ (when $q = 1$) coincides with the classical situation. \qed

Acknowledgments

The author gratefully acknowledges the NNSF of China (Grant 19731004, 10271047), the TRAPOYT and the FUKT and the SFUDP from the MOE of China, the SYVPST from the STSC, and the Shanghai Priority Academic Discipline from the SEC, and the Fellowship from the CSC for the supports. He also would like to thank Professor C. Kassel and his staff for their hospitality during his research stay at the Institut de Recherche Mathématique Avancée (IRMA – C.N.R.S), ULP, Strasbourg, in France.

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