

## On complex surfaces with 5 or 6 semistable singular fibers over $\mathbb{P}^1$ \*

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### Introduction

We denote by  $X$  a complex smooth projective surface, and by  $f : X \rightarrow C$  a fibration over a curve  $C$  whose generic fiber is a curve of genus  $g$ .  $f$  is called *isotrivial* if all smooth fibers are isomorphic to a fixed curve.  $f$  is called *semistable* if all of the singular fibers are reduced nodal curves. If there is no  $(-1)$ -curve contained in the fibers, then we call  $f$  *relatively minimal*. The projection  $X = F \times C \rightarrow C$  is called a trivial fibration.  $s$  is always the number of singular fibers of  $f$ .

In a well-known paper [3], Beauville proved that a non-isotrivial fibration  $f : X \rightarrow \mathbb{P}^1$  admits at least 3 singular fibers, and if  $f$  is semistable, then  $s \geq 4$ . The first author proved in [11] that if  $f$  is a semistable fibration of genus  $g \geq 2$ , then it admits at least 5 singular fibers.

Beauville [4] gave a beautiful explicit classification of all semistable elliptic fibrations ( $g = 1$ ) over  $\mathbb{P}^1$  with 4 singular fibers. More precisely, there are exactly 6 non-isotrivial such families, and all of them are modular families of elliptic curves. A very interesting natural problem is to classify semistable fibrations over  $\mathbb{P}^1$  of genus  $g \geq 2$  with 5 singular fibers. In the survey ([13], §1 and Ex. 5.9), Viehweg considered this problem. The purpose of this note is to try to get some information on the structure of the surface  $X$  when  $f$  has 5 or 6 singular fibers. Our main results are the following two theorems.

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**Theorem 0.1.** *Assume that  $f : X \rightarrow \mathbb{P}^1$  is a non trivial semistable fibration of genus  $g \geq 2$  with  $s$  singular fibers. Assume also that  $f$  is relatively minimal. Then we have:*

- (1) *If  $s = 5$ , then  $X$  is birationally rational or ruled.*
- (2) *If  $s = 6$  and  $g = 2, 3$  or  $4$ , then  $X$  is not of general type.*
- (3) *If  $s = 6, g = 5$  and  $X$  is of general type, then the minimal model  $S$  of  $X$  satisfies*

$$K_S^2 = 1, \quad p_g(S) = 2, \quad q(S) = 0.$$

*The fibration  $f$  comes from a pencil on  $S$  with 5 simple base points (including infinitely near base points).*

Thus if the Kodaira dimension of  $X$  is non negative, then  $s \geq 6$ . We have no example of case (3), and we conjecture that  $s$  is at least 7 when  $X$  is of general type. We shall give two examples to show that the two bounds are sharp.

- A)  $s = 6, g = 3, K_X^2 = -4$ , and  $X$  is birationally a K3 surface.
- B)  $s = 7, g = 4, K_X^2 = -3, p_g(X) = 2, q(X) = 0$ , and  $X$  is of general type containing 4  $(-1)$ -curves.

Our next purpose is to give some inequalities for genus  $g \geq 2$  fibrations over  $\mathbb{P}^1$  (not necessarily semistable), which imply the lower bounds on  $s$ .

**Theorem 0.2.** *Let  $f : X \rightarrow \mathbb{P}^1$  be a relatively minimal fibration of genus  $g \geq 2$ , and let  $K_f = K_{X/\mathbb{P}^1} = K_X + 2F$  be its relative canonical divisor. Assume that  $f$  is not locally trivial. Then*

$$K_f^2 \geq \begin{cases} 4g - 4, & \text{if } \kappa(X) = -\infty, \\ 6g - 6, & \text{if } \kappa(X) = 0, \\ 6g - 5, & \text{if } \kappa(X) = 1, \\ 6g - 6 + \frac{1}{2} \left( K_S^2 + \sqrt{K_S^2 \sqrt{K_S^2 + 8g - 8}} \right), & \text{if } \kappa(X) = 2, \end{cases}$$

where  $S$  is the unique minimal model of  $X$  when  $\kappa(X) \geq 0$ .

The bounds are optimal for infinitely many  $g$ . The example given in [11] satisfies  $K_f^2 = 4(g - 1)$ ,  $g = 2$  and  $s = 5$ . The example A) above satisfies  $K_f^2 = 6(g - 1)$ ,  $g = 3$  and  $s = 6$ , where  $X$  is a K3 surface. In Theorem 2.1, we will classify the fibrations  $f : X \rightarrow \mathbb{P}^1$  with minimal  $K_f^2$  according to its Kodaira dimension.

The proof of Theorem 0.1 is to use some inequalities, particularly the following strict canonical class inequality and its refinement for a non trivial semistable fibration  $f : X \rightarrow C$  of genus  $g \geq 2$  with  $s \neq 0$  [11] (see also [9] for a differential geometric proof):

$$K_{X/C}^2 < (2g - 2)(2g(C) - 2 + s). \tag{0.1}$$

We use Reider’s method to prove Theorem 0.2. Note that the strict canonical class inequality and the inequality  $K_f^2 \geq 6(g - 1)$  (resp.  $K_f^2 \geq 4(g - 1)$ ) in Theorem 0.2 imply that  $s \geq 6$  (resp. 5).

## 1. Preliminaries

Let  $f : X \rightarrow C$  be a fibration of genus  $g \geq 2$ , namely  $X$  (resp.  $C$ ) is a nonsingular complex surface (resp. curve) and the generic fiber  $F$  of  $f$  is a nonsingular curve of genus  $g$ . We always assume that  $f$  is relatively minimal, i.e., there is no  $(-1)$ -curve contained in the fibers.  $f$  is called *semistable* if all of the singular fibers are reduced nodal curves. Denote by  $K_f = K_{X/C} = K_X - f^*K_C$  the relative canonical divisor of  $f$ , and by  $\omega_{X/C}$  its corresponding invertible sheaf. The relative invariants of  $f$  are defined as follows:

$$\begin{aligned}\chi_f &= \deg f_*\omega_{X/C} = \chi(\mathcal{O}_X) - (g-1)(g(C)-1), \\ K_f^2 &= K_{X/C}^2 = K_X^2 - 8(g-1)(g(C)-1), \\ e_f &= \chi_{\text{top}}(X) - 4(g-1)(g(C)-1) = \sum_{i=1}^s (\chi_{\text{top}}(F_i) - (2-2g)).\end{aligned}$$

These invariants are nonnegative, and  $K_f^2 = 0$  (equivalently,  $\chi_f = 0$ ) if and only if  $f$  is locally trivial.  $e_f = 0$  iff  $f$  is smooth. Let

$$e_{F_i} = \chi_{\text{top}}(F_i) - (2-2g).$$

If  $F_i$  is semistable, then  $e_{F_i}$  is equal to the number of nodes of  $F_i$ .

Arakelov [1] and Beauville [5] proved that  $K_f$  is a nef divisor and the curves  $E$  with  $EK_f = 0$  are those  $(-2)$ -curves in the fibers. The map defined by  $|nK_f|$  for large  $n$  is the contraction morphism  $\sigma : X \rightarrow X^\#$  of the vertical  $(-2)$ -curves (cf. [12]), we get the relative canonical model  $f^\# : X^\# \rightarrow C$  of  $f$ .

In what follows, we assume that  $f : X \rightarrow C$  is semistable with  $s$  singular fibers. Denote by  $q$  a singular point of  $X^\#$ . Then  $(X^\#, q)$  is a rational double point of type  $A_{\mu_q}$ , here  $\mu_q$  is the number of  $(-2)$ -curves in  $X$  over  $q$ . We also denote by  $q$  the singular point of the fibers on the smooth part of  $X^\#$ , in this case  $\mu_q = 0$ . Then

$$e_f = \sum_q (\mu_q + 1)$$

is the number of nodes in the singular fibers. For convenience, we let

$$r_f := \sum_q \frac{1}{\mu_q + 1}.$$

It is obvious that

$$r_f \leq e_f. \tag{1.1}$$

In [11], we proved the strict canonical class inequality when  $s \neq 0$ :

$$K_f^2 < (2g-2)(2g(C)-2+s).$$

The inequality follows from the following

$$K_f^2 - (2g-2)(2g(C)-2+s) \leq -\frac{(2g-2)s}{e} + \frac{3 \cdot r_f}{e^2}, \tag{1.2}$$

where  $e \geq 2$  is any integer (see [11], p.594, (5)). Equivalently,

$$r_f \geq \frac{1}{3}e^2 \left( K_X^2 - (2g-2) \left( 6g(C) - 6 + s - \frac{s}{e} \right) \right). \quad (1.3)$$

**Lemma 1.1.** *Let  $q_1, \dots, q_r$  be the points such that  $\mu_{q_i} \neq 0$ . Let  $\ell' = \sum_{\mu_q \neq 0} \mu_q$  be the number of  $(-2)$ -curves contained in the fibers of  $f$ . Then*

$$r_f \leq e_f - \ell' - \frac{r}{2}. \quad (1.4)$$

*Proof.* Note that the number of points  $q$  such that  $\mu_q = 0$  is  $e_f - \sum_i (\mu_{q_i} + 1)$ . Hence

$$\begin{aligned} \sum_q \frac{1}{\mu_q + 1} &= e_f - \sum_i (\mu_{q_i} + 1) + \sum_i \frac{1}{\mu_{q_i} + 1} \\ &\leq e_f - \ell' - r + \frac{r}{2} \\ &= e_f - \ell' - \frac{r}{2}. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 1.2.**  *$e_{F_i} = g - 1 + c_i - g(\tilde{F}_i)$ , where  $c_i$  is the number of components of  $F_i$  and  $\tilde{F}_i$  is the normalization of  $F_i$ . So*

$$e_f \leq s(g-1) + \sum_{i=1}^s c_i. \quad (1.5)$$

*Proof.* Let  $\sigma : \tilde{F}_i \rightarrow F_i$  be the normalization of  $F_i$ , and  $\Delta$  be the subscheme of the singular locus of  $F_i$ . Then we have

$$0 \rightarrow \mathcal{O}_{F_i} \rightarrow \sigma_* \mathcal{O}_{\tilde{F}_i} \rightarrow \mathcal{O}_{\Delta} \rightarrow 0.$$

Since  $F_i$  has only nodes as its singular points, we have  $e_{F_i} = \deg \Delta = h^0(\mathcal{O}_{\Delta}) = \chi(\mathcal{O}_{\Delta})$ . Thus

$$\begin{aligned} e_{F_i} &= \chi(\mathcal{O}_{\tilde{F}_i}) - \chi(\mathcal{O}_{F_i}) \\ &= c_i - g(\tilde{F}_i) - (1 - g) \\ &\leq g - 1 + c_i. \end{aligned}$$

This completes the proof.  $\square$

**Corollary 1.3.** *Denote by  $\ell = \sum_i c_i - \ell'$  the number of curves in the singular fibers different from  $(-2)$ -curves. Then we have*

$$r_f \leq s(g-1) + \ell - \frac{r}{2}. \quad (1.6)$$

*Proof.* This follows from (1.4) and (1.5).  $\square$

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**Lemma 1.4.** *If  $e^2 > \frac{6g+3}{g-1}$ , then*

$$K_f^2 \leq \frac{2e(g-1)^2((2g(C)-2+s)e-s)}{e^2(g-1)-3(2g+1)}. \quad (1.7)$$

*Proof.* By Cornalba-Harris-Xiao's inequality [6, 14],

$$K_f^2 \geq \frac{4g-4}{g} \chi_f,$$

we get

$$e_f \leq \frac{2g+1}{g-1} K_f^2.$$

Then from (1.1) and (1.2), we obtain

$$K_f^2 - (2g-2)(2g(C)-2+s) \leq -\frac{(2g-2)s}{e} + \frac{3(2g+1)}{e^2(g-1)} K_f^2,$$

it implies (1.7). □

## 2. The proof of Theorem 0.2

In this section, we let  $f : X \rightarrow \mathbb{P}^1$  be a relatively minimal fibration of genus  $g \geq 2$ . If  $\kappa(X) \neq -\infty$ , we denote by  $S$  the unique minimal model of  $X$ .

**Theorem 2.1.** *Assume that  $f$  is not locally trivial. Then*

$$K_f^2 \geq \begin{cases} 4g-4, & \text{if } \kappa(X) = -\infty, \\ 6g-6, & \text{if } \kappa(X) = 0, \\ 6g-5, & \text{if } \kappa(X) = 1, \\ 6g-6 + \frac{1}{2} \left( K_S^2 + \sqrt{K_S^2 \sqrt{K_S^2 + 8g-8}} \right), & \text{if } \kappa(X) = 2. \end{cases}$$

- (1)  $K_f^2 = 4g-4$  if and only if  $X$  is the minimal resolution of the singularities of a double covering surface  $Z \xrightarrow{\pi} \mathbb{P}^1 \times C$  ramified over a curve of numerical type  $2F_1 + (2g+2-4g(C))F_2$ , and the fibration  $f$  is induced by the first projection  $\mathbb{P}^1 \times C \rightarrow \mathbb{P}^1$ . Here  $F_i$  is a fiber of the  $i$ -th projection of  $\mathbb{P}^1 \times C$ .
- (2) If  $\kappa(X) = 0$ , then  $K_f^2 = 6g-6$  iff the fibration is induced by a pencil  $\Lambda \subset |C|$  on its minimal model  $S$  with  $C^2 = 2g-2$  simple base points (including infinitely near base points).
- (3) If  $\kappa(X) = 1$ , then  $K_f^2 = 6g-5$  iff the fibration is induced by a pencil  $\Lambda \subset |C|$  of genus  $g$  on its minimal model  $S$  with  $C^2 = 2g-3$  simple base points.
- (4) If  $\kappa(X) = 2$ , then  $K_S^2 \geq 1$ .  $K_f^2 = 6(g-1) + \frac{1}{2} \left( K_S^2 + \sqrt{K_S^2 \sqrt{K_S^2 + 8g-8}} \right)$  iff the fibration is induced by a pencil  $\Lambda \subset |C|$  on  $S$  with only simple base points and  $C \sim rK_S$ . In particular, if  $g = 2$ , then  $K_f^2 \geq 8$ , with equality iff  $K_S^2 = 1$  and  $C \sim K_S$ . If  $g = 3$ , then  $K_f^2 \geq 15$ .

*Proof.* (1) Let  $A = K_X + F$ . We claim that  $|2A|$  (resp.  $|3A|$ ) is base point free if  $g \geq 3$  (resp.  $g = 2$ ). In particular,  $A$  is nef and hence

$$K_f^2 - 4(g - 1) = A^2 \geq 0.$$

Indeed, by Ramanujan's vanishing theorem ([2], p.131),  $H^1(-F) = H^0(-F) = 0$ . So

$$h^0(A) = h^2(-F) = \chi(-F) = \chi_f > 0.$$

Hence we can assume that  $A$  is an effective divisor. Since  $AF = 2g - 2 \geq 2$ ,  $A$  admits at least one horizontal component. Hence  $AK_f \geq 1$ .

Note that  $|2A| = |K_X + K_f|$ . If  $g \geq 3$ ,  $K_f^2 = AK_f + 2(g - 1) \geq 1 + 4 = 5$ . Suppose  $|2A|$  has a base point  $p$ , by Reider's theorem [10], there is a curve  $E$  passing through  $p$  such that

(i)  $K_f E = 0$  and  $E^2 = -1$ ; or (ii)  $K_f E = 1$  and  $E^2 = 0$ .

On the other hand,  $K_f E = K_X E + 2FE \equiv K_X E \equiv E^2 \pmod{2}$ . Thus the two cases can not exist. This proves that  $|2A|$  has no base point.

Now we consider the case  $g = 2$ . We first prove that  $A$  is nef.

Suppose  $\Gamma$  is an irreducible and reduced curve with  $A\Gamma < 0$ . It is easy to see that  $\Gamma$  must be a horizontal curve. Hence  $(A + F)\Gamma = K_f \Gamma \geq 1$ . Then we have  $\Gamma F \geq 2$ . On the other hand,  $|A|$  is non empty, so  $\Gamma$  is the fixed part of  $|A|$ . Hence  $2 = 2g - 2 = AF \geq \Gamma F \geq 2$ . We get  $\Gamma F = 2$ , and  $A = \Gamma + E$ , where  $E$  consists of vertical curves. It implies that  $\Gamma^2 = A\Gamma - E\Gamma < 0$ . Since  $A\Gamma = K_X \Gamma + 2 < 0$ , we have  $K_X \Gamma \leq -3$ . So  $p_a(\Gamma) < 0$ , impossible. Therefore,  $A$  is nef and  $A^2 \geq 0$ .

In this case,  $|3A| = |K_X + 2A + F|$ .  $L = 2A + F$  is nef and  $L^2 = 4A^2 + 4AF \geq 8(g - 1) > 4$ . If  $|3A|$  has a base point  $p$ , then there is a curve  $E$  passing through  $p$  such that

(i)  $LE = 0$ ,  $E^2 = -1$ ; or (ii)  $LE = 1$ ,  $E^2 = 0$ .

Since  $A$  and  $F$  are nef, (i) implies  $AE = FE = 0$ . Note that  $A = K_X + F$ , we have  $K_X E = 0$  and  $K_X E \not\equiv E^2 \pmod{2}$ , a contradiction. So case (i) is impossible. In case (ii), we have  $AE = 0$  and  $FE = 1$ , so  $K_X E = -1$  which is also impossible since  $E^2 = 0$ . This proves that  $|3A|$  is base point free.

Now we consider the case when  $K_f^2 = 4g - 4$ , i.e.,  $A^2 = 0$ .

In this case, the base point free linear system  $|6A|$  is composed with a fibration  $\varphi : X \rightarrow C$  over a smooth curve  $C$ . Denote by  $F'$  a generic fiber of  $\varphi$ . Then  $AF' = 0$ . Since  $AF = 2g - 2 \geq 2$ ,  $F'$  can not be the fiber of  $f$ . So  $FF' \geq 1$ . From  $AF' = 0$ , we get  $K_X F' = -FF' \leq -1$ . Since  $F'^2 = 0$ , we have  $K_X F' = -2$  and  $FF' = 2$ . So  $F' \cong \mathbb{P}^1$ . Now we get a generally double cover  $\pi : X \rightarrow \mathbb{P}^1 \times C$  defined by  $\pi(x) = (f(x), \varphi(x))$ . The pullback  $\pi^*F_2$  of a generic fiber  $F_2$  is still isomorphic to  $\mathbb{P}^1$ , and a generic  $\pi^*F_1$  has genus  $g$ . By Hurwitz formula, we see that the branch curve  $B_\pi$  of  $\pi$  has numerical type  $2F_1 + (2g + 2 - 4g(C))F_2$ .

Conversely, if the branch curve  $B_\pi$  has numerical type  $2F_1 + (2g + 2 - 4g(C))F_2$ , then  $B_\pi$  is nonsingular or admits at most  $ADE$  singularities. Thus the canonical resolution is the minimal one, and the fibration induced is relatively minimal. By easy computations ([2], p.183, or [8]), we have  $K_X^2 = -4(g - 1)$  and hence  $K_f^2 = 4g - 4$ . This completes the proof of (1).

(2) Suppose that the Kodaira dimension of  $X$  is non negative, and let  $S$  be the unique minimal model of  $X$  obtained by contracting  $(-1)$ -curves. So the fibers  $F$  of  $f$  are contracted to a pencil  $\Lambda$  in  $|C|$  on  $S$  with base points  $p_1, \dots, p_m$  (including infinitely near base points). We consider a generic curve  $C$  in  $\Lambda$ , and let  $n_i$  be the multiplicity of  $C$  at  $p_i$ . Then  $n_i \geq 1$ . Note that  $F^2 = 0$ ,  $K_X F = 2g - 2$ . One can prove easily that

$$C^2 = F^2 + \sum_{i=1}^m n_i^2 = \sum_{i=1}^m n_i^2, \quad (2.1)$$

$$K_S C = K_X F - \sum_{i=1}^m n_i = 2g - 2 - \sum_{i=1}^m n_i. \quad (2.2)$$

Since  $S$  is neither ruled nor rational,  $K_S$  is nef and  $K_S^2 \geq 0$ . Hence  $K_S C \geq 0$ . Thus  $m \leq \sum_i n_i \leq 2g - 2 - K_S C$ .

$$K_f^2 = K_X^2 + 8(g - 1) = K_S^2 - m + 8(g - 1) \geq K_S^2 + K_S C + 6(g - 1). \quad (2.3)$$

Hence  $K_f^2 \geq 6g - 6$ . If  $K_f^2 = 6g - 6$ , i.e.,  $K_X^2 = -(2g - 2) < 0$ , then  $K_S C = 0$ ,  $K_S^2 = 0$  and  $n_i = 1$  for any  $i$ . In particular,  $m = K_S^2 - K_X^2 = 2g - 2 > 0$  and  $C^2 > 0$ . By Hodge index theorem,  $K_S$  is numerically trivial, thus  $\kappa(X) = 0$ . By (2.1),  $C^2 = m = 2g - 2$ . This completes the proof of (2).

(3) We have proved that if  $\kappa(X) \geq 0$  and  $K_f^2 = 6g - 6$ , then  $\kappa(X) = 0$ . So if  $\kappa(X) = 1$ , we obtain  $K_f^2 \geq 6g - 5$  and  $K_S^2 = 0$ . Suppose  $K_f^2 = 6g - 5$ , i.e.,  $K_X^2 = -(2g - 3) < 0$ , then  $m = 2g - 3 > 0$  and  $C^2 > 0$ . As in the proof of (2),  $K_S C \neq 0$  (otherwise, by Hodge index theorem,  $K_S \sim 0$  and  $\kappa(X) = 0$ ). So  $K_S C \geq 1$ . From our assumption  $K_f^2 = 6g - 5$  and (2.3), we get  $K_S C = 1$ . Now (2.2) implies that  $n_i = 1$  for any  $i$ , hence  $C^2 = 2g - 3$  and  $C$  is smooth. The fibration is induced by a pencil in  $|C|$  with  $2g - 3$  simple base points.

(4) In the case when  $\kappa(X) = 2$ , we let  $x = C K_S$ ,  $y = C^2$ . Then

$$x + y \geq 2g - 2, \quad K_S^2 \cdot y \leq x^2. \quad (2.4)$$

It is easy to prove that

$$x \geq \frac{1}{2} \left( -K_S^2 + \sqrt{K_S^2 \sqrt{K_S^2 + 8g - 8}} \right).$$

Now from (2.3), we have

$$K_f^2 \geq 6g - 6 + \frac{1}{2} \left( K_S^2 + \sqrt{K_S^2 \sqrt{K_S^2 + 8g - 8}} \right).$$

If the equality holds, then the equalities in (2.4) hold. We obtain easily the desired characterization.  $\square$

**Corollary 2.2.** *Let  $f : X \rightarrow \mathbb{P}^1$  be a relatively minimal semistable fibration of genus  $g \geq 2$  with  $s$  singular fibers. If  $f$  is non trivial and  $s = 5$ , then  $X$  is birationally ruled or rational.*

*Proof.* By the strict canonical class inequality, we have  $K_f^2 < 6g - 6$ . The corollary follows from Theorem 2.1.  $\square$

Note that if  $s = 5$ , then  $S$  is either  $\mathbb{P}^2$  or a geometrically ruled surface. So

$$m = K_S^2 - K_X^2 > K_S^2 + 2g - 2 = \begin{cases} 2g + 7, & \text{if } S \cong \mathbb{P}^2, \\ 2g + 6 - 8q(X), & \text{otherwise.} \end{cases}$$

### 3. The proof of Theorem 0.1 for $s = 6$

In this section, we assume that  $f : X \rightarrow \mathbb{P}^1$  is semistable and  $s = 6$ . We use freely the notations in the previous sections, including those in the proof. Denote by  $F_1, \dots, F_6$  the 6 singular fibers.

We recall the inequalities in Sect. 1 for our  $f$ :

$$(A) \quad K_X^2 < 0, \quad \text{by (0.1)}$$

$$(B) \quad r_f \geq \frac{1}{3}e \left( K_X^2 e + 12g - 12 \right), \quad \text{by (1.3)}$$

$$(C) \quad r_f \leq 6g - 6 + \ell - \frac{r}{2}, \quad \text{by (1.6)}$$

$$(D) \quad K_X^2 \leq 4 \frac{(g-1)^2(2e-3)e}{e^2(g-1) - 6g - 3} - 8(g-1), \quad \text{if } e^2 > \frac{6g+3}{g-1}, \quad \text{by (1.7)}$$

where  $r_f = \sum_q \frac{1}{\mu_{q+1}}$ , and  $\ell$  is the number of non  $(-2)$ -curves in the 6 singular fibers.

In what follows, we assume that  $X$  is of general type, thus  $K_S^2 \geq 1$ . Denote respectively by  $C_1, \dots, C_6 \subset S$  the image curves of  $F_1, \dots, F_6$ . By (A),

$$m = K_S^2 - K_X^2 \geq 2. \quad (3.1)$$

We see that  $K_S C \neq 1$ . Otherwise, by Hodge index theorem,  $K_S^2 C^2 \leq (K_S C)^2 = 1$ , so  $C^2 \leq 1$ , which implies  $m \leq 1$ . Hence

$$2 \leq K_S C \leq 2g - 2 - m. \quad (3.2)$$

**Theorem 3.1.** *If  $g \leq 4$ , then  $X$  is not of general type.*

*Proof.* We shall rule out the cases  $g = 2, 3$  and 4 under the assumption  $\kappa(X) = 2$ .

*Case  $g = 2$ :* (3.2) implies  $m = 0$ , which contradicts (3.1).

*Case  $g = 3$ :* By (2.2), (3.1) and (3.2),  $m = K_S C = 2$ ,  $K_S^2 = 1$ ,  $K_X^2 = -1$ ,  $n_1 = n_2 = 1$ , hence  $C^2 = 2$ . By Noether inequality  $K_S^2 \geq 2p_g(S) - 4$  and Debarre's inequality [7]  $K_S^2 \geq 2p_g(S) \geq 2q(S)$  if  $q(S) \geq 1$ , we obtain

$$p_g(S) \leq 2, \quad q(S) = 0. \quad (3.3)$$

Note that

$$e_f = 12\chi_f - K_f^2 = 12(p_g(S) + 1) - K_X^2 + 4(g-1) = 12p_g(S) + 21 \leq 45. \quad (3.4)$$

On the other hand, from (B) for  $e = 12$ , we get  $r_f \geq 48$ . This contradicts (1.1):  $r_f \leq e_f$ .

Case  $g = 4$ : (D) for  $e = 12$  implies that  $K_X^2 \leq -8/5$ . Thus  $K_X^2 \leq -2$  and  $m \geq 3$ . By (3.2), we see that  $m \leq 4$ .

If  $m = 3$ , then  $K_S^2 = 1$  and  $K_X^2 = -2$ , we see that (3.3) holds. As in (3.4), we obtain  $e_f \leq 50$ . (1.3) for  $e = 9$  implies that  $r_f \geq 54$ , which contradicts  $r_f \leq e_f$ .

If  $m = 4$ , then (2.2) and (3.2) imply that  $K_S C = 2$ ,  $n_1 = \dots = n_4 = 1$ , and  $C^2 = 4$ . By Hodge index theorem, we can see that  $K_S^2 = 1$  and  $K_S^2 C^2 = (K_S C)^2$ . Hence  $C \sim 2K_S$ . This means that any  $(-2)$ -curve  $E$  on  $S$  does not pass through any one of the 4 base points  $p_1, \dots, p_4$  because  $CE = 2K_S E = 0$ . Hence  $(-2)$ -curves in  $C_i$  must be  $(-2)$ -curves in  $F_i$ .

Note that  $K_S C_i = 2$ , so  $C_i$  has at most two components different from  $(-2)$ -curves. This implies that  $F_i$  has at most two components which are neither  $(-2)$ -curves nor the exceptional curves of the 4 base points.

On the other hand, among the 4 exceptional curves on  $X$ , at least one of them is a horizontal  $(-1)$ -curve. If the remaining three exceptional curves are vertical, then at least one of them is a  $(-2)$ -curve. Hence the total number  $\ell$  of non  $(-2)$ -curves in the 6 singular fibers  $\leq 6 \times 2 + 2 = 14$ .

Now by (C),  $r_f \leq 32$ . On the other hand, (B) for  $e = 6$  implies that  $r_f \geq 36$ , a contradiction.  $\square$

**Theorem 3.2.** *If  $s = 6$ ,  $g = 5$  and  $X$  is of general type, then the minimal model  $S$  of  $X$  satisfies*

$$K_S^2 = 1, \quad p_g(S) = 2, \quad q(S) = 0.$$

*The fibration comes from a pencil on  $S$  with 5 simple base points.*

*Proof.* In this case, (D) for  $e = 10$  implies that  $K_X^2 \leq -\frac{864}{367} < -2$ , thus  $K_X^2 \leq -3$  and  $m \geq 4$ . On the other hand,  $m \leq 2g - 2 - 2 = 6$ . If  $m = 6$ , then  $C^2 \geq 6$  and  $K_S C \leq 2$ , by Hodge index theorem,  $K_S^2 C^2 \leq (K_S C)^2 \leq 4$ , and hence  $K_S^2 \leq 0$ , a contradiction. Thus  $m = 4$  or  $5$ .

If  $m = 4$ , then  $K_S^2 = 1$ ,  $K_X^2 = -3$ ,  $p_g(S) \leq 2$ ,  $q(S) = 0$ ,  $C^2 \geq 4$ , and  $K_S C \leq 4$ . Thus

$$e_f = 12(p_g + 1) - K_X^2 + 4(g - 1) \leq 55.$$

From (1.3) for  $e = 8$ , we have

$$r_f \geq \frac{1}{3}e(-3e + 48) = e(16 - e) = 64 > e_f.$$

It contradicts (1.1).

So  $m = 5$ . By (2.1) and (2.2), we have  $C^2 \geq 5$ ,  $K_S C \leq 3$ . From (3.2), we get  $K_S C \geq 2$ . Since  $K_S^2 \geq 1$ , by Hodge index theorem,  $K_S^2 C^2 \leq (K_S C)^2$ , we have  $K_S C = 3$  and  $K_S^2 = 1$ . So  $K_X^2 = -4$ ,  $p_g \leq 2$  and  $q = 0$ . By (2.1) and (2.2), we obtain  $n_1 = \dots = n_5 = 1$  and  $C^2 = 5$ . Generic  $C$  in the pencil is smooth. Let  $e = 6$ . Then (B) gives us that  $r_f \geq \frac{1}{3}e(-4e + 48) = 48$ .

Now we prove that  $p_g = 2$ . Otherwise  $p_g \leq 1$ , by (3.4),  $e_f \leq 12(1+1) - (-4) + 16 = 44 < r_f$ , a contradiction. Then we have

$$K_S^2 = 1, \quad p_g = 2, \quad q = 0, \quad m = 5.$$

So  $e_f = \sum_q (\mu_q + 1) = 56$ ,  $r_f = \sum_q \frac{1}{\mu_q + 1} \geq 48$ .

Let  $m_i = \#\{q \mid \mu_q = i\}$ . Then

$$e_f = m_0 + 2m_1 + 3m_2 + \cdots = 56,$$

$$r_f = m_0 + \frac{1}{2}m_1 + \frac{1}{3}m_2 + \cdots \geq 48.$$

We see that  $e_f - m_0 \geq 4(r_f - m_0)$ , hence  $m_0 \geq 46$ . If  $m_0 = 46$ , one can prove that  $m_1 = 5$  and  $m_i = 0$  for  $i > 1$ . So there are at least 46 points  $q$  such that  $\mu_q = 0$ , and there are at most  $(56 - 46)/2 = 5$  points  $q$  with  $\mu_q \neq 0$ . By (1.4) we get

$$\ell' + \frac{r}{2} \leq 8.$$

It implies that  $r \leq 5$  and  $\ell' \leq 7$ . □

#### 4. Two examples

Note that for any two points  $p_1, p_2$  on  $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ , there exists a cyclic cover  $\varphi_{p_1, p_2} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  ramified exactly over  $p_1$  and  $p_2$ . For example,  $\tau_n : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  defined by  $x \mapsto x^n$  is totally ramified over 0 and  $\infty$ . Now we are going to construct some covers of  $\mathbb{P}^1$ .

Let  $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be the double cover ramified over 1 and  $\infty$ , and let  $\psi_e : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be the cyclic cover of degree  $e$  totally ramified over the two points  $\pi^{-1}(0)$ . Then the composition

$$\varphi_{2e} : \mathbb{P}^1 \xrightarrow{\psi_e} \mathbb{P}^1 \xrightarrow{\pi} \mathbb{P}^1$$

is a covering of degree  $2e$  ramified uniformly over 0, 1,  $\infty$ . The map  $\varphi_{2e}$  is the quotient map of  $\mathbb{P}^1$  for the standard action of the dihedral group of order  $2e$  on  $\mathbb{P}^1$ . The cover has two ramification points over 0 with ramification index  $e$ , and  $e$  ramification points over 1 (resp.  $\infty$ ) with index 2. Now we consider the fiber product of  $\tau_n : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  and  $\varphi_{2e} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ , we get a curve

$$\Gamma = \{ (x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid \tau_n(x) = \varphi_{2e}(y) \}.$$

This curve is of type  $(2e, n)$ . The first projection  $p_1 : \Gamma \rightarrow \mathbb{P}^1$  can be viewed as the pullback of  $\varphi_{2e}$  under the base change  $\tau_n$ . So the set  $\Sigma$  of critical points of  $p_1$  is  $\tau_n^{-1}\{0, 1, \infty\}$ .  $\Sigma$  contains  $n + 2$  points. In fact,  $\Gamma$  is locally defined by  $\varphi_{2e}(y) = \tau_n(x)$ . Thus we can see that  $\Gamma$  admits two singular points defined by

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$y^e = x^n$  over  $\tau_n^{-1}(0)$  and  $e$  singular points of type  $y^2 = x^n$  over  $\tau_n^{-1}(\infty)$ .  $p_1$  has  $e$  simple ramification points over each point of  $\tau_n^{-1}(1)$ .

$$\begin{array}{ccc} \Gamma & \subset & \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\text{pr}_2} \mathbb{P}^1 \\ & \searrow & \text{pr}_1 \downarrow \qquad \qquad \qquad 2e:1 \downarrow \varphi_{2e} \\ & & \mathbb{P}^1 \xrightarrow[\tau_n]{n:1} \mathbb{P}^1 \end{array}$$

Let  $e = n = 4$ . Then  $\Gamma$  is of type (8, 4). Let  $X$  be the minimal resolution of the double cover  $\Sigma \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  ramified over  $\Gamma$ .  $X$  can be obtained by the canonical resolution  $\sigma : Y \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ , where  $\sigma$  is the blowing-up of  $\mathbb{P}^1 \times \mathbb{P}^1$  at the two singular points of  $\Gamma$  over  $\tau_n^{-1}(0)$ . Denote by  $E_1$  and  $E_2$  the two exceptional curves, and by  $\bar{\Gamma}$  the strict transform of  $\Gamma$  on  $Y$ . Then  $\bar{\Gamma} \equiv \sigma((8, 4)) - 4E_1 - 4E_2 = 2\bar{\delta}$ , where  $\bar{\delta} = \sigma^*((4, 2)) - 2E_1 - 2E_2$ . Let  $\bar{\pi} : \bar{X} \rightarrow Y$  be the double cover ramified over  $\bar{\Gamma}$ . Note that  $\bar{\Gamma}$  admits only ADE singularities, so  $\bar{X}$  admits at most rational double points, and  $X$  is the minimal resolution of  $\bar{X}$ . Now by the formulas ([2], p.183), we have

$$K_{\bar{X}} = \bar{\pi}^*(K_Y + \bar{\delta}) = \bar{\pi}^*(\sigma^*(2, 0) - E_1 - E_2).$$

Denote by  $C_1$  and  $C_2$  the two horizontal sections of  $\text{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  passing through the two points blown up by  $\sigma$ , and by  $\bar{C}_i$  their strict transforms on  $Y$ . Since  $\bar{C}_1$  and  $\bar{C}_2$  are disjoint with  $\bar{\Gamma}$ , it is easy to see that their pullback on  $\bar{X}$  are 4  $(-1)$ -curves. Note that

$$\bar{C}_1 + \bar{C}_2 = \sigma^*(2, 0) - E_1 - E_2,$$

so  $K_{\bar{X}} = \bar{\pi}^*(\bar{C}_1 + \bar{C}_2)$  is the sum of the 4  $(-1)$ -curves. Let  $S$  be the surfaces obtained by contracting the 4 horizontal  $(-1)$ -curves. Then we see that  $K_S \equiv 0$ . By easy computation, we have

$$\begin{aligned} \chi(\mathcal{O}_S) &= \chi(\mathcal{O}_{\bar{X}}) \\ &= 2\chi(\mathcal{O}_Y) + \frac{1}{2}(\bar{\delta}^2 + K_Y \bar{\delta}) \\ &= 2 + 0 = 2. \end{aligned}$$

So  $S$  is a  $K3$  surface. The first projection of  $\mathbb{P}^1 \times \mathbb{P}^1$  induces a semistable fibration  $f : X \rightarrow \mathbb{P}^1$  of genus 3 with  $n + 2 = 6$  singular fibers. Because  $K_X^2 = -4$ ,  $K_f^2 = 12 = 6g - 6$ .

Similarly, let  $e = n = 5$ , and let  $F_\infty$  be the vertical fiber of  $\mathbb{P}^1 \times \mathbb{P}^1$  passing through the two singular points of type  $x^5 = y^5$ . Then consider the double cover of  $\mathbb{P}^1 \times \mathbb{P}^1$  ramified over  $F_\infty + \Gamma$ . The canonical resolution is similar to the above case. We have  $\bar{\delta} \equiv \sigma^*((5, 3)) - 3E_1 - 3E_2$  and

$$\begin{aligned} K_Y + \bar{\delta} &= \sigma^*((3, 1)) - 2E_1 - 2E_2 \\ &= \sigma^*((1, 1)) - E_1 - E_1 + \bar{C}_1 + \bar{C}_2 \\ &= \sigma^*((1, 0)) + \bar{F}_\infty + \bar{C}_1 + \bar{C}_2, \end{aligned}$$

where  $\bar{F}_\infty \equiv \sigma^*((0, 1) - E_1 - E_2)$  is the strict transform of  $F_\infty$  on  $Y$ . The pullback of  $\bar{F}_\infty + \bar{C}_1 + \bar{C}_2$  on  $\bar{X}$  consists of 5  $(-1)$ -curves, the one from  $\bar{F}_\infty$  is a vertical  $(-1)$ -curve. After blowing-down this vertical  $(-1)$ -curve, we get a relatively minimal semistable fibration  $f : X \rightarrow \mathbb{P}^1$  of genus  $g = 4$  with  $n + 2 = 7$  singular fibers. We see easily that  $|K_S|$  is a genus 2 pencil with one simple base point. The pencil is in fact induced by  $\text{pr}_2$ . So  $X$  is of general type. By easy computation, we have  $\chi(\mathcal{O}_S) = 3$ , hence

$$K_S^2 = 1, \quad p_g(S) = 2, \quad q(S) = 0.$$

In this case,  $K_X^2 = -3$  and  $K_f^2 = 6g - 3$ .

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