On nilpotent Lie triple systems of maximal rank

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Abstract

For a Lie triple system $T$ over the complex field $\mathbb{C}$, we prove that a nilpotent Lie triple system $T$ has a decomposition of subspaces as $T = V + [T, T, T]$, and that a basis of $V$ is a minimal system of generators of $T$. We also show that if $t$ is a maximal torus on $T$, then $\dim t \leq \dim V$. By the above results, we naturally call a nilpotent Lie triple system $T$ satisfying $\dim t = \dim V$ being of maximal rank. We find an interesting relation between nilpotent Lie triple systems of maximal rank and the Kac-Moody algebra, by which we construct a kind of Lie triple systems of maximal rank with some universal property.

Introduction

A Lie triple system $T$ over a field $\mathbf{F}$ is a vector space and a mapping denoted by $[x, y, z]$ of $T \times T \times T$ into $T$ satisfying the following conditions:

1) the mapping is trilinear;
2) $[x, y, z] = -[y, x, z]$;
3) $[x, y, z] + [y, z, x] + [z, x, y] = 0$;
4) $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$

where $x, y, z, u, v \in T$.

For $x, y \in T$, we can define $L(x, y) \in \text{End}T$ by $L(x, y)z = [x, y, z]$ and set $L(T, T)$ to be the linear span of $L(x, y)$’s. Then the condition 4) above can be written using the bracket in $\text{End}T$ as

$$[L(u, v), L(x, y)] = L([u, v, x], y) + L(x, [u, v, y]).$$

A mapping $f$ of a Lie triple system $T$ into a Lie algebra $L$ is called an imbedding of $T$ in $L$ if and only if

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1) $f$ is linear;
2) $f([a, b, c]) = ([f(a), f(b)], f(c))$ holds for all $a$, $b$, $c$ in $T$;
3) the enveloping Lie algebra of the image set $f(T)$ is $L$.

$L$ is called an imbedding Lie algebra of $T$, some free Lie algebra over the vector space $T$, which is examined the relationship between the nil-radical of a Lie triple system algebra of $T$ given by Nora. C. Hopkins in [5] as follows:

$$\begin{align*}
&\text{let } \phi \in \text{Aut}(T) \text{ defined by } \theta(x + l) := -x + l, \text{ for } x \in T, \ l \in L(T, T). \ \theta \text{ is called the main involution automorphism of } L(T). \\
&\text{An imbedding } U \text{ of a Lie triple system } T \text{ is called universal if and only if for any imbedding } R \text{ of } T, \text{ the mapping } U(x) \rightarrow R(x) \text{ is single-valued and can be extended to a Lie algebra homomorphism of } L_U(T) := U(T) + L(U(T), U(T)) \text{ onto } L_R(T). \text{ If an imbedding } U \text{ of a Lie triple system } T \text{ is universal, then the Lie algebra } L_U(T) \text{ is called a universal imbedding Lie algebra of } T. \text{ We can prove the existence of a universal Lie algebra of } T \text{ using the method employed for universal associative algebras. Let } \mathfrak{U} \text{ be the free Lie algebra over the vector space } T \text{ and } \mathfrak{A} \text{ be the ideal in } \mathfrak{U} \text{ generated by the elements } [[a, b, c] - [a, b, c], a, b, c \in T. \text{ Then letting } \bar{a} \text{ be the coset of } a \in T, \text{ we can see that } a \mapsto \bar{a} \text{ is a universal imbedding of } T \text{ (cf.}[6]).}
\end{align*}$$

In [8], William G. Lister developed notions of radical, semi-simplicity and solvability as defined for Lie triple systems including proofs of the existence of a semi-simple subsystem complementary to the radical and of the decomposition of a semi-simple system into the direct sum of simple ideals. Therefore, similar to the theory of finite dimensional Lie algebras, the natural problem of determining all the Lie triple systems of finite dimension was broken in two parts:

1) the classification of simple Lie triple systems;
2) the classification of solvable Lie triple systems.

In [8], all simple Lie triple systems over an algebraically closed field $F$ with $\text{ch}F = 0$ were given by reducing the problem to the study of automorphism of period 2 in simple Lie algebras. In the research of solvable Lie triple systems, some works have been done on nilpotent Lie triple systems. The concept of nilpotent ideal of Lie triple systems was given by Nora. C. Hopkins in [5] as follows:

**Suppose $\varphi$ is an ideal of a Lie triple system $T$, then $\varphi^0 := \varphi$ and for $n \geq 0$, $\varphi^{n+1} := [\varphi^n, T, \varphi] + [\varphi, T, \varphi^n]$ defines a lower central series for $\varphi$. $\varphi$ is $T$-nilpotent if $\varphi^m = 0$ for some $m$.**

**$T$ is nilpotent if it is $T$-nilpotent.**

In [5], Nora. C. Hopkins proved the existence of a nil-radical (the unique maximal nilpotent ideal) and examined the relationship between the nil-radical of a Lie triple system and that of its standard imbedding Lie algebra. Let $N(T)$ be the nil-radical of a Lie triple
A Lie triple system $T$ is nilpotent if and only if the standard imbedding Lie algebra $L(T)$ is nilpotent.

In [4], L. J. Santharoubane gave the definition of nilpotent Lie algebras of maximal rank and have shown some relationships between Kac-Moody algebras and such special kind of nilpotent Lie algebras as follows. For any nilpotent Lie algebra $g$ of rank $l$ and have shown some relationships between Kac-Moody algebras and such special kind of nilpotent Lie algebras as follows. For any nilpotent Lie algebra $g$ of rank $l$ and have shown some relationships between Kac-Moody algebras and such special kind of nilpotent Lie algebras as follows. For any nilpotent Lie algebra $g$ of rank $l$ and have shown some relationships between Kac-Moody algebras and such special kind of nilpotent Lie algebras as follows. For any nilpotent Lie algebra $g$ of rank $l$ and have shown some relationships between Kac-Moody algebras and such special kind of nilpotent Lie algebras as follows.

In Section 1, we prove that a nilpotent Lie triple system $T$ has a decomposition of subspaces as $T = V + [T, T, T]$, where a basis of $V$ is a minimal system of generators of $T$. We also show that if $t$ is a maximal torus on $T$, then $\dim t \leq \dim V$. By the above results, we naturally call a nilpotent Lie triple system $T$ satisfying $\dim t = \dim V$ being of maximal rank. We prove that the standard imbedding Lie algebra of such a Lie triple system is a nilpotent one of maximal rank.

In Section 2, we associate any finite dimensional nilpotent Lie triple system of maximal rank with a general Cartan matrix. Let $A = (a_{ij})$ be a $l \times l$ general Cartan matrix, $g(A)$ be the Kac-Moody algebra on Chevalley generators $e_i$ and $f_i$ with $i = 1, 2, \cdots, l$. From $g(A)$ we construct a Lie triple system denoted by $N(A)$ and one of its ideals denoted by $N(A)_{2p+3}$ for a positive integer $p$ satisfying some condition. Then we discuss some properties of the quotient Lie triple system

$$\widetilde{N}(A) := N(A)/N(A)_{2p+3}.$$ 

In Theorem 2.1, we prove that $\widetilde{N}(A)$ is a nilpotent Lie triple system of nilpotency $p$ and of maximal rank $l$. Moreover, we verify that the general Cartan matrix associated to $\widetilde{N}(A)$ coincides with $A$. In Theorem 2.2, we give a universal property of $\widetilde{N}(A)$ as following: for any nilpotent Lie triple system $T'$ of maximal rank $l$ with nilpotency $q$ such that $q \leq p$, whose associated general Cartan matrix $B = (b_{ij})$ satisfies that for any $i, j$, $b_{ij} \geq a_{ij}$, if a map $\phi: X = \{e_1, \cdots, e_l\} \rightarrow T'$ satisfies that $\{\phi(e_1) \cdots \phi(e_l)\}$ is a $t'$-mssq relative to $B(t'$ is a maximal torus on $T'$), then there exists a unique homomorphism $\psi$ such that $\psi \cdot \rho|_X = \phi$, where $\rho$ is the canonical map of $N(A)$ onto $\widetilde{N}(A)$. We introduce the notion of free Lie triple system at the beginning of Section 2 for the proof Theorem 2.2.
1 Definition of nilpotent Lie triple system of maximal rank

**Lemma 1.1** Let \( \mathfrak{g} \) be a nilpotent Lie algebra, \( \mathfrak{c} \) a subspace of \( \mathfrak{g} \) such that \( \mathfrak{g} = \mathfrak{c} + [\mathfrak{g}, \mathfrak{g}] \), and \( \mathfrak{c} \cap [\mathfrak{g}, \mathfrak{g}] = \{0\} \), then \( \mathfrak{c} \) generates \( \mathfrak{g} \).

**Proof.** Let \( V \) be a subspace of \( \mathfrak{g} \). Put \( V^0 = V \), and \( V^k = [V, V^{k-1}] \). Since \( \mathfrak{g} \) is nilpotent, there exists \( n \in \mathbb{Z} \) such that \( \mathfrak{g}^{n-1} \neq \{0\} \), \( \mathfrak{g}^n = \{0\} \). From

\[
\mathfrak{g} = \mathfrak{c} + [\mathfrak{g}, \mathfrak{g}] = \mathfrak{c} + \mathfrak{g}^1,
\]

we get

\[
\mathfrak{g}^k = \mathfrak{c}^k + \mathfrak{g}^{k+1}
\]

by induction on \( k \), and then \( \mathfrak{g}^{n-1} = \mathfrak{c}^{n-1} \). So, using induction on \( k \) again, we get

\[
\mathfrak{g} = \mathfrak{c} + \mathfrak{c}^1 + \cdots + \mathfrak{c}^{n-1}.
\]

\( \square \)

**Lemma 1.2** For a nilpotent Lie algebra \( \mathfrak{g} \), the intersection between any minimal system of generators and \([\mathfrak{g}, \mathfrak{g}]\) is empty.

**Proof.** Since \( \mathfrak{g} \) is nilpotent, there exists some \( n \in \mathbb{Z} \) and \( n \geq 0 \) such that \( \mathfrak{g}^n \neq \{0\} \) and \( \mathfrak{g}^{n+1} = \{0\} \).

**Step 1** We prove that the intersection between any minimal system of generators and \( \mathfrak{g}^n \) is empty by contradiction argument.

Let \( A = \{x_1, x_2, \ldots, x_l\} \subset \mathfrak{g} \) be a minimal system of generators. For convenience, we denote \([\cdots [[a_1, a_2], a_3], \ldots, a_m] \) by \( [a_1, a_2, \ldots, a_m] \). Since \( \mathfrak{g} \) is nilpotent, an element \( a \) in \( \mathfrak{g} \) can be expressed by a finite sum of the products of elements in \( A \). We say \( a \) is an element with \( x_i \) if \( x_i \) appears in a summand of \( a \) at least once, otherwise we say \( a \) is an element without \( x_i \). So, if we choose \( x_i \in A \), then for any \( x \in \mathfrak{g} \), we have \( x = I_x(x_i) + II_x \), where \( I_x(x_i) \) is the sum of \( x \)'s summands with \( x_i \), and \( II_x \) is the sum of those without \( x_i \).

Suppose that there exists \( x_j \in A \cap \mathfrak{g}^n \). Then we have

\[
x_j = \sum k_{i_1,i_2,\ldots,i_n}[x_{i_1}, x_{i_2}, \ldots, x_{i_n}], \quad x_{i_1}, x_{i_2}, \ldots, x_{i_n} \in A = I_{x_j}(x_j) + II_{x_j}.
\]

Replacing \( x_j \) in \( I(x_j) \) by \( I_{x_j}(x_j) + II_{x_j} \), we have

\[
x_j = I_{x_j}(I_{x_j}(x_j) + II_{x_j}) + II_{x_j}.
\]

Since \( x_j \in \mathfrak{g}^n \), we have \( I_{x_j}(I_{x_j}(x_j) + II_{x_j}) \in \mathfrak{g}^{n+1} = \{0\} \) and

\[
x_j = II_{x_j}.
\]

Therefore \( A \setminus \{x_j\} \) forms a minimal system. Contradiction!

**Step 2** We prove the following statement by contradiction argument: for \( k \geq 1 \), that the intersection between every minimal system of generators and \( \mathfrak{g}^{k+1} \) is empty implies that the intersection between every minimal system of generators and \( \mathfrak{g}^k \) is empty.
Suppose that there exists a minimal system \( A = \{x_1, x_2, \ldots, x_l\} \) such that \( A \cap g^k \neq \emptyset \). If \( x_j \in A \cap g^k \), then by the argument in Step 1, we have
\[
x_j = I_{x_j}(x_j) + II_{x_j},
\]
where \( I_{x_j}(x_j) \) is a sum of the elements with \( x_j \), and \( I_{x_j}(x_j) \) is in \( g^{k+1} \), \( II_{x_j} \) is a sum of the elements without \( x_j \). If \( I_{x_j}(x_j) = 0 \), then we have the contradiction by the proof in Step 1. Otherwise,
\[
A' = \{x_j - II_{x_j}\} \bigcup \{x_i | x_i \in A, i \neq j\}
\]
is another minimal system of generators such that \( I_{x_j}(x_j) \in A' \cap g^{k+1} \). This contradicts the assumption that every minimal system has no intersection with \( g^{k+1} \).

\[ \square \]

Corollary 1.1 If \( A = \{x_1, x_2, \ldots, x_l\} \) is a minimal system of generators of a nilpotent Lie algebra \( g \), then \( \{x_1 + [g, g], x_2 + [g, g], \ldots, x_l + [g, g]\} \) is a basis of \( g/[g, g] \).

The similar results as above hold for Lie triple systems.

Lemma 1.3 Let \( T \) be a nilpotent Lie triple system. Then the following results hold.

1) If \( V \) is a subspace of \( T \) such that \( T = V + [T, T, T] \) and \( V \cap [T, T, T] = \{0\} \), then \( V \) generates \( T \) and a basis of \( V \) is a minimal system of generators of \( T \).

2) If \( A = \{x_1, x_2, \ldots, x_l\} \) is a minimal system of generators of \( T \), then \( \{x_1 + [T, T, T], x_2 + [T, T, T], \ldots, x_l + [T, T, T]\} \) is a basis of \( T/[T, T, T] \).

Proof. Since \( T \) is nilpotent, there exists \( n \in \mathbb{Z} \) and \( n \geq 0 \) such that \( T^n \neq \emptyset \), \( T^{n+1} = \{0\} \).

1) The standard imbedding Lie algebra \( L(T) \) equals to \( T + L(T, T) = (V + [T, T, T]) + L(T, T) \). Since the derived Lie algebra \( [L(T), L(T)] = [T, T, T] + L(T, T) \), we have
\[
L(T) = V + [L(T), L(T)],
\]
and
\[
V \cap [L(T), L(T)] = \{0\}.
\]

By Lemma 1.1, the nilpotent Lie algebra \( L(T) \) is generated by \( V \), and then \( T = V + V^1 + \cdots + V^n \), where \( V^k = [V^{k-1}, V, V] + [V, V, V^{k-1}] \), for \( k = 1, 2 \cdots n \), which implies that \( T \) is generated by \( V \). Since \( V \cap [T, T, T] = \{0\} \), we have that the basis of \( V \) is a minimal system of generators.

2) By the assumption, \( A \) is also a minimal system of the standard imbedding Lie algebra \( L(T) = T + L(T, T) \). By Lemma 1.2, we have
\[
A \cap [L(T), L(T)] = \emptyset.
\]

Since \( [L(T), L(T)] = [T, T, T] + L(T, T) \), \( A \cap [T, T, T] = \emptyset \), which implies that \( \{x_1 + [T, T, T], x_2 + [T, T, T], \cdots, x_l + [T, T, T]\} \) is a basis of \( T/[T, T, T] \).

\[ \square \]
Definition 1.1 Let $T$ be a Lie triple system, a subalgebra $t$ of $\text{Der} T$ is called a torus on $T$ if $t$ is abelian and consists of semisimple derivations. A torus is said to be maximal if it is not contained strictly in any other torus.

Therefore, all of the minimal systems of generators of $T$ have the same number $l = \dim T/[T,T,T]$, which is called the type of $T$. If $g$ is an algebraic Lie algebra, it is known that $g$ contains a semi-simple Lie subalgebra $s$ and an abelian subalgebra $a$ of fully reducible endomorphism such that $g = s + a + n$, $[s,a] = 0$, and the radical $r = a + n$(semi-direct)(cf. [2]). Since the derivation algebra $\text{Der} T$ of a Lie triple system $T$ is an algebraic Lie algebra of linear transformations, $\text{Der}T$ has the same decomposition of subspaces

\[(1) \quad \text{Der}T = s + a + n.\]

We can obtain a maximal torus on $T$ as follows.

Lemma 1.4 If $\mathfrak{h}$ be a Cartan subalgebra of $s$, then $t = \mathfrak{h} + a$ is a maximal torus on $T$.

Proof. $t = \mathfrak{h} + a$ is a torus since $[a,s] = 0$. Let $D$ be a semi-simple derivation of $T$ such that $[D,t] = 0$. By the decomposition (1), we have

\[D = D_1 + D_2 + D_3, \quad D_1 \in s, \quad D_2 \in a, \quad D_3 \in n.\]

If $h \in \mathfrak{h}$, and $a \in a$, then $0 = [h + a,D] = [a,D_1] + [h + a,D_3]$ which results in $[h,D_1] = 0$ and $[h + a,D_3] = 0$. So we have $D_1 \in \mathfrak{h}$ and $[D_1 + D_2,D_3] = 0$. Thus $D_1 + D_2$ and $D_3$ are the semi-simple part and the nilpotent part of $D$ respectively. Since $D$ is semi-simple, $D_3 = 0$, and then $D = D_1 + D_2 \in t$. Therefore, we have that $t$ is maximal. □

It is well known that any two maximal fully reducible subalgebras of a linear Lie algebra $g$ are conjugate under an inner automorphism from the radical of $[g,g]$(cf. [2]). For any maximal torus $t$ on a nilpotent Lie triple system $T$, there exists a maximal fully reducible subalgebra of $\text{Der} T$ whose center is $t$. The fact implies the conclusion as follows:

Lemma 1.5 Any two maximal tori on a Lie triple system $T$ are conjugate under an inner automorphism from the radical of $[\text{Der} T,\text{Der} T]$.

Therefore, all of the maximal tori on a nilpotent Lie triple system $T$ have the same dimension which we can call the rank of $T$.

Let $T$ be a Lie triple system with a torus $t$ on it. As a vector space, $T$ is a $t$-module if a representation is defined as follows:

\[t \times T \to T : (h,x) \mapsto h(x)\]

Since $t$ is a commutative family of semisimple endomorphisms, $t$ is simultaneously diagonalizable. In another words, $T$ is the direct sum of subspaces $T_\beta = \{x \in T|h \cdot x = \beta(h)x, \forall h \in t\}$. The set of all nonzero $\beta \in t^*$ for which $T_\beta \neq 0$ is denoted by $\Delta$. The elements of $\Delta$ are called the roots of $T$ relative to $t$, and for any root $\beta$ in $\Delta$, a vector $x$ in $T$ such that for any $h \in t$, $h \cdot x = \beta(h)x$, is called a root vector of $\beta$. With this notion we have a root space decomposition:

\[T = \sum_{\beta \in \Delta} T_\beta\]
Definition 1.2 Let $t$ be a torus on a nilpotent Lie triple system $T$. A minimal system of generators is called a $t$−msg if and only if it consists of root vectors for $t$.

Lemma 1.6 For any torus $t$ on a nilpotent Lie triple system $T$, there exists a $t$−msg.

Proof. Since $t$ is a torus, every $t$-module is completely reducible. Considering $t$-module $T$ and its submodule $[T,T,T]$, we can find another submodule $V$ such that $T = V + [T,T,T]$. As a module of $t$, there exists a basis $\{x_1, x_2, \ldots, x_n\}$ of $V$ consisting of root vectors for $t$. By Lemma 1.3, $\{x_1, x_2, \ldots, x_n\}$ is a $t$−msg. \qed

Lemma 1.7 Let $\{x_1, x_2, \ldots, x_l\}$ be a $t$−msg, $\beta_i$ the root of $x_i$, then the dimension of $t$ is equal to the rank of $\{\beta_1, \beta_2, \ldots, \beta_l\}$, and $\dim t \leq \dim T/[T,T,T]$.

Proof. Let $\{h_1, h_2, \ldots, h_k\}$ be a basis of $t$. The rank of $\{\beta_1, \beta_2, \ldots, \beta_l\}$ is equal to the rank of the matrix $(\beta_i(h_j))_{1 \leq i \leq l, 1 \leq j \leq k}$, whose value is $k$. \qed

Definition 1.3 If $\dim t = \dim T/[T,T,T] = l$, then $T$ is called a nilpotent Lie triple system of maximal rank $l$.

Example 1.1 Let $g$ be a semisimple Lie algebra with root decomposition relative to a Cartan subalgebra $h$
\[ g = h + \sum_{\alpha \in \Delta} g_{\alpha}, \]
where $\Delta$ is the root system.

Let $\pi$ be the simple root system, $\Delta_+$ be the positive root system, then
\[ g_+ = \sum_{\alpha \in \Delta_+} g_{\alpha} \]
is a subalgebra of $g$. Define a transformation $\theta$ on $g_+$: $\theta|_{g_{\alpha}} = (-1)^{ht\alpha}\text{id}$. Obviously, $\theta \in \text{Aut}g_+$ and $\theta^2 = \text{id}$.

Let $T = \{g_{\alpha}|\alpha \in \Delta_+, \text{ and } ht\alpha \equiv 1(\text{mod }2)\}$. $T$ is a nilpotent Lie triple system under the product that $[a,b,c] = [[a,b],c]$, $\forall a, b, c \in T$. Moreover, we have
\[ T = \sum_{\alpha \in \Pi} g_{\alpha} + \sum_{\alpha \in \Delta_+ \setminus \Pi, \text{ht} \alpha \equiv 1(\text{mod }2)} g_{\alpha} = \sum_{\alpha \in \Pi} g_{\alpha} + [T,T,T] \]
and $adh$ is a maximal torus acting on $T$. Thus $T$ is a nilpotent Lie triple system of maximal rank.

Proposition 1.1 Let $T$ be a nilpotent Lie triple system such that $T = T_1 \oplus T_2 \oplus \cdots \oplus T_i$ (direct sum of ideals), then $T$ is of maximal rank if and only if $T_i$ is of maximal rank for all $i$.
Proof. \( t \) is a maximal torus on \( T \) if and only if \( t = \bigoplus t_i \) is maximal torus on \( T \). Since
dim t = \sum_{i=1}^l \dim t_i \leq \sum_{i=1}^l \dim(T_i/[T_i,T_i]) = \dim(T/[T,T,T])
then we get the conclusion at once. \( \square \)

Proposition 1.2 Let \( T \) be a nilpotent Lie triple system of maximal rank \( l \), \( t \) a maximal torus on \( T \), \( \Delta \) the associated root system. If \( \{x_1, x_2, \ldots, x_l\} \) and \( \{y_1, y_2, \ldots, y_l\} \) are two \( t - \text{msg} \), then there exists a unique \( \sigma \in S_l \) and \( (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathbb{C}^l \) such that \( y_i = \lambda_i x_{\sigma_i} \).

Proof. Let
\[
\{x_1, x_2, \ldots, x_l\} \cup \{[x_{i_1}, x_{i_2}, \ldots, x_{i_{2n+1}}], n \geq 0, n \in \mathbb{Z}\}
\]
(where \( [x_{i_1}, x_{i_2}, \ldots, x_{i_{2n+1}}] = [[\cdots [[x_{i_1}, x_{i_2}], x_{i_3}], x_{i_4}], \ldots]x_{i_{2n}}, x_{i_{2n+1}}] \))
be a basis of the vector space \( T \) generated by \( \{x_1, x_2, \ldots, x_l\} \). There exist \( k_{j_i}, k_{i_1, i_2, \ldots, i_r} \in \mathbb{C} \) such that
\[
y_i = \sum_{j_i=1}^l k_{j_i} x_{j_i} + \sum_{r \geq 3} k_{i_1, i_2, \ldots, i_r} [x_{i_1}, x_{i_2}, \ldots, x_{i_r}].
\]
Let \( \beta_i \) be the root of \( x_i \) and \( \gamma_i \) the root of \( y_i, 1 \leq i \leq l \). Since \( T \) is of maximal rank,
\( \{\beta_1, \beta_2, \ldots, \beta_l\} \) and \( \{\gamma_1, \gamma_2, \ldots, \gamma_l\} \) are two bases of \( t^* \). For any \( h \) in \( t \), we have that
\[
h \cdot y_i = \gamma_i(h) y_i = \sum_{j_i=1}^l k_{j_i} \gamma_i(h) x_{j_i} + \sum_{i_1, i_2, \ldots, i_r} k_{i_1, i_2, \ldots, i_r} \gamma_i(h) [x_{i_1}, x_{i_2}, \ldots, x_{i_r}].
\]
On the other hand, by identity (3)
\[
h \cdot y_i = \sum_{j_i=1}^l k_{j_i} \beta_{j_i}(h) x_{j_i} + \sum_{i_1, i_2, \ldots, i_r} k_{i_1, i_2, \ldots, i_r} (\beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_r})(h) [x_{i_1}, x_{i_2}, \ldots, x_{i_r}].
\]
Comparing the two identities above, we have the following relations:
\[
k_{j_i} (\beta_{j_i} - \gamma_i) = 0,
\]
\[
k_{i_1, i_2, \ldots, i_r} (\gamma_i - (\beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_r})) = 0,
\]
\[\forall j_i = 1, 2 \ldots, l, \ \forall r = 2n + 1, \ n \geq 0, \ n \in \mathbb{Z}.
\]
Since the intersection between any minimal system of generators of \( T \) and \( [T, T, T] \) is empty (by the proof in Lemma 1.3), we have the fact that for any \( i = 1, 2, \ldots, l \), there exists \( j_i \in \{1, 2, \ldots, l\} \) such that \( k_{j_i} \neq 0 \), which results in \( \beta_{j_i} = \gamma_i \). We can find the integer \( j_i \) is unique. Assume there exists \( \{i_1, i_2, \ldots, i_r\} \) such that \( k_{i_1, i_2, \ldots, i_r} \neq 0 \). Then by identity (5), we have
\[
\beta_{j_i} = \gamma_i = \beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_r}
\]
which is impossible since \( \{\beta_1, \cdots, \beta_l\} \) is a basis of \( t^* \). Thus
\[
k_{i_1, i_2, \ldots, i_r} = 0, \ \text{for all} \ i_1, i_2, \ldots, i_r.
\]
Therefore we have that $y_i = k_j x_j$, and $k_j \neq 0$. Since the $(x_j)$’s are all distinct, we can define a map:

$$\sigma : \{1, 2, \cdots, l\} \rightarrow \{1, 2, \cdots, l\},$$

by setting

$$\sigma(i) = j_i.$$ 

Obviously, $\sigma \in S_l$. Letting $\lambda_i = k_j$, we have $y_i = \lambda_i x_{\sigma(i)}$. $\square$

For any derivation $D$ of $T$, $D$ can be extended to be one of $L(T)$ by

$$D(L(x, y)) = L(D(x), y) + L(x, D(y)), \quad x, y \in T$$

By the fact, we can say that a torus on $T$ is also a torus on $L(T)$. Furthermore, we have

**Proposition 1.3** If $T$ is a nilpotent Lie triple system of maximal rank, then $L(T)$ is a nilpotent Lie algebra of maximal rank.

**Proof.** Let $t$ be a maximal torus on $T$. Since $T$ is of maximal rank, $\dim t = \dim T/[T, T, T]$. By the fact that the standard imbedding Lie algebra $L(T)$ is a nilpotent algebra with $\dim L(T)/[L(T), L(T)] = \dim T/[T, T, T]$, we have that as a torus on $L(T)$, $\dim t = \dim L(T)/[L(T), L(T)]$, which implies that $t$ is maximal. Therefore, $L(T)$ is a nilpotent Lie algebra of maximal rank. $\square$

## 2 Free Lie triple systems, Kac-Moody algebras and nilpotent Lie triple systems of maximal rank

**Definition 2.1** Let $\tilde{T}(X)$ be a Lie triple system generated by a set $X$ over a field $F$. We say that $\tilde{T}(X)$ is free on $X$ if, given any mapping $\phi$ of $X$ into a Lie triple system $M$, there exists a unique homomorphism $\psi: \tilde{T}(X) \rightarrow M$ extending $\phi$.

It is easy to verify the uniqueness of such a Lie triple system $\tilde{T}(X)$. As to its existence, we begin with a vector space $V$ having $X$ as a basis, which forms the tensor algebra $\mathcal{L}(V)$ (viewed as a Lie algebra via the blanket operation, a Lie triple system via the $[x, y, z] = xyz - yxz - zyx + zyx$). Let $\tilde{T}(X)$ be the Lie triple subsystem of $\mathcal{L}(V)$ generated by $X$. Given any map $\phi$: $X \rightarrow M$, let $\hat{\phi}$ be extended first to a linear map $V \rightarrow M$ is, then to Lie algebra homomorphism $\mathcal{L}(V) \rightarrow L_U(M)$ is the universal embedding Lie algebra of $M$, or a Lie triple system homomorphism (whose restriction to $\tilde{T}(X)$ is the desired $\psi$: $\tilde{T}(X) \rightarrow M$, since $\psi$ maps the generators $X$ into $M$).

**Remark 2.1** Since any homomorphism $f$ between two Lie triple systems $T_1$ and $T_2$ can be extended to a Lie algebra homomorphism between $L(T_1)$ and $L(T_2)$ by $f(L(t_1, t_2)) = L(f(t_1), f(t_2))$, we have the universal property of $L(\tilde{T}(X))$ as follows:

Given any mapping $\phi$ from $X$ into a Lie triple system $M$, there exists a unique Lie algebra homomorphism $\psi$: $L(\tilde{T}(X)) \rightarrow L(M)$ extending $\phi$.  

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Let $T$ be nilpotent Lie triple system of maximal rank $l$, $t$ a maximal torus on $T$, $\{y_1, y_2, \cdots, y_l\}$ a $t - msg$. By the results in Section 1, we have that $\{y_1, y_2, \cdots, y_l\}$ is also a $t - msg$ of the nilpotent Lie algebra $L(T)$ of maximal rank $l$. Therefore we have that for all $i \neq j$, there exists $a_{ij} \in \mathbb{Z}_{\leq 0}$ satisfying

$$ (ad y_i)^{-a_{ij}} y_j \neq 0, \quad (ad y_i)^{-a_{ij} + 1} y_j = 0, $$

where $ad$ is the adjoint representation of $L(T)$. Letting $a_{ii} = 2$, we have an $l \times l$ generalized Cartan matrix $A = (a_{ij})$.

Let $\{x_1, x_2, \cdots, x_l\}$ is another $t - msg$ of $T$. By Lemma 2.1, there exists $\sigma \in S_l$ and $(\lambda_1, \lambda_2, \cdots, \lambda_l) \in \mathbb{C}^l$ such that $y_i = \lambda_i x_{\sigma i}$. Therefore,

$$ (ad x_{\sigma i})^{-a_{ij}} x_{\sigma j} \neq 0, \quad (ad x_{\sigma i})^{-a_{ij} + 1} x_{\sigma j} = 0. $$

Let $t'$ be another maximal torus on $T$. By Lemma 1.5, there exists $\omega \in \text{Aut} T$ such that $\omega \omega^{-1} = t'$. It is clear that $\{\theta(y_1), \theta(y_2), \cdots, \theta(y_l)\}$ is also a $t' - msg$. Therefore there exists a general Cartan matrix $A' = (a'_{ij})$ such that for all $i \neq j$

$$ (ad \omega(y_i))^{-a'_{ij}} \omega(y_j) \neq 0, \quad (ad \omega(y_i))^{-a'_{ij} + 1} \omega(y_j) = 0. $$

This implies that

$$ (ad y_i)^{-a_{ij} + 1} y_j \neq 0, \quad (ad y_i)^{-a_{ij} + 1} y_j = 0. $$

Hence, we have $A' = A$.

The discussion above derives the following assertion:

**Lemma 2.1** For any nilpotent Lie triple system $T$ of maximal rank $l$, there exists an $l \times l$ general Cartan matrix $A = (a_{ij})$ satisfying the following property: to any torus $t$ and any $t - msg = \{x_1, x_2, \cdots, x_l\}$, there exists $\sigma \in S_l$ such that for all $i \neq j$,

$$ (ad x_{\sigma i})^{-a_{ij}} x_{\sigma j} \neq 0, \quad (ad x_{\sigma i})^{-a_{ij} + 1} x_{\sigma j} = 0. $$

**Definition 2.2** We call the general Cartan matrix $A$ obtained in Lemma 2.1 the general Cartan matrix associated to $T$, and the $t - msg \{y_1, y_2, \cdots, y_l\}$ satisfying the condition (7) the $t - msg$ ordered relative to $A$.

**Example 2.1** $T_3 = \sum_{i=1}^{3} \mathbb{C} x_i$, $[x_1, x_2, x_1] = x_3$.

Type: 2.

Nilpotency: $p = 1$.

Maximal torus: $t = \mathbb{C} t_1 + \mathbb{C} t_2$, where $t_i(x_j) = \delta_{ij} x_j$, for $i, j = 1, 2$.

$t - msg$: $\{x_1, x_2\}$.

Root spaces decomposition relative to $t$:

$$ T = T_{\beta_1} + T_{\beta_2} + T_{2\beta_1 + \beta_2}, $$

$$ T_{\beta_1} = \mathbb{C} x_1, \quad T_{\beta_2} = \mathbb{C} x_2, \quad T_{2\beta_1 + \beta_2} = \mathbb{C} x_3. $$

$$ L(T_3) \cong \sum_{i=1}^{4} \mathbb{C} x_i$: $[x_1, x_2] = x_4$, $[x_4, x_1] = x_3$. 

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Example 2.2 \( T_4 = \sum_{i=1}^{4} C x_i = [x_1, x_2, x_1] = x_3, [x_1, x_2, x_2] = x_4. \)

Type: 2.

Nilpotency: \( p = 1. \)

Maximal torus: \( t = Ct_1 + Ct_2, \) where \( t_i(x_j) = \delta_{ij}x_j, \) for \( i, j = 1, 2. \)

\( t \)-msg: \( \{x_1, x_2\}. \)

Root spaces decomposition relative to \( t: \)

\( T = T_{\alpha} + T_{\beta} + T_{2\alpha} + T_{2\beta}, \)

\( T_{\alpha} = C x_1, T_{\beta} = C x_2, T_{2\alpha} = C x_3, T_{2\beta} = C x_4. \)

\( L(T_4) \cong \sum_{i=1}^{4} C x_i: [x_1, x_2] = x_5, [x_5, x_2] = x_4, [x_5, x_1] = x_3. \)

\( L(T_4) = L(T_4)_{\alpha} + L(T_4)_{\beta} + L(T_4)_{2\alpha} + L(T_4)_{2\beta}, \)

\( L(T_4)_{\alpha} = T_{\alpha}, L(T_4)_{\beta} = T_{\beta}, \)

\( L(T_4)_{2\alpha} = C x_5, L(T_4)_{2\beta} = T_{2\alpha}, L(T_4)_{2\beta} = T_{2\beta}. \)

\( (\text{ad}x_1)^2x_2 \neq 0, (\text{ad}x_1)^3x_2 = 0; \)

\( (\text{ad}x_2)^2x_1 \neq 0, (\text{ad}x_2)^3x_1 = 0. \)

General Cartan matrix: \( A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \) is the general Cartan matrix of affine Lie algebra \( A_1^{(1)}. \)

Let \( (\mathfrak{h}, \Pi, \Pi^\vee) \) be a realization of a general Cartan matrix \( A = (a_{ij}) \), where \( \Pi = \{a_1, \cdots, a_l\} \) included in \( \mathfrak{h}^* \) is root basis, \( \Pi^\vee = \{a_1^\vee, \cdots, a_l^\vee\} \) included in \( \mathfrak{h} \) is a coroot basis.

We set \( Q = \sum_{i=1}^{l} \mathbf{Z}a_i. \) For \( \alpha = \sum_{i=1}^{l} k_i a_i \in Q, \) the number \( \text{ht}(\alpha) := \sum_{i=1}^{l} k_i \) is called the height of \( \alpha. \)

Let \( g(A) \) be the Kac-Moody Lie algebra on the Chevally generators \( e_i, f_i (i = 1, \cdots l) \)
and \( h \), with the following relations:

\[
[h, h'] = 0 \quad (h, h' \in h),
\]

\[
[e_i, f_j] = \delta_{ij}h_i \quad (i, j = 1, \ldots, l),
\]

\[
[\alpha_i^\vee, e_j] = \alpha_j(\alpha_i^\vee)e_j = a_{ij}e_j, \\
[\alpha_i^\vee, f_j] = -\alpha_j(\alpha_i^\vee)f_j = -a_{ij}f_j \quad (i \neq j, \ \alpha_i^\vee \in \Pi^\vee, \ \alpha_j \in \Pi),
\]

\[
(\text{ad}e_i)^{-a_{ij}+1}e_j = 0, \\
(\text{ad}f_i)^{-a_{ij}+1}f_j = 0, \quad (\text{all } i \neq j)
\]

Then \( g(A) \) has the root space decomposition with respect to \( h \):

\[
g(A) = \sum_{\alpha \in Q} g(A)_\alpha.
\]

Here, \( g_\alpha = \{ x \in g(A) | [h, x] = \alpha(h)x \text{ for all } h \in h \} \) is the root space attached to \( \alpha \), and any element \( \alpha \in Q \) is called a root if \( \alpha \neq 0 \) and \( \dim g_\alpha \neq 0 \). A root \( \alpha > 0 \) (resp. \( \alpha < 0 \)) is called positive (resp. negative). It is well known that every root is either positive or negative. Denote by \( \Delta^+ \) the set of all positive roots.

Let

\[
n_+ = \sum_{\alpha \in \Delta^+} g(A)_\alpha.
\]

It follows immediately that \( n_+ \) is a subalgebra generated by \( e_1, \ldots, e_l \). Set

\[
N(A) = \sum_{\alpha \in \Delta'} g(A)_\alpha.
\]

\[
\Delta' = \{ \alpha \in \Delta^+, \text{and } h\alpha \equiv 1(\text{mod } 2) \}
\]

Given a triple product \([ , , ]\) on \( N(A) \) such that \([a, b, c] = [[a, b], c], N(A) \) forms a Lie triple system generated by \( e_1, \ldots, e_l \) satisfying only the relations

\[
(\text{ad}e_i)^{-a_{ij}+1}e_j = 0, \ \forall i \neq j.
\]

Let \( p \in \mathbb{N}^+ \) such that

\[
2p + 1 \geq \max\{-a_{ij} + 1, i \neq j\}.
\]

If \( A \) is of finite type, we further require that \( p \leq p_A \), where \( 2p_A + 1 \) is the height of the highest root of \( \Delta' \).

Setting

\[
N(A)_{2p+3} = \sum_{\substack{\alpha \in \Delta' \\ h\alpha \geq 2p + 3}} g(A)_\alpha,
\]

we have \( N(A)_{2p+3} \) is an ideal of \( N(A) \).

**Theorem 2.1** Let \( \tilde{N}(A) = N(A)/N(A)_{2p+3} \) and \( \rho: N(A) \longrightarrow \tilde{N}(A), x \mapsto \pi \) the canonical map, we have the following:

(i) The Lie triple system \( \tilde{N}(A) \) is nilpotent, and its nilpotency is \( p \).
(ii) The set of \( \{ \tilde{e}_1 \cdots \tilde{e}_l \} \) is a minimal system of generators of \( \tilde{N}(A) \) satisfying
\[
(ad\tilde{e}_i)^{-a_{ij}}\tilde{e}_j \neq 0, \quad (ad\tilde{e}_i)^{-a_{ij}+1}\tilde{e}_j = 0, 
\]
(15)

(iii) Let \( t_i \in \text{Der} \tilde{N}(A) \) defined by \( t_i(\tilde{e}_j) = \delta_{ij}\tilde{e}_j \). Then \( t = \bigoplus_{i=1}^{l} F t_i \) is a maximal torus on \( \tilde{N}(A) \), \( N(A) \) is of maximal rank. Furthermore, \( \{ \tilde{e}_1 \cdots \tilde{e}_l \} \) is a t-msg.

(iv) \( A \) is the general Cartan matrix associated to \( \tilde{N}(A) \).

**Proof.** (i) Obviously, the Lie triple system \( \tilde{N}(A) \) is nilpotent and of nilpotency \( \leq p \). It is clear that the nilpotency of \( N(A) \) is \( p \) if and only if
\[
(\tilde{N}(A))^p = N(A)^p = \sum_{hta=2p+1} g(A)_a \neq \{0\},
\]
and
\[
(\tilde{N}(A))^{(p+1)} = N(A)^{(p+1)} = \sum_{hta=2p+3} g(A)_a = \{0\}.
\]
Therefore, it is sufficient to verify that \( N(A)^p \neq \{0\} \). If \( g(A) \) is finite semi-simple, then \( N(A) \) is of nilpotency \( p_A \), thus \( N(A)^p \neq \{0\} \) by the fact \( p \leq p_A \); If \( A \) is not of finite type, \( g(A) \) is infinite dimensional. By the property of roots system that for all \( \alpha \in \Delta_+ \setminus \{\alpha_1 \cdots \alpha_l\} \) there exists \( i \in \{1 \cdots l\} \) such that \( \alpha - \alpha_i \in \Delta_+ \), we have \( N(A)^p \neq \{0\}, \forall p \geq 1 \).

(ii) We have that
\[
\tilde{N}(A)/[\tilde{N}(A), \tilde{N}(A), \tilde{N}(A)] \cong \sum_{hta=1}^{l} g(A)_a = \sum_{i=1}^{l} C\tilde{e}_i.
\]
By Lemma 1.3, \( \{ \tilde{e}_1 \cdots \tilde{e}_l \} \) is a minimal system of generators. From the construction of \( \tilde{N}(A) \), we have that \( \{ \tilde{e}_1 \cdots \tilde{e}_l \} \) satisfies the condition in (15).

(iii) Obviously, \( \mathfrak{t} \) is a torus on \( \tilde{N}(A) \) and
\[
\text{dim} \mathfrak{t} = \dim \tilde{N}(A)/[\tilde{N}(A), \tilde{N}(A), \tilde{N}(A)] = l.
\]
By Definition 1.3, we have that \( \mathfrak{t} \) is a maximal torus, \( \tilde{N}(A) \) is of maximal rank and \( \{ \tilde{e}_1 \cdots \tilde{e}_l \} \) is a t-msg.

From the results in (i), (ii), (iii), we can draw (iv) immediately. \( \square \)

**Remark 2.2** We can find that for all \( i \neq j \), \( (ad\tilde{e}_i)^{-a_{ij}+1}\tilde{e}_j \notin N(A)_{2p+1}/\{0\} \). In fact, if it is not true, then we have that \( 2p+1 \leq -a_{ij}+1 \), which contradicts the condition in (13).

**Example 2.3**

\[
sl(2, C) = C\mathbb{H}C = C\mathbb{H}C \oplus Cf
\]

with Cartan subalgebra \( \mathfrak{h}_0 = C\mathbb{H}C \), the root system \( \Delta_0 = \{\pm \alpha\} \).

\[
g = g(A_1^{(1)}) = C[t, t^{-1}] \otimes sl(2, C) \oplus Cc \oplus Cd
\]
is the affine algebra associated to the general Cartan maximal \( A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \) with generators \( e \) and \( f \). \( g \) has a root space decomposition relative \( h = \mathfrak{ch} + \mathfrak{c} + \mathfrak{d} \)

\[
g = h + \sum_{\alpha \in \Delta} g_{\alpha},
\]

\[
\Delta = \{ j\delta + \alpha, j\delta - \alpha \mid j \in \mathbb{Z} \} \cup \{ j\delta \mid j \in \mathbb{Z}, j \neq 0 \},
\]

\[
d(k_0 h + k_1 c + k_2 d) = k_2,
\]

\[
\Pi = \{ \alpha_0, \alpha \}, \quad \alpha_0 = \delta - \alpha.
\]

Let

\[
N(A_1^{(1)}) = \sum_{\beta \in \Delta_+} g_{\beta}, \quad \text{ht}\beta \equiv 1(\text{mod } 2)
\]

and

\[
N(A_1^{(1)})_5 = \sum_{\beta \in \Delta_+} g_{\beta}, \quad \text{ht}\beta \geq 5 \quad \text{ht}\beta \equiv 1(\text{mod } 2)
\]

we have that

\[
\tilde{N}(A_1^{(1)}) \cong g_{\alpha} + g_{\alpha_0} + g_{2\alpha_0} + g_{\alpha_0 + 2\alpha}
\]

and \( \tilde{N}(A_1^{(1)}) \) is isomorphic to \( T_4 \) in Example 2.2 which is a nilpotent Lie triple system of maximal rank 2, and with nilpotency \( p = 1 \).

Let \( X = \{ e_1 \cdots e_l \} \). Thanks to the universal property of \( \tilde{T}(X) \), we have the claim as follows.

**Lemma 2.2** Let \( i \) be the imbedding map of \( X \) into \( \tilde{T}(X) \). To the imbedding map \( i' \) of \( X \) into \( N(A) \), there exists a unique homomorphism \( f \) from \( \tilde{T}(X) \) onto \( N(A) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{i} & \tilde{T}(X) \\
\downarrow{i'} & & \downarrow{f} \\
N(A) & & 
\end{array}
\]

**Theorem 2.2** \( \tilde{N}(A) \) with the canonical map \( \rho \) in Theorem 2.1 satisfies the following universal property: for any nilpotent Lie triple system \( T' \) of maximal rank \( l \), of nilpotency \( q \) such that \( q \leq p \), whose associated general Cartan matrix \( B = (b_{ij}) \) satisfies that for any \( i, j, b_{ij} \geq a_{ij} \), if a map \( \phi: X = \{ e_1 \cdots e_l \} \rightarrow T' \) satisfies that \( \{ \phi(e_1) \cdots \phi(e_l) \} \) is a \( \text{t}' \)-msg relative to \( B(\text{t}' \text{ is a maximal torus on } T') \), then there exists a unique homomorphism \( \psi \) such that \( \psi \cdot \rho|_X = \phi \).
\textit{Proof.} For convenience we still use the notation in the discussion above as follows:
i: the imbedding map of $X$ to $\tilde{T}(X)$;
i': the imbedding map of $X$ to $N(A)$;
f: the homomorphism of $\tilde{T}(X)$ onto $N(A)$ such that $f \cdot i = \sigma$.

For $\phi: X = \{e_1 \cdots e_l\} \longrightarrow T'$, there exists a unique homomorphism
$$\mu: \tilde{T}(X) \longrightarrow T'$$
such that the following diagram commutes:

Since $\{\phi(e_1) \cdots \phi(e_l)\}$ is ordered relative to $B = (b_{ij})$, we have that for any $i \neq j$
$$(\text{ad}\mu(e_i))^{-b_{ij}+1}\mu(e_j)(\text{ad}\phi(e_i))^{-b_{ij}+1}\phi(e_j) = 0.$$  

Hence,
$$\mu(\text{ad}(e_i)^{-b_{ij}+1}e_j) = 0.$$  

From Lemma 2.2, $\ker f$ is generated by
$$(\text{ad}e_i)^{-a_{ij}+1}e_j = 0, \forall i \neq j.$$  

Since we assume that $b_{ij} \geq a_{ij}$ for all $i, j$, we have that $\ker f \subseteq \ker \mu$. Therefore, there exists a unique homomorphism
$$\tau: N(A) \longrightarrow T'$$
such that
$$\tau \cdot f = \mu. \tag{16}$$

Thus the diagram

is a commutative diagram. Since $f \cdot i$ is equal to the imbedding map $i'$ of $X$ into $N(A)$ in Lemma 2.2, we have that
$$\tau \cdot i' = \phi. \tag{17}$$

Considering the assumption that the nilpopency $q$ of $T'$ is not larger than $p$, we have that
$$\tau(\sum_{h\alpha \geq q} g(A_{\alpha}) = \{0\}, \tag{18}$$
which implies that $N(A)_{p+1} \subseteq \ker \tau$. Thus there exists a unique homomorphism $\psi$ of $\tilde{N}(A)$ to $T$ such that

$$\psi \cdot \rho = \tau,$$

and the following commutative diagram is obtained, from which we have that $\psi \cdot \rho|_X = \phi$.

![Diagram](image)

**Remark 2.3** Let $T$ be a nilpotent Lie triple system of maximal rank $l$, of nilpotency $p$, $A$ be an associated general Cartan matrix, $\{x_1 \cdots x_l\}$ be the $t-\text{msg}$ ordered relative to $A$. Let $h : \{e_1 \cdots e_l\} \mapsto \{x_1 \cdots x_l\}$ be a map defined by $h(x_i) = e_i$. By Theorem 2.2, there exists a homomorphism $H$ from $T$ onto $N(A)$ such that $H(x_i) = x_i$. Setting $a = \ker H$, we have that $T \cong \tilde{N}(A)/a$. Since $T$ is of nilpotency $p$, $(\tilde{N}(A))^p$ is not contained in $a$. Furthermore, by (15), we have

$$(\text{ad} \bar{e}_i)^{a_{ij}} \bar{e}_j \notin a.$$  

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**References**


