Lie triple systems associated with $F[t, t^{-1}]$ *

Yiqian Shi  
Department of mathematics, East China Normal University  
Shanghai, P. R. China, 200062

Daoji Meng  
Department of Mathematics & LPMC, Nankai University,  
Tianjin P. R. China, 300071

Abstract  
In this paper, we will give two methods to construct Lie triple systems from the Laurent-polynomial algebra $F[t, t^{-1}]$.

0. Introduction  
Throughout this paper, the vector spaces we discuss are all over a field $F$ of characteristic 0.

A Lie triple system $T$ is a vector space with a trilinear product $[ , , ]$ satisfying
\begin{align*}
\forall x, y, z, u, v \in T \\
1) \quad [x, y, z] = -[y, x, z]; \\
2) \quad [x, y, z] + [y, z, x] + [z, x, y] = 0; \\
3) \quad [u, v, [x, y, z]] = [[u, v, x, y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]].
\end{align*}

Any Lie algebra $(l, [ , ])$ forms a Lie triple system with the trilinear product
\begin{equation*}
[x, y, z] = [[x, y], z], \quad \forall x, y, z \in l.
\end{equation*}

If $\theta$ is an involutive automorphism of $l$, then the eigenspace $E_{-1}(\theta)$ is a Lie triple system since $E_{-1}(\theta)$ is closed under the trilinear product. We begin the present paper with

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the involutive automorphisms of $F[t, t^{-1}]$ and then construct some Lie triple systems precisely.

In section 1, we find all the automorphisms of $F[t, t^{-1}]$ (Proposition 1.1), and then obtain all the involutive ones (Proposition 1.2). In section 2, we proceed to discuss the derivation algebra $m$ of $F[t, t^{-1}]$ (Theorem 2.1) and use the deduced involutive automorphisms of $m$ to construct some Lie triple systems (Lemma 2.1, Theorem 2.2). We find the infinite dimensional Lie triple systems obtained in this way are simple. From the relations between the induced automorphisms of $m$, we give some automorphisms of the simple Lie triple systems obtained in Theorem 2.2 (Lemma 2.2, Theorem 2.3).

Since a Novikov algebra can be formed with derivation and an element of $F[t, t^{-1}]$ which is fixed, then we can construct Lie algebras by the fact that there exists a natural Lie algebraic structure on any Novikov algebra (Proposition 3.2). In section 3, we discuss the involutive automorphisms of such Lie algebras and get more Lie triple systems (Theorem 3.1(a) and (b)).

1. $F[t, t^{-1}]$ and its automorphisms

Let $A = F[x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_m]$ be a polynomial algebra in $x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_m$ over a field $F$, $J$ an ideal of $A$ generated by $\{y_1z_1 - 1, \ldots, y_mz_m - 1\}$. Let $\pi$ be the natural homomorphism of $A$ to the quotient algebra $A/J$. We have

$$\pi(y_jz_j) = \pi(y_j)\pi(z_j) = 1, \quad 1 \leq j \leq m.$$ 

If we denote $\pi(y_j)$ by $t_j$, then $\pi(z_j) = t_j^{-1}$ and $A/J = F[x_1, \ldots, x_n, t_1^{-1}, \ldots, t_m^{-1}]$. When $n = 0$ and $m = 1$, $A/J = F[t, t^{-1}]$ is the laurent-polynomial algebra of $t$.

Let $\theta$ be a transformation of $F[x, y]$ satisfying

$$\theta(f(x, y)) = f(y, x), \quad \forall f(x, y) \in F[x, y],$$
Then $\theta$ is an involutive automorphism of $F[x, y]$, and

$$F[x, y] = E_1(\theta) + E_{-1}(\theta),$$

where

$$E_1(\theta) = \{ f(x, y) \in F[x, y] | f(x, y) = f(y, x) \},$$

$$E_{-1}(\theta) = \{ f(x, y) \in F[x, y] | f(x, y) = -f(y, x) \}.$$

Since the ideal $J = \{ (xy - 1)f(x, y) | f(x, y) \in F[x, y] \}$ is fixed by $\theta$, an involutorial automorphism of $F[t, t^{-1}]$ is deduced, denoted by $\nu$, satisfying $\nu \pi = \pi \theta$, where $\pi$ is the natural homomorphism of $F[x, y]$ to $F[t, t^{-1}] = F[x, y]/J$.

It is not difficult to find that $\{ at^n | a \in F, a \neq 0; n \in \mathbb{Z} \}$ is the set of all the inverse elements in $F[t, t^{-1}]$.

In fact, for two arbitrary elements in $F[t, t^{-1}]$,

$$f(t) = a_n t^n + \cdots + a_m t^m, \quad n < m, \quad a_n a_m \neq 0,$$

and

$$g(t) = b_l t^l + \cdots + b_k t^k, \quad l < k, \quad b_l b_k \neq 0,$$

we have

$$f(t)g(t) = a_n b_l t^{n+l} + \cdots + a_m b_k t^{m+k}, \quad n + l < m + k.$$

Since

$$at^n a^{-1} t^{-n} = 1, \quad \forall a \in F, \ a \neq 0, \ n \in \mathbb{Z},$$

all the inverse elements are obtained.

By the fact that $t$ and $t^{-1}$ generate $F[t, t^{-1}]$, it is sufficient to discuss the actions on $t$ and $t^{-1}$ when considering an automorphism of $F[t, t^{-1}]$. Suppose $\varphi$ is an automorphism. Then we have

$$1 = \varphi(1) = \varphi(tt^{-1}) = \varphi(t)\varphi(t^{-1}).$$
Here we can see that \( \varphi(t) \) is inverse. Hence there exists a nonzero element \( a \) in \( F \) and an integer \( n \) such that \( \varphi(t) = at^n \). Note that \( \varphi(t) \) and \( \varphi(t^{-1}) \) generate \( F[t, t^{-1}] \), then \( n = \pm 1 \). Therefore we have the following conclusion.

**Proposition 1.1** All the automorphisms of \( F[t, t^{-1}] \) consist of two classes:

1) \( \varphi_a(t) = at, \varphi_a(t^{-1}) = a^{-1}t^{-1}, \ a \in F, \ a \neq 0; \)
2) \( \psi_a(t) = at^{-1}, \psi_a(t^{-1}) = a^{-1}t, \ a \in F, \ a \neq 0. \)

From the conclusion above, we can obtain the results as follows.

**Proposition 1.2**

1) \( \varphi_a \cdot \nu = \psi_{a^{-1}}, \ \nu = \psi_1. \)
2) \( \varphi_1 \) and \( \psi_a (\forall a \neq 0) \) are involutive, and \( F[t, t^{-1}] \) has the corresponding eigenspace decompositions respectively as follows:

**case i)**

\[
F[t, t^{-1}] = E_1(\varphi_1) + E_{-1}(\varphi_1),
\]

\[
E_1(\varphi_1) = \left\{ f(t) = \sum_{i=-\infty}^{+\infty} a_i t^i \left| a_{2k+1} = 0 \right. \right\},
\]

\[
E_{-1}(\varphi_1) = \left\{ f(t) = \sum_{i=-\infty}^{+\infty} a_i t^i \left| a_{2k} = 0 \right. \right\} ;
\]

**case ii)**

\[
F[t, t^{-1}] = E_1(\psi_a) + E_{-1}(\psi_a)
\]

\[
E_1(\psi_a) = \left\{ f(t) = \sum_{i=-\infty}^{+\infty} a_i t^i \left| a_{-i} = a_i a^i \right. \right\},
\]

\[
E_{-1}(\psi_a) = \left\{ f(t) = \sum_{i=-\infty}^{+\infty} a_i t^i \left| a_{-i} = -a_i a^i \right. \right\} .
\]

2. Derivations of \( F[t, t^{-1}] \) and Lie triple system

Let us recall some results of associative algebras. Let \( (A, \cdot) \) be an associative algebra over \( F \). The set of all derivations of the associative algebra, denoted by \( \text{Der} A \), constructs
a Lie algebra \((A, [ , ])\) with

\[
[D_1, D_2] = D_1D_2 - D_2D_1, \quad \forall D_1, D_2 \in \text{Der}A.
\] (2.1)

For any automorphism \(\varphi\) of \(A\), we have

\[
\varphi D \varphi^{-1} \in \text{Der}A, \quad \forall D \in \text{Der}A
\] (2.2)

and the transformation \(\Phi\) of \(\text{Der}A\), defined by

\[
\Phi(D) = \varphi D \varphi^{-1}, \quad \forall D \in \text{Der}A,
\] (2.3)

is an automorphism of \(\text{Der}A\).

In this section we will apply these general results to Laurent-polynomial algebra \(F[t, t^{-1}]\) to construct Lie triple systems.

We denote by \(m\) the set of all the derivations of \(F[t, t^{-1}]\) and obtain result as follows.

**Theorem 2.1**

1) \(m = \{ f(t) \frac{d}{dt} | f(t) \in F[t, t^{-1}] \} \); (2.4)

2) \(m\) is a simple Lie algebra.

**Proof**

1) For any \(D\) in \(m\), there exists \(f_1(t) \in F[t, t^{-1}]\) such that \(D(t) = f_1(t)\). Obviously, \(f_1(t) \frac{d}{dt}\) is a derivation of \(F[t, t^{-1}]\). Notice that \(t \cdot t^{-1} = 1\), we have

\[
D(t^{-1}) = -t^{-2}D(t) = -t^{-2}f_1(t)
\] (2.5).

Hence, the following identities

\[
(D - f_1(t) \frac{d}{dt})(t) = 0 \quad \text{and} \quad (D - f_1(t) \frac{d}{dt})(t^{-1}) = 0,
\] (2.6)

are obtained. Thus we have

\[
D = f(t) \frac{d}{dt}
\]
and
\[
m = \left\{ f(t) \frac{d}{dt} \middle| f(t) \in F[t, t^{-1}] \right\}.
\]

2) Set \( d_n = t^{n+1} \frac{d}{dt} \) \((n \in \mathbb{Z})\). Then \( m = \sum_{n \in \mathbb{Z}} F d_n \). It is easy to verify \( m \) is a Lie algebra with the product such as

\[
[ d_n, d_m ] = \left[ t^{n+1} \frac{d}{dt}, t^{m+1} \frac{d}{dt} \right] \\
= (m - n) t^{n+m+1} \frac{d}{dt} \\
= (m - n) d_{m+n}.
\]

(2.7)

Let \( n \) be an ideal of \( m \) and \( n \neq \{0\} \). For any nonzero element \( D \) in \( n \), there exist \( c_i \in F \), and \( n_i \in \mathbb{Z} \) such that
\[
D = \sum_{i=1}^{n} c_i d_{n_i}.
\]

We can obtain
\[
d_{n_i} \in n,
\]
from the fact that for any \( k \in \mathbb{Z} \), and \( k \geq 0 \),
\[
(\text{ad} d_0)^k(D) = \sum_{i=1}^{n} c_i n_i^k d_{n_i} \in n.
\]

(2.8)

If \( n_i \neq 0 \), then \( [d_{n_i}, d_{-n_i}] = -2n_i d_0 \in n \). Thus \( d_0 \in n \) and \( [d_0, d_{n}] = nd_{n}, \forall n \neq 0 \).
Therefore we have \( n = m \), which implies \( m \) is simple. \( \square \)

**Corollary 2.1** Let \( \rho \) be an involutive automorphism of \( m \). The eigenspace \( E_{-1}(\rho) \) of \( m \) as a Lie triple system is simple.

**proof** Let \( g \) be a simple Lie algebra, and \( \rho \) be an involutive automorphism of \( g \).
Then \( g \) has the eigenspace decomposition \( g = g_+ \oplus g_- \) relative to \( \rho \). \( g_- \) forms a Lie triple system, and \( [g_-, g_-] \subseteq g_+, [g_-, g_+] \subseteq g_- \). \( g_- \oplus [g_-, g_-] \) is an ideal of \( g \) since the
fact that
\[
([g_− ⊕ [g_−, g_−]), (g_− ⊕ g_+)] \subseteq [g_−, g_−] + g_− + [g_−, g_+] + [g_−, g_−, g_+]
\]
\[
\subseteq g_− + [g_−, g_−] + [g_+, g_−, g_−]
\]
\[
\subseteq g_− ⊕ [g_−, g_−].
\]

If \( a \neq \{0\} \) is an ideal of \( g_- \), then \( a ⊕ [a, g_-] \) is an ideal in \( g_- \), which implies that \( a ⊕ [a, g_-] = g_- \). Hence we have \( a = g_- \) and \( g_- \) is simple. Since \( m \) is simple, the result is obtained immediately. \( \square \)

From Proposition 1.1 we have \( φ_a \) and \( ψ_a \) (\( a ∈ F \) and \( a \neq 0 \)) are automorphisms. Here we denote by \( Φ_a \) and \( Ψ_a \) the corresponding automorphisms of \( m \) respectively, i.e.
\[
Φ_a(D) = φ_aDφ_a^{-1}, \quad Ψ_a(D) = ψ_aDψ_a^{-1}, \quad ∀D ∈ m.
\] (2.9)

Since \( m = \{ f(t) \frac{d}{dt} | f(t) ∈ F[t, t^{-1}] \} \), the actions of \( Φ_a \) and \( Ψ_a \) on \( m \) can be described more precisely.

**Lemma 2.1**

1) \( Φ_a(f(t) \frac{d}{dt}) = a^{-1}f(at) \frac{d}{dt} \) \hspace{1cm} (2.10)

2) \( Ψ_a(f(t) \frac{d}{dt}) = -a^{-1}t^2f(at^{-1}) \frac{d}{dt} \). \hspace{1cm} (2.11)

**proof** Since \( φ_a^{-1} = φ_{a^{-1}} \) and \( ψ_a^{-1} = ψ_a \), we have
\[
Φ_a(f(t) \frac{d}{dt})(t) = φ_a(f(t) \frac{d}{dt})φ_{a^{-1}}(t)
\]
\[
= φ_a(f(t) \frac{d}{dt})(a^{-1}t)
\]
\[
= φ_a(a^{-1}f(t)) = a^{-1}f(at),
\]
and
\[
Ψ_a(f(t) \frac{d}{dt})(t) = ψ_a(f(t) \frac{d}{dt})ψ_a(t)
\]
\[
\psi_a(f(t) \frac{d}{dt})(at^{-1}) = \psi_a(-at^{-2}f(t)) = -a^{-1}t^2 f(at^{-1}),
\]

and then (2.10) and (2.11) follows respectively.

Since \(\Phi_{-1}\) and \(\Psi_a(\forall a \in \mathbb{F} \text{ and } a \neq 0)\) are involutive, \(m\) has the corresponding eigenspace decompositions respectively as follows:

\[
m = E_1(\Phi_{-1}) + E_{-1}(\Phi_{-1}),
\]

\[
E_1(\Phi_{-1}) = \left\{ f(t) \frac{d}{dt} \mid f(-t) = -f(t) \right\},
\]

\[
E_{-1}(\Phi_{-1}) = \left\{ f(t) \frac{d}{dt} \mid f(-t) = f(t) \right\};
\]

and

\[
m = E_1(\Psi_a) + E_{-1}(\Psi_a),
\]

\[
E_1(\Psi_a) = \left\{ f(t) \frac{d}{dt} \mid t^2 f(at^{-1}) = -af(t) \right\},
\]

\[
E_{-1}(\Psi_a) = \left\{ f(t) \frac{d}{dt} \mid t^2 f(at^{-1}) = af(t) \right\}.
\]

From the discussion for the derivation algebra \(m\) of \(\mathbb{F}[t, t^{-1}]\) above, we draw the result as follows.

**Theorem 2.2** Let

\[
\tilde{T} = \left\{ f(t) \frac{d}{dt} \mid f(-t) = f(t), f(t) \in \mathbb{F}[t, t^{-1}] \right\},
\]

\[
\hat{T}_a = \left\{ f(t) \frac{d}{dt} \mid t^2 f(at^{-1}) = af(t), f(t) \in \mathbb{F}[t, t^{-1}] \right\} \quad (\forall a \in \mathbb{F} \text{ and } a \neq 0).
\]

Defining a product on \(\text{Der}[t, t^{-1}]\) such that

\[
\begin{bmatrix}
    f_1(t) \frac{d}{dt}, & f_2(t) \frac{d}{dt}, & f_3(t) \frac{d}{dt}
\end{bmatrix}
\]

\[
= (f_1(t)f_2'(t)f_3'(t) - f_2(t)f_1'(t)f_3'(t) - f_1(t)f_2'(t)f_3'(t) + f_1'(t)f_2(t)f_3(t)) \frac{d}{dt}. \quad (2.12)
\]
we have that \( \tilde{T} \) and \( \tilde{T}_a \) is infinite dimensional simple Lie triple system respectively. \( \square \)

An automorphism \( \lambda \) of a Lie triple system \( T \) is an automorphism of vector spaces satisfying

\[
\lambda([x_1, x_2, x_3]) = [\lambda(x_1), \lambda(x_2), \lambda(x_3)], \quad \forall x_1, x_2, x_3 \in T. \tag{2.13}
\]

Suppose \( T \) is the Lie triple system of \( E_{-1}(\theta) \) of a Lie algebra \( g \), where \( \theta \) is an involutive automorphism of \( g \). An automorphism \( \tau \) of \( g \) is an automorphism of \( T \) if and only if \( \tau(T) = T \), if and only if \( \tau\theta = \theta\tau \). In fact, if \( \tau \) is an automorphism of \( T \), then \( \theta\tau(x) = -\tau(x) = \tau\theta(x), \forall x \in T \), thus \( \theta\tau = \tau\theta \). On the other hand, if \( \theta\tau = \tau\theta \), then \( \theta(\tau(x)) = \tau\theta(x) = -\tau(x), \forall x \in T \), which implies \( \tau(T) = T \).

For the Lie algebra \( m \), we have obtained the automorphisms in the forms \( \Phi_a \) and \( \Psi_a \) (\( \forall a \in F \), and \( a \neq 0 \)). We list some relations between them as follows.

**Lemma 2.2** For \( \forall a, b \in F \), and \( a \neq 0, b \neq 0 \), we have

1) \( \Phi_a\Phi_b = \Psi_a\Psi_b = \Phi_{ab} \);

2) \( \Psi_a\Phi_b = \Psi_{ab} \);

3) \( \Psi_a\Phi_b = \Phi_b\Psi_a \) if and only if \( b = \pm 1 \).

**proof** By the identities (2.10) in Lemma 2.1, for any element \( f(t)\frac{d}{dt} \in m \),

\[
\Phi_a\Phi_b(f(t)\frac{d}{dt}) = \Phi_a(b^{-1}f(bt)\frac{d}{dt}).
\]

Since

\[
(\Phi_a(b^{-1}f(bt)\frac{d}{dt}))(t) = b^{-1}(\varphi_a(f(bt)\frac{d}{dt}))\varphi^{-1}(t)
= b^{-1}\varphi_a(f(bt))\frac{d}{dt}(a^{-1}t)
= b^{-1}a^{-1}\varphi_a(f(bt))
= b^{-1}a^{-1}f(abt)
= \Phi_{ab}(f(t)\frac{d}{dt})(t),
\]
we have
\[ \Phi_a \Phi_b (f(t) \frac{df}{dt}) = \Phi_{ab}(f(t) \frac{df}{dt}), \]
which implies that
\[ \Phi_a \Phi_b = \Phi_{ab}. \]
Using the same method, the relations \( \Psi_a \Psi_b = \Phi_{ab}, \Psi_a \Phi_b = \Psi_{ab} \) can be verified. Therefore, the identities in 1) and 2) are true.

Comparing the following two identities
\[ \Phi_b \Psi_a (f(t) \frac{df}{dt}) = -a^{-1} b f(ab^{-1} t^{-1}) t^2 \frac{dt}{dt}, \]
and
\[ \Psi_a \Phi_b (f(t) \frac{df}{dt}) = \Psi_{ba}(f(t) \frac{df}{dt}) = -a^{-1} b^{-1} f(ab^{-1} t^{-1}) t^2 \frac{dt}{dt}, \]
we have that \( \Psi_a \Phi_b = \Phi_b \Psi_a \) if and only if \( b = \pm 1. \)

From the relations above, we draw the following conclusion immediately.

**Theorem 2.3** Let \( \tilde{T} \) and \( \hat{T}_a \) be the Lie triple systems in Theorem 2.2. Then we have

1) \( \forall a \in F, \) and \( a \neq 0, \) \( \Phi_a, \Psi_a \) are automorphisms of \( \tilde{T}; \)

2) \( \forall b \in F, \) and \( b \neq 0, \) \( \Psi_b \) is automorphism of \( \hat{T}_a. \)

**3. Novikov algebra, \( F[t, t^{-1}] \) and Lie triple system**

A Novikov algebra \( N \) is an algebra with a product \(*\) satisfying the following conditions:

\[ \forall x, y, z \in N, \]

1) \( (x, y, z) = (y, x, z), \) where \( (x, y, z) = (x * y) * z - x * (y * z); \) \hspace{1cm} (3.1)

2) \( (x * y) * z = (x * z) * y. \) \hspace{1cm} (3.2)
Let $A$ be a commutative associative algebra over $F$, $D$ a derivation of $A$. If we fix an element $a$ of $A$, and define a product $*$ on $A$ such as

$$x * y = xD(y) + axy, \quad \forall x, y \in A,$$

then $(A, *)$ is a Novikov algebra which is denoted by $(A, D, a)$.

**Proposition 3.1** Let $A$ be a commutative associative algebra, $D$ a derivation of $A$, and $a \in A$. If $\varphi$ is an automorphism of $A$, then the Novikov algebras $(A, D, a)$ and $(A, \varphi D \varphi^{-1}, \varphi(a))$ are isomorphic.

**proof** In order to avoid confusion, we denote by $*$ (resp. $*_1$) the product of $(A, D, a)$ (resp. $(A, \varphi D \varphi^{-1}, \varphi(a))$). By (3.3), we have

$$\varphi(x) *_1 \varphi(y)$$

$$= \varphi(x) (\varphi D \varphi^{-1} \varphi(y)) + \varphi(a) \varphi(x) \varphi(y)$$

$$= \varphi(x) \varphi (D(y)) + \varphi(axy)$$

$$= \varphi(xD(y) + axy),$$

$$= \varphi(x * y), \quad \forall x, y \in (A, D, a),$$

then the conclusion follows. \qed

For a Novikov algebra $N$, if we define a Lie product such that

$$[x, y] = x * y - y * x, \quad x, y \in N,$$

then $N$ constructs a Lie algebra which is called associated Lie algebra of $N$. We can see that $(A, D, a)$ and $(A, D, b)$ have the same associated Lie algebra. It is obvious that an automorphism $\phi$ of a Novikov algebra is also one of the associated Lie algebra.

Let us return to $F[t, t^{-1}]$.

**Proposition 3.2** For any element $f(t)$ in $F[t, t^{-1}]$, $F[t, t^{-1}]$ forms a Lie algebra
denoted by \((F[t, t^{-1}], f(t))\) under such a product that
\[
[g(t), h(t)] = f(t)(g(h)h'(t) - g'(t)h(t)), \quad g(t), h(t) \in F[t, t^{-1}],
\] (3.5)
where \(g'(t) = \frac{dg(t)}{dt}\).

**proof** Since \(f(t) \frac{d}{dt} \in \mathfrak{m}\), the Novikov algebra \((F[t, t^{-1}], f(t) \frac{d}{dt}, g(t)), \forall 0 \neq g(t) \in F[t, t^{-1}]\), can be obtained by (3.3). Hence, from (3.4) we can define the Lie product of the associated Lie algebra of \(N\) in (3.5). \(\Box\)

Obviously, \((F[t, t^{-1}], 0)\) is an abelian Lie algebra.

When we define a linear map \(\rho\) of \((F[t, t^{-1}], 1)\) to \(\mathfrak{m}\) such that \(\rho(f(t)) = f(t) \frac{d}{dt}, \forall f(t) \in (F[t, t^{-1}], 1)\), it is easy to verify \(\rho\) is an isomorphism. Then \((F[t, t^{-1}], 1)\) is isomorphic to \(\mathfrak{m}\).

**Proposition 3.3** Let \(g(t) = a^{-1}f(at), h(t) = -a^2t^2f(at^{-1})\). Then \(\varphi_a\) is an isomorphism of \((F[t, t^{-1}], f(t))\) to \((F[t, t^{-1}], g(t))\), and \(\psi_a\) is that of \((F[t, t^{-1}], f(t))\) to \((F[t, t^{-1}], h(t))\).

**proof** From the fact that
\[
\Phi_a(D) = \varphi_a D \varphi_a^{-1} \in \mathfrak{m},
\]
\[
\Phi_a(f(t) \frac{d}{dt}) = a^{-1}f(at) \frac{d}{dt} = g(t),
\]
by Proposition (3.1) we have that the Novikov algebra \((F[t, t^{-1}], f(t) \frac{d}{dt}, b)\) is isomorphic to \((F[t, t^{-1}], a^{-1}f(at) \frac{d}{dt}, \varphi_a(b))\).

Note that \((F[t, t^{-1}], a^{-1}f(at) \frac{d}{dt}, \varphi_a(b))\) and \((F[t, t^{-1}], a^{-1}f(at) \frac{d}{dt}, b)\) have the same associated Lie algebra, then as Lie algebra \((F[t, t^{-1}], f(t))\) is isomorphic to \((F[t, t^{-1}], g(t))\). \(\Box\)

The case of \(\psi_a\) is similar.

**Lemma 3.1**
\[
[F[t, t^{-1}], F[t, t^{-1}]] = f(t)F[t, t^{-1}],
\] (3.6)
proof  Note the fact
\[ t^m(t^n)' - (t^m)'t^n = (n - m)t^{m+n-1}, \]
we have
\[
[F[t, t^{-1}], F[t, t^{-1}]] = \{ f(t)(g(t)h'(t) - g'(t)h(t))|g(t), h(t) \in F[t, t^{-1}]\}
\supset \{(n - m)f(t)t^{m+n+1}|m, n \in \mathbb{Z}\}.
\]
Hence the identity (3.6) is true. \qed

As a corollary of lemma above, we have

**Corollary 3.1**
\[
\{g(t)h'(t) - g'h(t)|g(t), h(t) \in F[t, t^{-1}]\} = F[t, t^{-1}]. \tag{3.7}
\]

**Lemma 3.2**

1) \( \varphi_a \) is an automorphism of Lie algebra \((F[t, t^{-1}], f(t))\) if and only if \( a \) is a root of unity of \( k \)-degree and \( f(t) = \sum_{m \in \mathbb{Z}} a_{mk+1}t^{mk+1}; \)

2) \( \psi_a \) is an automorphism of Lie algebra \((F[t, t^{-1}], f(t))\) if and only if the coefficients of \( f(t) = \sum_{m \in \mathbb{Z}} a_t t^i \) satisfy
\[
a_i = -a^{-i+1}a_{i-1}, \quad \forall i \in \mathbb{Z}.
\]

**proof** 1) \( \forall g(t), h(t) \in (F[t, t^{-1}], f(t)) \), we have
\[
[\varphi_a(g(t)), \varphi_a(h(t))] = [g(at), h(at)]
\]  
\[ = f(t)(ag(at)h'(at) - ag'(at)h(at)) \]
\[ = af(t)(g(at)h'(at) - g'(at)h(at)), \]

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and
\[
\varphi_a([g(t), h(t)]) = \varphi_a(f(t)(g(t)h'(t) - g'(t)h(t))) = f(at)(g(at)h'(at) - g'(at)h(at)).
\]

Therefore for any elements \(g(t)\) and \(h(t)\) in \(F[t, t^{-1}]\)
\[
a f(t)(g(at)h'(at) - g'(at)h(at)) = f(at)(g(at)h'(at) - g'(at)h(at)),
\]
and by (3.7),
\[
a f(t) = f(at).
\]

Set \(f(t) = \sum_{i \in \mathbb{Z}} a_i t^i\). Then
\[
aa_i t^i = a_i (at)^i, \quad \forall i \in \mathbb{Z}.
\]

Hence if \(a_i \neq 0\), we have \(a = a_i^i\). By this fact, \(a\) should be a root of unity of \(k\)-degree \((k \in \mathbb{Z})\), and
\[
f(t) = \sum_{m \in \mathbb{Z}} a_{mk+1} t^{mk+1}.
\]

2) \(\forall g(t), h(t) \in (F[t, t^{-1}], f(t))\), we have
\[
[\psi_a(g(t)), \psi_a(h(t))] = f(t)(-at^{-2}g(at^{-1})h'(at^{-1}) - at^{-2}g'(at^{-1})h(at^{-1})) = -at^{-2} f(t)(g(at^{-1})h'(at^{-1}) - g'(at^{-1})h(at^{-1})),
\]
and
\[
\psi_a([g(t), h(t)]) = \psi_a(f(t)g(t)h'(t) - g'(t)h(t)) = f(at^{-1})(g(at^{-1})h'(at^{-1}) - g'at^{-1}h(at^{-1})).
\]
Then for any elements \( g(t) \) and \( h(t) \) in \( F[t, t^{-1}] \), we have

\[
- \alpha t^{-2} f(t)(g(\alpha t^{-1})h'(\alpha t^{-1}) - g'(\alpha t^{-1})h(\alpha t^{-1}))
\]

\[
= f(\alpha t^{-1})(g(\alpha t^{-1})h'(\alpha t^{-1}) - g'(\alpha t^{-1})h(\alpha t^{-1}))
\]

and by (3.7 ),

\[
- \alpha t^{-2} f(t) = f(\alpha t^{-1}).
\]

Set \( f(t) = \sum_{i \in \mathbb{Z}} a_i t^i \). Then

\[
- \alpha a_i t^{-2} = a_{-i} a^2 - i t^{i-2}, \quad \forall i \in \mathbb{Z}.
\]

Thus we have

\[
a_i = -a^{-i+1} a_{-i+2}, \quad \forall i \in \mathbb{Z}.
\]

**Theorem 3.1(a)** Let \( f(t) = \sum_{m \in \mathbb{Z}} a_{2m+1} t^{2m+1} \),

\[
T = \left\{ g(t) \in F[t, t^{-1}] \middle| g(t) = \sum_{m \in \mathbb{Z}} b_{2m+1} t^{2m+1} \right\}.
\]

\( T \) forms a Lie triple system with the product: \( \forall a(t), b(t), c(t) \in T \)

\[
[a(t), b(t), c(t)] = (a(t)b'(t) - a'(t)b(t))c'(t).
\]

**proof** Let \( a = -1 \), and \( f(t) = \sum_{m \in \mathbb{Z}} a_{2m+1} t^{2m+1} \). Then \( \varphi_a = \varphi_{-1} \) is an involutive automorphism of \( (F[t, t^{-1}], f(t)) \), and by Proposition 1.2 \( E_{-1}(\varphi_{-1}) = T \). Therefore, \( T \) forms a Lie triple system with the product

\[
[a(t), b(t), c(t)] = [[a(t), b(t)], c(t)]
\]

\[
= f^2(t)(a(t)b'(t) - a'(t)b(t))c'(t)
\]

\[
- f(t)(f(t)(a(t)b'(t) - a'(t)b(t)))'c(t).
\]

\( \Box \)
The following result can be proved by using the same method as in 2) of Lemma 3.2. So the proof is omitted.

**Theorem 3.1(b)** Let \( f(t) = \sum_{i \in \mathbb{Z}} a_i t^i \) and \( a_i = -a^{-i+1} a_{-i+2}, \forall i \in \mathbb{Z}, \)

\[
T = \left\{ g(t) \in F[t, t^{-1}] \mid g(t) = \sum_{m \in \mathbb{Z}} b_i t^i, \text{and } b_i = a_i b_i, \ i \in \mathbb{Z} \right\}.
\]

\( T \) forms a Lie triple system with the product in (3.9).

**References**


