Automorphism Groups of a Class of
Solvable Complete Lie Algebras *

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Abstract

In this paper we study the automorphism group of solvable complete Lie algebra
whose nilpotent radical is a quasi Heisenberg algebra.

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1 Introduction

A Lie algebra is called complete if its center is zero, and all its derivations are inner.
The definition of complete Lie algebra was given by N. Jacobson in 1962 [2]. Since
then, especially in recent years, many structural properties of complete Lie algebras
were found [3-5]. For example, if a solvable Lie algebra $L$ is complete, then $L = H + N$,
where $H$ is the maximal torus subalgebra of $L$, $N$ is the maximal nilpotent ideal of
$L$, and $\text{ad} H|_N$ is a maximal torus on $N$. On the other hand, for any nilpotent Lie
algebra $N$, if $T$ is a torus on $N$, we can define the bracket in $T + N$ such that $T + N$ is
a solvable Lie algebra. Ref.[5] shows that if $T + N$ is complete, then $T + N$ is unique up
to isomorphism. At this time we also call $N$ a completable nilpotent Lie algebra.

Obviously the study of solvable complete Lie algebras plays an important role
in the study of complete Lie algebras. But there is few result about automorphism
group of solvable complete Lie algebras. Ref.[8] discussed the automorphism group of
a solvable complete Lie algebra whose nilpotent radical is a Heisenberg algebra. Since
Ref.[7] gave the concept of quasi Heisenberg algebra and shows that quasi Heisenberg
algebra is completable. It is a natural idea to determine the automorphism group of
solvable complete Lie algebra whose nilpotent radical is a quasi Heisenberg algebra. In
this paper we finish this work.

Throughout this paper, all Lie algebras discussed are finite dimensional and over
the complex field $\mathbb{C}$.

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2 Main results

A Lie algebra $N$ which satisfies $[N, N] = C(N)$ (the center of $N$) and $\dim C(N) = 1$ is called a Heisenberg algebra. It is easy to know that it has a basis $\{x_1, x_2, \ldots, x_{2n}, z\}$ such that $[x_{2i-1}, x_{2i}] = z$ $(i = 1, 2, \ldots, n)$, the undefined brackets being zero or obtained by antisymmetry.

**Definition 1** [7] A nilpotent Lie algebra $N$ is called a quasi Heisenberg algebra if

$$N = N_1 + N_2 + \ldots + N_k,$$

where $N_i$ is a Heisenberg subalgebra of $N$, and $[N_i, N_j] = N_i \cap N_j = 0$, $i \neq j$. The decomposition (2.1) is called a Heisenberg decomposition of $N$.

If $N$ has a Heisenberg decomposition (2.1), let $\{x_{i1}, x_{i2}, \ldots, x_{i2n_i}\}$ be a minimal system of generators of $N_i$, $z_i = [x_{i(2j-1)}, x_{i2j}] \neq 0$ $(1 \leq j \leq n_i)$. Assume that $\{z_1, z_2, \ldots, z_r\}$ is a basis of $[N, N]$, and $z_i = \sum a_{ij} z_j$. If $N$ is indecomposable (i.e., $N$ cannot be decomposed into direct sum of its ideals), by Lemma 12 in [7], there exists a maximal torus $H_0$ on $N$ such that $\{h_0, h_{11}, h_{12}, \ldots, h_{1n_1}, \ldots, h_{k1}, h_{k2}, \ldots, h_{kn_k}\}$ is a basis of $H_0$, where

$$h_0(x_{ij}) = x_{ij},$$

$$h_{ij}(x_{i2j-1}) = x_{i2j-1}, \quad h_{ij}(x_{i2j}) = -x_{i2j},$$

$$h_{ij}(x_{st}) = 0, \quad \text{for } (s, t) \notin \{(i, 2j-1), (i, 2j)\}.$$

Let $L = H_0 + N$, define the bracket in $L$ by $[h_1 + y_1, h_2 + y_2] = h_1(y_2) - h_2(y_1) + [y_1, y_2]$, where $h_i \in H_0, y_i \in N$, then $L$ is a solvable complete Lie algebra.

**Lemma 1** $H_0$ is a maximal torus subalgebra of $L$, and also a Cartan subalgebra of $L$.

**Lemma 2** [1] Let $\text{Int} L$ be the inner automorphism group of $L$, if $H_1, H_2$ are both maximal torus subalgebras of $L$, then there exists $\sigma \in \text{Int} L$ such that $\sigma(H_1) = H_2$.

We now determine the automorphism group of $L = H_0 + N$.

**Lemma 3** If $\tau \in \text{Aut} L$, then $\tau(N) \subseteq N$.

Let $G_0 = \{\tau \in \text{Aut} L \mid \tau(H_0) \subseteq H_0\}$. By Lemma 3, $G_0$ is a subgroup of $\text{Aut} L$.

Let $E = \{\exp(\text{ad} x) \mid x \in N\}$. Since $N$ is 2-step nilpotent Lie algebra, Proposition of Section 1.11 in [5] shows that for any $x, y \in N$,

$$\exp(\text{ad} x) \exp(\text{ad} y) = \exp(\text{ad}(x + y + \frac{1}{2}[x, y])).$$

This implies that $E$ is closed under multiplication. Note that $E$ is closed under inversion ($\exp(\text{ad} x)^{-1} = \exp(-\text{ad} x)$), so $E$ is a subgroup of $\text{Aut} L$, i.e., we have the following lemma.

**Lemma 4** $\text{Int} L = E = \{\exp(\text{ad} x) \mid x \in N\}$.
Now we have the following theorem.

**Theorem 1** Aut$L$ is the semidirect product of Int$L$ and $G_0$.

**Proof.** Since Int$L$ is a normal subgroup of Aut$L$, what we only have to prove is Aut$L = \text{Int}L \cdot G_0$ and $G_0 \cap \text{Int}L = \{id\}$.

Let $\sigma \in \text{Aut}L$, then $\sigma(H_0)$ is a maximal torus subalgebra of $L$. So there exists $\sigma_1 \in \text{Int}L$, such that $\sigma_1(H_0) = \sigma(H_0)$, hence $\sigma_1^{-1}\sigma(H_0) = H_0$. This means $\sigma^{-1}\sigma \in G_0$.

So $\sigma \in \text{Int}L \cdot G_0$. Then we have Aut$L = \text{Int}L \cdot G_0$.

If $\sigma \in \text{Int}L \cap G_0$, by Lemma 4, $\exists x \in N$ such that $\sigma = \exp(\text{ad}x)$. Set

$$x = \sum_{i=1}^{m} \sum_{j=1}^{2n_i} a_{ij}x_{ij} + \sum_{i=1}^{r} b_i z_i,$$

then for any $h_{ij}$,

$$\exp(\text{ad}x)(h_{ij}) = (1 + \text{ad}x + \frac{1}{2!} (\text{ad}x)^2)(h_{ij})$$

$$= h_{ij} - a_{i,2j-1}x_{i,2j-1} + a_{i,2j}x_{i,2j} + 2a_{i,2j}a_{i,2j-1}z_i.$$  

Note that $\sigma(H_0) \subseteq H_0$, we have $a_{i,2j-1} = a_{i,2j} = 0$, hence $x = \sum_{i=1}^{r} b_i z_i$.

On the other hand, as $x = \sum_{i=1}^{r} b_i z_i$,

$$\exp(\text{ad}x)(h_0) = (1 + \text{ad}x + \frac{1}{2!} (\text{ad}x)^2)(h_0)$$

$$= h_0 - \sum_{i=1}^{r} 2b_i z_i,$$

$b_i = 0$, hence $x = 0$. Then we have

$$\sigma = \exp(\text{ad}x) = \exp(\text{ad}0) = \text{id}.$$  

By Theorem 1, if we can explicitly determine the subgroup $G_0$ of Aut$L$, the structure of Aut$L$ is clear. Next we do this work.

**Lemma 5** Let $\tau \in \text{Aut}L$ such that $\tau(H_0) \subseteq H_0$, then

$$\tau(h_{ij}) = \sum_{s=1}^{k} \sum_{t=1}^{n_s} a_{st}^{ij} h_{st}, \quad \tau(h_0) = h_0, \quad \tau(x_{ij}) = \sum_{s=1}^{k} \sum_{t=1}^{2n_s} d_{st}^{ij} x_{st}.$$  

**Proof.** By Lemma 3, may assume that

$$\tau(h_{ij}) = \sum_{s=1}^{k} \sum_{t=1}^{n_s} a_{st}^{ij} h_{st} + a_{ij} h_0,$$

$$\tau(h_0) = \sum_{s=1}^{k} \sum_{t=1}^{n_s} b_{st} h_{st} + b_0 h_0.$$
Theorem 2

Let \( \tau \) be a linear transformation of \( L \) such that

\[
\tau(h_{ij}) = \sum_{s=1}^{k} \sum_{t=1}^{n_s} a_{ij}^{st} h_{st}, \quad \tau(h_0) = h_0, \quad \tau(x_{ij}) = \sum_{s=1}^{k} \sum_{t=1}^{n_s} d_{ij}^{st} x_{st},
\]

and

\[
\tau(z_i) = [\tau(x_{i1}), \tau(x_{i2})], \quad 1 \leq i \leq r.
\]

Then \( \tau(H_0) \subseteq H_0 \), and we have the following theorem.

**Theorem 2** Let \( \tau \) be as above, \( (z_1, z_2, \ldots, z_k) = (z_1, z_2, \ldots, z_r)R \), then \( \tau \in \text{Aut}L \) if and only if the following conditions hold:

1. For any \( h_{ij} \), \( \exists a_{ij}^{st} \in \pm 1 \) such that

   \[
   \tau(h_{ij}) = a_{ij}^{st} h_{st}, \quad (2.2)
   \]

   \[
   \tau(x_{i2j-1}) = d_{ij}^{j,2j-1} x_{ij,2tj-1} - d_{ij}^{j,2j} x_{ij,2tj}, \quad (2.3)
   \]

   \[
   \tau(x_{i2j}) = d_{ij}^{j,2j} x_{ij,2tj-1} - d_{ij}^{j,2j-1} x_{ij,2tj}, \quad (2.4)
   \]

   and

   \[
   \begin{pmatrix}
   d_{ij}^{j,2j-1} & d_{ij}^{j,2j} \\
   d_{ij}^{j,2j-1} & d_{ij}^{j,2j}
   \end{pmatrix}
   =
   \begin{pmatrix}
   d_{ij}^{j,2j-1} & 0 \\
   0 & d_{ij}^{j,2j}
   \end{pmatrix}, \quad \text{if } a_{ij}^{st} = 1, \quad (2.5)
   \]

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and there exists a permutation matrix $D$ hence (2.3) holds, and then we have
\[
(a^i_j = 2, \quad \text{Similarly, } [2.9] \quad \text{and } (2.10), \quad \exists i, j, \text{ then for any } 1 \leq i, j \leq n_i,
\]
\[
s_{ij} = s_{i1} := s_i, \quad D_i := D_i \neq 0,
\]
and there exists a permutation matrix $T$ such that $(s_1, s_2, \cdots, s_k) = (1, 2, \cdots, k)T$.

(3)
\[
RTD = RTD_1R,
\]
where $D = \text{diag}(D_1, D_2, \cdots, D_k), \quad D_1 = (D_1, D_{11}), \quad D_1$ is a $k \times r$ matrix.

Proof. (⇒) As $\tau \in \text{Aut}L$, for any $h_{ij}, \exists a_{s_{ij}t_{ij}}^{ij} \neq 0$, then for any $x_{uv}$ (if $u = i, v \neq 2j - 1, 2j$), by
\[
0 = \tau[h_{ij}, x_{uv}] = \left[ \sum_{s=1}^{k} \sum_{t=1}^{n_s} a_{s_{ij}}^{ij} h_{st}, \sum_{s=1}^{k} \sum_{t=1}^{2n_s} d_{s_{ij}}^{uv} x_{st} \right],
\]
we have
\[
a_{s_{ij}t_{ij}}^{ij} \left( d_{s_{ij},2t_{ij}}^{ij} x_{s_{ij},2t_{ij}} - d_{s_{ij},2t_{ij}}^{ij} x_{s_{ij},2t_{ij}} \right) = 0,
\]
then
\[
d_{s_{ij},2t_{ij}}^{ij} = d_{s_{ij},2t_{ij}}^{uv} = 0. \tag{2.9}
\]
If $\exists h_{i'j'}, (i', j') \neq (i, j)$ such that $a_{s_{ij}t_{ij}}^{ij} \neq 0$, then we have
\[
d_{s_{ij},2t_{ij}}^{ij} = d_{s_{ij},2t_{ij}}^{ij} = d_{s_{ij},2t_{ij}}^{ij} = d_{s_{ij},2t_{ij}}^{ij} = 0. \tag{2.10}
\]
By (2.9) and (2.10), $\{x_{s_{ij},2t_{ij}} - x_{s_{ij},2t_{ij}}\} \not\subseteq \tau(L).$ This contradicts to $\tau \in \text{Aut}L$.
Hence $a_{s_{ij}t_{ij}}^{ij} = 0$ for any $i', j'$, $(i', j') \neq (i, j)$. This implies that (2.2) holds.
Since $[h_{ij}, x_{i,2j-1}] = x_{i,2j-1}$, we have
\[
a_{s_{ij}t_{ij}}^{ij} \left( d_{s_{ij},2t_{ij}}^{ij} x_{s_{ij},2t_{ij}} - d_{s_{ij},2t_{ij}}^{ij} x_{s_{ij},2t_{ij}} \right) = \sum_{s=1}^{k} \sum_{t=1}^{2n_s} d_{s_{ij}}^{ij} x_{st},
\]
hence (2.3) holds, and
\[
a_{s_{ij}t_{ij}}^{ij} d_{s_{ij},2t_{ij}}^{ij} = d_{s_{ij},2t_{ij}}^{ij} \tag{2.11},
\]
\[
-a_{s_{ij}t_{ij}}^{ij} d_{s_{ij},2t_{ij}}^{ij} = d_{s_{ij},2t_{ij}}^{ij}. \tag{2.12}
\]
Similarly, $[h_{ij}, x_{i,2j}] = -x_{i,2j}$ implies that (2.4) holds, and
\[
a_{s_{ij}t_{ij}}^{ij} d_{s_{ij},2t_{ij}}^{ij} = -d_{s_{ij},2t_{ij}}^{ij} \tag{2.13},
\]
\[
a_{s_{ij}t_{ij}}^{ij} d_{s_{ij},2t_{ij}}^{ij} = d_{s_{ij},2t_{ij}}^{ij}. \tag{2.14}
\]
By (2.11)-(2.14), we have $a_{s_i,t_i}^{ij} = \pm 1$, and (2.5), (2.6) hold. Since $\tau \in \text{Aut} L$, and (2.3), (2.4) hold, for any $z_i$, we have

$$\tau(z_i) = [\tau(x_{i,2j-1}), \tau(x_{i,2j})] = (d_{s_i,2j-1}d_{s_i,2j} - d_{s_i,2j-1}d_{s_i,2j})z_s.$$ 

Since $[x_{i1}, x_{i2}] = [x_{i,2j-1}, x_{i,2j}]$, we have $D_{i1}z_{s1} = D_{ij}z_{sij}$. By the definition of quasi Heisenberg algebra, $N_s \cap N_t = 0$ ($s \neq t$), hence

$$s_{ij} = s_{i1} := s_i, \quad D_{ij} = D_{i1} := D_i.$$ 

As $\tau \in \text{Aut} L$, there exists a permutation matrix $T$ such that

$$\begin{bmatrix} s_1 & s_2 & \cdots & s_k \end{bmatrix} = \begin{bmatrix} 1 & 2 & \cdots & k \end{bmatrix} T.$$ 

Let $D = \text{diag}(D_1 \quad D_2 \quad \cdots \quad D_k)$, we have

$$\tau(z_1, z_2, \cdots, z_k) = (z_{s_1}, z_{s_2}, \cdots, z_{s_k})D = (z_1, z_2, \cdots, z_k)TD = (z_1, z_2, \cdots, z_r)RTD.$$ 

But on the other hand, let $D_I = (D_1 \quad D_{II})$, $D_I$ is a $k \times r$ matrix, then

$$\tau(z_1, z_2, \cdots, z_k) = \tau(z_1, z_2, \cdots, z_r)R = (z_{s_1}, z_{s_2}, \cdots, z_{s_k})D_IR = (z_1, z_2, \cdots, z_k)TD_IR = (z_1, z_2, \cdots, z_r)RTD_IR.$$ 

Hence $RTD = RTD_IR$, i.e., (2.8) holds.

$(\Leftarrow)$: We only prove that $\tau[x, y] = [\tau x, \tau y]$ for any $x, y \in L$. Set

$$x = \sum_{i=1}^{k} \sum_{j=1}^{2n_i} a_{ij}x_{ij} + \sum_{i=1}^{k} \sum_{j=1}^{n_i} b_{ij}h_{ij} + \sum_{i=1}^{r} p_i z_i + qh_0,$$

$$y = \sum_{i=1}^{k} \sum_{j=1}^{2n_i} c_{ij}x_{ij} + \sum_{i=1}^{k} \sum_{j=1}^{n_i} d_{ij}h_{ij} + \sum_{i=1}^{r} s_i z_i + th_0.$$ 

It is easy to know that $\tau[x, y] = [\tau x, \tau y]$ holds if the following equations hold:

$$\tau[x_{ij}, x_{st}] = [\tau(x_{ij}), \tau(x_{st})], \quad \tau[h_{ij}, x_{st}] = [\tau(h_{ij}), \tau(x_{st})],$$

$$\tau[h_0, x_{st}] = [\tau(h_0), \tau(x_{st})], \quad \tau[0, z_i] = [\tau(0), \tau(z_i)].$$ 

We only prove the equation: $\tau[x_{ij}, x_{st}] = [\tau(x_{ij}), \tau(x_{st})]$. The latter three equations can be proved similarly, we omit their proofs.
Therefore we have the following corollary.

Let \( \tau \) be a linear transformation of \( L \) as in Theorem 2, if \( n_i \neq n_j \) (\( i \neq j \)), then \( \forall i, j \), \( s \neq i \), \( s \neq j \), \( s_i = s_j \), i.e., \( n_i = n_s \).

Remark 1  Since \( \tau \in \text{Aut} L \), \( s \neq i \) implies that \( \text{dim} N_i = \text{dim} N_s \), i.e., \( n_i = n_s \).

Remark 2  If \( n_i \neq n_j \) (\( i \neq j \)), then \( s_i = s_j \), i.e., \( T \) is an identity matrix. At this time we also have \( D_1 = D_2 = \cdots = D_k \).

In fact, if \( D_1, D_2, \ldots, D_r \) are not all same, we may assume that \( D_i = D_1 \), \( 1 \leq i \leq p \), \( D_p \neq D_1 \), \( p < s \leq r \). Note that \( z_i = \sum_{j=1}^{r} a_{ij} z_j \) and \( N \) is indecomposable, there exists a \( z \) (\( t > r \)) such that

\[
\tau(z) = a_{ut} z_u + a_{vt} z_v + \cdots,
\]

where \( a_{ut} \neq 0 \), \( 1 \leq u \leq p \), \( a_{vt} \neq 0 \), \( p < v \leq r \). Then we have

\[
D_{lt} z_l = a_{ut} z_u + a_{vt} D_v z_v + \cdots,
\]

so \( D_l = D_u = D_v \), a contradiction. Hence \( D_1 = D_2 = \cdots = D_r \). Our assertion holds. Therefore we have the following corollary.

Corollary  Let \( \tau \) be a linear transformation of \( L \) as in Theorem 2, if \( n_i \neq n_j \) (\( i \neq j \)), then \( \tau \in \text{Aut} L \) if and only if the following conditions hold:

1. as in Theorem 2.
2. For any \( 1 \leq i \leq k \), \( 1 \leq j \leq n_i \), \( s_i = j \), \( D_{ij} = D_{11} \).

If \( N = N_1 \) is a Heisenberg algebra and \( n_1 = 2 \), then it’s easy to find the following fact. Let \( \tau \in G_0 \), if the matrix of \( \tau \) relative to the set \( \{h_{11}, h_{12}, x_{11}, x_{12}, x_{13}, x_{14}\} \) is

\[
\text{diag}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}), \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & c & 0 & 0 \\ \frac{ab}{c} & 0 & 0 & 0 \end{pmatrix}),
\]

then we have

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & c & 0 & 0 \\ \frac{ab}{c} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & \frac{ab}{c} \end{pmatrix},
\]

\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\]
It’s easy to know that the similar decomposition above holds for each element of $G_0$ when $N = N_1 + N_2 + \ldots + N_k$, $k > 1$.

**Remark 3** $G_0$ has a decomposition: $G_0 = G_1G_2G_3$, where

$$G_1 = \{\tau \in G_0 \mid \tau(h_{ij}) = a_{ij}^ih_{ij}, \ a_{ij}^i = 1\},$$

$$G_2 = \{\tau \in G_0 \mid \tau(h_{ij}) = (\pm 1)h_{ij}, \ d_{pq}^{st} = 0, \ or \ 1\},$$

$$G_3 = \{\tau \in G_0 \mid a_{sij}^j = 1, \ d_{sij}^{2j} = d_{sij}^{2j}, \ d_{sij}^{2j} - 1 = 1\}.$$

Obviously $G_1$, $G_2$, $G_3$ are subgroups of $G_0$, and $G_i \cap G_j = \{id\}$, $i \neq j$.

It’s easy to find that $G_1 \cong (C^*)^{(\sum_{i=1}^k n_i + 1)}$, $G_2$ is a finite subgroup, its order is $2^{\sum_{i=1}^k n_i}$.

When $n_1 = n_2 = \ldots = n_k = n$, it’s easy to know that

$$G_1 \cong (C^*)^{(kn+r)}, \ G_2 \cong S_2^{kn}, \ G_3 \cong S_n^k \cdot S_k,$$

where $S_n$ denotes the order $n$ permutation group, $C^*$ denotes the set of nonzero complex number, $A^n$ denotes $n$ times product of $A$ as $A \cdot A \cdot \cdot \cdot A$.

**References**


