Asymptotic stability for the Navier-Stokes equations in the marginal class

Yong Zhou
Department of Mathematics, East China Normal University
Shanghai 200062, CHINA
yzhou@math.ecnu.edu.cn

Abstract: In this paper we consider the Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 3$. We prove the asymptotic stability for weak solutions in the marginal class $u \in L^2(0, \infty; BMO)$ with arbitrary initial and external perturbations.

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1 Introduction

We consider the Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 3$,

$$\begin{cases}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla \bar{p} = \Delta u + \bar{f} & \text{in } x \in \mathbb{R}^n, t > 0, \\
\text{div} u = 0, & \text{in } x \in \mathbb{R}^n, t > 0, \\
u(x, 0) = u_0(x), & \text{in } x \in \mathbb{R}^n,
\end{cases} \quad (1.1)$$

where $u(x, t) \in \mathbb{R}^n$ is the velocity field, $\bar{p}(x, t)$ is a scalar pressure, $\bar{f}$ is the external force, and $u_0(x)$ with $\text{div} u_0 = 0$ in the sense of distribution is the initial velocity field.

The study of the incompressible Navier-Stokes equations has a long history. In the pioneering work [16] and [8], Leray and Hopf proved the existence of its weak solutions $u(x, t) \in L^\infty(0, T; L^2(\mathbb{R}^n)) \cap L^2(0, T; H^1(\mathbb{R}^n))$ for given $u_0(x) \in L^2(\mathbb{R}^n)$. 
But the uniqueness and regularity of the Leray-Hopf weak solutions are still big open problems. In [19], Scheffer began to study the partial regularity theory of the Navier-Stokes equations. Deeper results were obtained by Caffarelli, Kohn and Nirenberg in [4]. Further result can be found in [24] and references there in.

On the other hand, the regularity of a given weak solution $u$ can be shown under additional conditions. In 1962, Serrin [21] proved that if $u$ is a Leray-Hopf weak solution belonging to $L^{\alpha, \gamma}(0, T; L^{\gamma}(\mathbb{R}^n))$ with $\frac{2}{\alpha} + \frac{n}{\gamma} \leq 1$, $3 < \gamma < \infty$, (1.2) then the solution $u(x, t) \in C^\infty(\mathbb{R}^n \times (0, T))$. Later, Beirão da Veiga [1] proposed the regularity criterion on $\nabla u$, which states that if a weak solution $u(x, t)$ satisfies $\nabla u \in L^{\alpha, \gamma}$, with $\frac{2}{\alpha} + \frac{n}{\gamma} \leq 2$, $3/2 < \gamma < \infty$, (1.3) then $u(x, t) \in C^\infty(\mathbb{R}^n \times (0, T))$.

The point is that $\|u_\lambda\|_{L^{\alpha, \gamma}} = \|u\|_{L^{\alpha, \gamma}}$ and $\|\nabla u_\lambda\|_{L^{\alpha, \gamma}} = \|\nabla u\|_{L^{\alpha, \gamma}}$ hold for all $\lambda > 0$ if and only if $2/\alpha + 3/\gamma = 1$ and $2/\alpha + n/\gamma = 2$ respectively, where $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$, $p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)$ and if $(u, p)$ solves the Navier-Stokes equations, then so does $(u_\lambda, p_\lambda)$ for all $\lambda > 0$. Usually we say that the norm $\|u\|_{L^{\alpha, \gamma}}$ and $\|\nabla u\|_{L^{\alpha, \gamma}}$ have the scaling dimension zero for $2/\alpha + n/\gamma = 1$ and $2/\alpha + n/\gamma = 2$ respectively [4].

Later, the regularity criteria for $u$ and $\nabla u$ in the marginal class, i.e., $u \in L^2(0, T; BMO)$ and $\nabla u \in L^1(0, T; BMO)$ respectively, were proved by Kozono and Taniuchi [14].

The Leray-Hopf type weak solution of the Navier-Stokes equations (1.1) is previously known that its energy $\|u(t)\|_{L^2}$ decays as $t \to \infty$ when $\bar{f}$ decays sufficiently fast, which is obtained in [17], [20] and [10]. The question here is the stability of the above weak solution when the initial data and the external force are perturbed. Let $v(x, t)$ be governed by the following perturbed equation

\[
\begin{cases}
\frac{\partial v}{\partial t} + v \cdot \nabla v + \nabla p = \Delta v + \bar{f} + f & \text{in } x \in \mathbb{R}^n, \ t > 0, \\
\text{div} v = 0, & \text{in } x \in \mathbb{R}^n, \ t > 0, \\
v(x, 0) = u_0(x) + a(x), & \text{in } x \in \mathbb{R}^n.
\end{cases}
\] (1.4)

To show the asymptotic stability, we should show that the difference of $u(x, t)$ and $v(x, t)$ goes to zero as time goes to infinity in some Sobolev norm. In what
follows, we assume that $\bar{f} \equiv 0$. However, our main theorem is true (see Remark 3.1 in section 3) for $\bar{f}$ with rough conditions. It is well-known that one open question about decay of solution is: If $\bar{f} = \bar{f}(x)$ is a function of $x$ alone, what can be said about the decay of the weak solution to (1.1)? The expectation is that $\|u - u_s\|_{L^2} \to 0$, where $u_s$ is the corresponding solution to the stationary equation. However, there is no proof for this expectation up to now. In [23] and reference there in, another interesting case is to consider the time periodic force case. Our main theorem (see Theorem 1.2 below) says that although we do not know whether $u$ goes to $u_s$ or $v$ goes to $v_s$ or not, we know that the difference of $u$ and $v$ goes to zero as time goes to infinity.

Before list the main results of this paper, we recall some previous (not all) results. If $f \equiv 0$, and $a \in L^1(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$, $r > 3$, with $\|a\|_{L^r}$ small, Beirão da Veiga and Secchi [2] considered $u(x, t) \in L^{\infty, r+3}$ and showed that there exist a unique solution $v(x, t)$ such that $\|v(t) - u(t)\|_{L^r} = O(t^{-3/4})$ as $t \to \infty$. Ponce et al. [18] treated the solution $u$ with $\nabla u \in L^4(0, \infty; L^2)$ and showed that if

$$
\|a\|_{H^1} + \int_0^\infty \left( \|f(t)\|_{L^2(\Omega)} + \|f(t)\|_{L^2(\Omega)}^2 \right) dt \leq \delta,
$$

then there exists a unique global strong solution $v(t)$ to (1.4) with $\sup_{t>0} \|u(t) - v(t)\|_{H^1(\Omega)} \leq M(\delta)$, where $M(\delta)$ is a constant with $\lim_{\delta \to 0} M(\delta) = 0$. Later, Kawanago [11] treated the class $u \in L^{n+2}(0, \infty; L^{n+2}(\mathbb{R}^n))$ with small perturbation both for the initial datum $a$ and the external force $f$. Kozono [12] showed asymptotic stability for the weak solution $u$ in the class (1.2) without the smallness restriction on the perturbation $a(x)$ and $f(t)$. Recently, the author [25] showed the asymptotic stability for the Navier-Stokes equations with $\nabla u$ in the class (1.3) with arbitrary initial and external perturbations. Therefore, the recent two results in [12] and [25] are significant improvements of the previous results, say [3, 17, 18].

The purpose of this paper is to prove asymptotic stability in the marginal class $u \in L^2(0, \infty; BMO)$.

Before writing down the main theorem, we would like to introduce some function spaces. Let $C_{0,\sigma}^\infty(\mathbb{R}^n)$ denote the set of all $C^\infty$ vector functions $\psi$ with compact support in $\mathbb{R}^n$, such that $\text{div}\psi = 0$. $L^r_\sigma$ is the closure of $C_{0,\sigma}^\infty$ with respect to the $L^r$-norm; $(.,.)$ denotes the duality pairing between $L^r$ and $L^{r'}$, with $1/r + 1/r' = 1$. 

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While $H^1_0(\mathbb{R}^n)$ denotes the closure of $C_0^\infty$ with respect to the standard $H^1$-norm. $BMO$ denotes the bounded mean oscillation functions (see [22]) and $H^1$ denotes the Hardy space in $\mathbb{R}^n$. It is well-known that $(H^1)^* = BMO$ [22].

Next we introduce the following definition for the weak solution to (1.1).

**Definition 1.1** Let $u_0 \in L^2_2(\mathbb{R}^n)$. A measurable function $u(x, t)$ on $\mathbb{R}^n \times (0, \infty)$ is called a weak solution to (1.1) if

(i) $u \in L^\infty(0, T; L^2_2) \cap L^2(0, T; H^1_0, \sigma)$ for all $T > 0$;
(ii) $u$ is weakly continuous from $[0, \infty)$ to $L^2_2(\mathbb{R}^n)$.
(iii)

$$\int_s^t (-u, \partial_t \Phi) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)) d\tau = -(u(t), \Phi(t)) + (u(s), \Phi(s)) + \int_s^t (\bar{f}, \Phi) d\tau$$

for all $0 \leq s \leq t < \infty$ and all $\Phi \in H^1([s, t]; H^1_0 \cap L^n)$.

Definition 1.1 is essentially due to J.-L. Lions [15], and in [17] it is proved that for the solutions defined in $\mathbb{R}^n$, the weak solution in the above sense is equivalent to the Leray-Hopf weak solution (see [6] for the definition).

Our main result reads

**Theorem 1.2** Let $a \in L^2_2$, $f \in L^1(0, \infty; L^2(\mathbb{R}^n))$. Assume that $u(x, t)$ is a weak solution of (1.1) with $u_0 \in L^2_2(\mathbb{R}^n)$ in the class $u \in L^2(0, \infty; BMO)$. Then for every weak solution $v(x, t)$ to (1.4) satisfying the energy inequality

$$\|v(t)\|^2_{L^2} + 2 \int_0^t \|\nabla v(\tau)\|^2_{L^2} d\tau \leq \|u_0 + a\|^2_{L^2} + 2 \int_0^t (f, v) d\tau, \quad 0 \leq t < \infty, \quad (1.5)$$

we have

$$\int_t^{t+1} \|v(\tau) - u(\tau)\|^2_{L^2} d\tau \to 0 \quad as \quad t \to \infty. \quad (1.6)$$

If, in addition, $v(x, t)$ satisfies the strong energy inequality

$$\|v(t)\|^2_{L^2} + 2 \int_s^t \|\nabla v(\tau)\|^2_{L^2} d\tau \leq \|v(s)\|^2_{L^2} + 2 \int_s^t (f, v) d\tau, \quad (1.7)$$

for almost every $s \geq 0$, including $s = 0$, then

$$\|u(t) - v(t)\|_{L^2} \to 0, \quad as \quad t \to \infty. \quad (1.8)$$
Remark 1.1 It should be mentioned that the existence of the weak solution that satisfies the strong energy inequality (1.7) is only known up to \( n \leq 5 \).

Remark 1.2 Up to now, we do not know whether the asymptotic stability of the Navier-Stokes equations for \( \nabla u \) in the marginal class, i.e., \( \nabla u \in L^1(0, \infty; BMO) \), can be proved. We wish we can explore this problem in the near future.

2 Preliminaries

Let us recall the Stokes operator \( A \) in \( L^2_\sigma(\mathbb{R}^n) \). Just as usual, we use \( P \) to denote the orthogonal projection from \( L^2 \) onto \( L^2_\sigma \), while the Stokes operator \( A \) is defined as \( A = -P \Delta \) with domain \( D(A) = \{ u \in H^2(\mathbb{R}^n) \} \cap L^2_\sigma \). Furthermore, we have the following estimate for \( L^p \) in terms of \( A^\alpha \) [6, 13]

\[
\|u\|_{L^p} \leq C\|A^\alpha u\|_{L^2}, \quad \text{with} \quad \frac{1}{p} = \frac{1}{2} - \frac{2\alpha}{3} \quad \text{for} \quad 0 \leq \alpha \leq \frac{1}{2},
\]

(2.1) holds for any \( u \in D(A^\alpha) \). \( A^{-\alpha} \) denotes the inverse operator of \( A^\alpha \).

Now we list two known technique lemmas

Lemma 2.1 [14] Let \( u \in L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; H^1_\sigma) \) and \( v \in L^2(0, T; H^1_\sigma \cap BMO) \). Then there holds

\[
\int_0^T (u \cdot \nabla v, v) d\tau = 0.
\]

Lemma 2.2 [17] Let \( B_0 \) be a dense subset of a Banach space \( B \). Then for any function \( \Psi \in H^1(s, t; B) \), there exists a sequence \( \{ \Psi^{(k)} \}^{\infty}_{k=1} \) of functions having the form

\[
\Psi^{(k)} = \sum_{j \in \text{finite}} \lambda_j^{(k)}(t) \psi_j^{(k)} \quad \text{with} \quad \lambda_j^{(k)} \in C^\infty(\mathbb{R}^1), \quad \psi_j^{(k)} \in B_0,
\]

such that

\[
\Psi^{(k)} \rightarrow \Psi \quad \text{in} \quad H^1(s, t; B) \quad \text{as} \quad k \rightarrow \infty.
\]

3 Proof of the main theorem

First, we follow the argument in [17]. Let \( \rho(t) \in C^\infty_0(-1, 1) \) such that \( \rho(t) \geq 0 \), \( \rho(t) = \rho(-t) \) for all \( t \in (-\infty, \infty) \) with \( \int_{-1}^1 \rho(t) dt = 1 \). We set \( \rho_h = h^{-1} \rho(t/h) \),
$h > 0$, and define

$$u_h(\tau) = \int_0^\tau \rho_h(\tau - \sigma)u(\sigma)d\sigma$$

and

$$v_h(\tau) = \int_0^\tau \rho_h(\tau - \sigma)v(\sigma)d\sigma, \text{ for } 0 \leq \tau \leq t.$$  

Since the interpolation theorem of Janson-Jones [9] implies $L^2 \cap BMO \subset L^n$, we have $u_h \in H^1(0, T; H^{1}_\sigma \cap L^n)$. Moreover, it was proved [7, 17] that $C_{0, \sigma}^\infty$ is dense in $H^{1}_\sigma \cap L^n$. Therefore it follows from Lemma 2.2 that there exists a sequence $\{v_h^{(k)}\}_{k=1}^\infty$ of functions having the form

$$v_h^{(k)} = \sum_{j, \text{finite}} \lambda_j^{(k)}(t)\phi_j^{(k)} \text{ with } \lambda_j^{(k)} \in C^\infty([0, T]), \phi_j^{(k)} \in C_{0, \sigma}^\infty, (3.1)$$

such that

$$v_h^{(k)} \to v_h \text{ in } H^1(0, T; H^1_\sigma \cap L^n) \text{ as } k \to \infty. (3.2)$$

So $v_h^{(k)}$ and $u_h$ belong to $H^1([0, t]; H^1_\sigma)$. By the definition of weak solution, if we let $u_h$ be the test function for $v$ and $v_h^{(k)}$ be the test function for $u$ respectively, it follows that

$$\int_0^t \left( -(u, \partial_\tau v_h^{(k)}) + (\nabla u, \nabla v_h^{(k)}) + (u \cdot \nabla u, v_h^{(k)}) \right) d\tau = -(u(t), v_h^{(k)}(t)) + (u_0, v_h^{(k)}(0)) \quad (3.3)$$

and

$$\int_0^t \left( -(v, \partial_\tau u_h) + (\nabla v, \nabla u_h) + (v \cdot \nabla v, u_h) \right) d\tau = -(v(t), u_h(t)) + (v_0, u_h(0)) + \int_0^t (f, u_h)d\tau. \quad (3.4)$$

Combining (3.3) and (3.4) and letting $k \to \infty$, due to symmetry of $\rho_h$, there holds

$$\int_0^t(-(u, \partial_\tau v_h) - (v, \partial_\tau u_h))d\tau = 0.$$
Hence one has
\[
\lim_{k \to \infty} \int_0^t \left( (\nabla u, \nabla v_h^{(k)}) + (u \cdot \nabla u, v_h^{(k)}) + (\nabla v, \nabla u_h) + (v \cdot \nabla, u_h) \right) d\tau
\]
\[= - (u(t), v_h(t)) + (u_0, v_h(0)) - (v(t), u_h(t)) + (v_0, u_h(0)) + \int_0^t (f, u_h) d\tau
\]
(3.5)

First, it is obvious that
\[
\nabla v_h \to \nabla v \text{ in } L^2(0, t; L^2) \text{ and } u_h \to u \text{ in } L^2(0, t; BMO), \text{ as } h \to 0, \quad (3.6)
\]
by standard approximation theorem, since the regularization is done only with respect to time.

Secondly, due to integration by parts,
\[
\int_0^t (u \cdot \nabla u, v_h^{(k)}) d\tau = - \int_0^t (u \cdot \nabla v_h^{(k)}, u) d\tau
\]
\[\to - \int_0^t (u \cdot \nabla v, u) d\tau \text{ as } k \to \infty, h \to 0. \quad (3.7)
\]

Since \( \text{div} v = 0 \), it follows from [5] that
\[v \cdot \nabla v \in \mathcal{H}^1 \text{ with } \|v \cdot \nabla v\|_{\mathcal{H}^1} \leq C \|v\|_{L^2} \|\nabla v\|_{L^2}.
\]

Hence,
\[
\left| \int_0^t (v \cdot \nabla, u_h - u) d\tau \right| \leq \int_0^t \|v \cdot \nabla v\|_{\mathcal{H}^1} \|u_h - u\|_{BMO} d\tau
\]
\[\leq C \int_0^t \|v\|_{L^2} \|\nabla v\|_{L^2} \|u_h - u\|_{BMO} d\tau
\]
\[\leq C \sup_{0 < \tau < t} \|v(\tau)\|_{L^2} \left( \int_0^t \|\nabla v\|_{L^2}^2 d\tau \right)^{1/2} \left( \int_0^t \|u_h - u\|_{BMO}^2 d\tau \right)^{1/2}
\]
\[\to 0 \text{ as } h \to 0. \quad (3.8)
\]

On the other hand, due to Hölder’s inequality
\[
\left| (u(t), v_h(t)) - \frac{1}{2} (u(t), v(t)) \right| = \left| \int_0^t \rho_h(\sigma)(u(t), v(t - \sigma) - v(t)) d\sigma \right|
\]
\[\leq \frac{1}{2} \sup_{0 < \sigma < h} \|(u(t), v(t - \sigma) - v(t))\|
\]
\[\to 0, \text{ as } h \to 0. \quad (3.9)
\]
Similarly, we have
\[
\begin{align*}
(u_0, v_h(0)) &\rightarrow \frac{1}{2}(u_0, v_0) = \frac{1}{2}(u_0, u_0 + a), \\
(v(t), u_h(t)) &\rightarrow \frac{1}{2}(v(t), u(t)), \\
(v_0, u_h(0)) &\rightarrow \frac{1}{2}(v_0, u_0) = \frac{1}{2}(u_0 + a, u_0),
\end{align*}
\] (3.10)
as $h \to 0$.

Letting $h \to 0$ in (3.5) and taking (3.6)–(3.10) into account, we obtain
\[
\int_0^t (2(\nabla v, \nabla u) - (u \cdot \nabla v, u) + (v \cdot \nabla v, u)) \, d\tau = - (u(t), v(t)) + (u_0, u_0 + a) + \int_0^t (f, u) \, d\tau \] (3.11)

Since $u$ is a weak solution which belongs to $L^2(0, \infty; BMO)$, $u$ actually is regular and $u$ satisfies the energy equality [14]
\[
\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau = \|u_0\|_{L^2}^2, \quad \text{for all } t \geq 0. \] (3.12)

Then from (1.5), (3.11) and (3.12), we get the estimate for $w = v - u$ as
\[
\|w(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla w\|_{L^2}^2 \, d\tau \leq 2 \int_0^t (w \cdot \nabla w, u) \, d\tau + \|a\|_{L^2}^2 + 2 \int_0^t (f, w) \, d\tau, \] (3.13)

where we used Lemma 2.1,
\[
\int_0^t (w \cdot \nabla u, u) \, d\tau = 0.
\]

**Remark 3.1** For general $\bar{f} \neq 0$ under the condition of existence theorem, we can get (3.13) similarly by the above argument.

Now we give a rough estimate for $w$ first, from the inequality (3.13) directly. Since
\[
\begin{align*}
\left| \int_0^t (w \cdot \nabla w, u) \, d\tau \right| &\leq \int_0^t \|w\|_{L^2} \|\nabla w\|_{L^2} \|u\|_{BMO} \, d\tau \\
&\leq \frac{1}{2} \int_0^t \|\nabla w\|_{L^2}^2 \, d\tau + C \int_0^t \|u\|_{BMO}^2 \|w\|_{L^2}^2 \, d\tau,
\end{align*}
\]
then from (3.13),
\[
\begin{align*}
\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w\|_{L^2}^2 \, d\tau &\leq C \int_0^t \|u\|_{BMO}^2 \|w\|_{L^2}^2 \, d\tau + \|a\|_{L^2}^2 + 2 \int_0^t (f, w) \, d\tau \\
&\leq C \int_0^t (\|u\|_{BMO}^2 + \|f\|_{L^2}^2) \|w\|_{L^2}^2 \, d\tau + \|a\|_{L^2}^2 + \int_0^t \|f(\tau)\|_{L^2} \, d\tau.
\end{align*}
\] (3.14)
Thanks to Gronwall’s inequality, an estimate derived from (3.14) is given by

\[
\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w\|_{L^2}^2 \, d\tau \\
\leq \left( \|a\|_{L^2}^2 + \|f\|_{L^{1.2}} \right) \exp \left( C(\|f\|_{L^{1.2}} + \int_0^t \|u(\tau)\|_{BMO}^2 \, d\tau) \right). \tag{3.15}
\]

By the condition, the right-hand-side of (3.15) is bounded, say \(M_0^2\).

In order to prove (1.6), we use the definition of weak solutions for some suitable test function. Since \(w = v - u\), one has

\[
\int_s^t \left(- (w, \partial_\tau \Phi) + (\nabla w, \nabla \Phi) + (w \cdot \nabla u, \Phi) + (w \cdot \nabla w, \Phi)\right) \, d\tau \\
= - (w(t), \Phi(t)) + (w(s), \Phi(s)) + \int_s^t (f, \Phi) \, d\tau, \tag{3.16}
\]

for all \(0 \leq s \leq t < \infty\) and all \(\Phi \in H^1([s, t]; H^1_\sigma)\). Just as in [17] (see also [12]), take the test function \(\Phi = \Phi_{\epsilon, h}\), \(\epsilon, h > 0\), with

\[
\Phi_{\epsilon, h}(\tau) = U_\epsilon(\tau) \int_s^\tau \rho_h(\tau - \sigma) U_\epsilon(\sigma) w(\sigma) \, d\sigma, \quad \text{where } U_\epsilon(\tau) = (1 + A)^{-1/4} e^{-(t+\epsilon-\tau)A},
\]

and \(e^{-tA}\) is the semigroup of the Stokes operator. By the aid of (2.1), the properties of \(\Phi_{\epsilon, h}\) are as follows

\[
\begin{cases}
\sup_{s < \tau < t} \|\Phi_{\epsilon, h}(\tau)\|_{L^n} \leq CM_0, \\
\partial_\tau \Phi_{\epsilon, h} - A \Phi_{\epsilon, h} = U_\epsilon(\tau) \int_s^\tau \partial_\tau \rho_h(\tau - \sigma) U_\epsilon(\sigma) w(\sigma) \, d\sigma, \quad s \leq \tau \leq t.
\end{cases}
\]

Now, we estimate the terms in (3.16) one by one. First

\[
\left| \int_s^t (u \cdot \nabla w, \Phi_{\epsilon, h}) \, d\tau \right| \leq \int_s^t \|u\|_{L^{\frac{2n}{n+2}}} \|\nabla w\|_{L^2} \|\Phi_{\epsilon, h}\|_{L^n} \, d\tau \\
\leq \sup_{s < \tau < t} \|\Phi_{\epsilon, h}\|_{L^n} \left( \frac{1}{2} \left( \int_s^t \|\nabla u\|_{L^2}^2 \, d\tau \right) \right)^{1/2} \left( \int_s^t \|\nabla w\|_{L^2}^2 \, d\tau \right)^{1/2} \\
\leq CM_0^2 \left( \int_s^t \|\nabla u\|_{L^2}^2 \, d\tau \right)^{1/2} \tag{3.17}
\]

Similarly, we can obtain

\[
\left| \int_s^t (u \cdot \nabla w, \Phi_{\epsilon, h}) \, d\tau \right| \leq CM_0^2 \left( \int_s^t \|\nabla u\|_{L^2}^2 \, d\tau \right)^{1/2} \tag{3.18}
\]
and

\[
\left| \int_s^t (w \cdot \nabla w, \Phi_{\epsilon,h}) \, d\tau \right| \leq CM_0 \int_s^t \| \nabla w \|_{L^2}^2 \, d\tau.
\]  

(3.19)

Due to the property of \( \Phi_{\epsilon,h} \) and the symmetry of \( \rho_h \), it follows that

\[
\int_s^t \left(-((w(\tau), \partial_\tau \Phi_{\epsilon,h}(\tau)) + (\nabla w(\tau), \nabla \Phi_{\epsilon,h}(\tau)))\right) \, d\tau \\
= \int_s^t \left( w(\tau), U_\epsilon(\tau) \int_s^\tau \frac{d}{d\tau} \rho_h(\tau - \sigma) U_\epsilon(\sigma) w(\sigma) \, d\sigma \right) \, d\tau \\
= \int_s^t \int_s^\tau \frac{d}{d\tau} \rho_h(\tau - \sigma) (U_\epsilon(\tau) w(\tau), U_\epsilon(\sigma) w(\sigma)) \, d\sigma \, d\tau = 0.
\]  

(3.20)

On the other hand,

\[
(w(t), \Phi_{\epsilon,h}(t)) = \int_s^t \rho_h(t - \sigma)(w(t), (1 + A)^{-1/2} e^{-(t+\epsilon-\sigma)A} w(\sigma)) \, d\sigma \\
\rightarrow \frac{1}{2} \left\| (1 + A)^{-1/4} w(t) \right\|_{L^2}^2, \quad \text{as } \epsilon, h \to 0.
\]  

(3.21)

Similarly,

\[
(w(t), \Phi_{\epsilon,h}(t)) \to \frac{1}{2} \left\| (1 + A)^{-1/4} w(t) \right\|_{L^2}^2, \quad \text{as } \epsilon, h \to 0.
\]  

(3.22)

Letting \( \epsilon, h \to 0 \) in (3.16) and taking (3.17)–(3.22) into account, it follows that

\[
\left\| (1 + A)^{-1/4} w(t) \right\|_{L^2}^2 \leq \left\| (1 + A)^{-1/4} e^{-(t-s)A} w(s) \right\|_{L^2}^2 \\
+ CM_0^2 \left( \int_s^t \| \nabla u \|_{L^2}^2 d\tau \right)^{1/2} + CM_0 \int_s^t \| \nabla w \|_{L^2}^2 d\tau \\
+ 2M_0 \int_s^t \| f \|_{L^2} d\tau,
\]  

(3.23)

with \( C \) independent of \( s, t \). Since

\[
\lim_{t \to \infty} \left\| (1 + A)^{-1/4} e^{-(t-s)A} w(s) \right\|_{L^2}^2 = 0,
\]

letting \( t \to \infty \), (3.23) is reduced to

\[
\lim_{t \to \infty} \left\| (1 + A)^{-1/4} w(t) \right\|_{L^2}^2 \leq CM_0^2 \left( \int_s^\infty \| \nabla u \|_{L^2}^2 d\tau \right)^{1/2} + 2M_0 \int_s^\infty \| f \|_{L^2} d\tau \\
+ CM_0 \int_s^\infty \| \nabla w \|_{L^2}^2 d\tau.
\]  

(3.24)
Note that the left-hand-side of (3.24) is independent of \( s \), then let \( s \to \infty \) in (3.24), due to the integrability of the right-hand-side, it follows that
\[
\limsup_{t \to \infty} \|(1 + A)^{-1/4} w(t)\|_{L^2} = 0. \tag{3.25}
\]
By the interpolation inequality
\[
\|u\|_{L^2}^2 \leq \|(1 + A)^{-1/4} u\|_{L^2}^{4/3} \|(1 + A)^{1/2} u\|_{L^2}^{2/3} \\
\leq \|(1 + A)^{-1/4} u\|_{L^2}^{4/3} \left( \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right)^{1/3},
\]
it follows that
\[
\int_t^{t+1} \|w\|_{L^2}^2 d\tau \leq \int_t^{t+1} \|(1 + A)^{-1/4} w\|_{L^2}^{4/3} \left( \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right)^{1/3} d\tau \\
\leq \left( \int_t^{t+1} \|(1 + A)^{-1/4} w\|_{L^2}^2 \right)^{2/3} \left( \int_t^{t+1} \left( \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right) d\tau \right)^{1/3} \\
\leq M_0^{2/3} \left( \int_t^{t+1} \|(1 + A)^{-1/4} w\|_{L^2}^2 \right)^{2/3} \\
\leq M_0^{2/3} \limsup_{t < \tau < t+1} \|(1 + A)^{-1/4} w(t)\|_{L^2}^2 \\
\to 0, \quad \text{as} \quad t \to \infty, \tag{3.26}
\]
where we used (3.25). (3.26) is what we desired.

If in addition, \( v \) satisfies the strong energy inequality (1.7), by a similar way, we get a similar inequality as (3.14) for \( w \),
\[
\|w(t)\|_{L^2}^2 + \int_s^t \|\nabla w\|_{L^2}^2 d\tau \\
\leq \|w(s)\|_{L^2}^2 + CM_0^2 \int_s^t (\|u\|_{BMO}^2 + \|f\|_{L^2}) d\tau + \int_s^t \|f(\tau)\|_{L^2} d\tau. \tag{3.27}
\]
Integrating (3.27) on \([t/2, t]\) with respect to \( s \), and dividing by \( t/2 \), we obtain
\[
\|w(t)\|_{L^2}^2 \leq \frac{2}{t} \int_{t/2}^t \|w(s)\|_{L^2}^2 ds + \frac{2}{t} \int_{t/2}^t \int_s^t \|f(\tau)\|_{L^2} d\tau ds \\
+ \frac{2}{t} CM_0^2 \int_{t/2}^t \int_{t/2}^t (\|u\|_{BMO}^2 + \|f\|_{L^2}) d\tau ds \\
\leq \frac{2}{t} \int_{t/2}^t \|w(s)\|_{L^2}^2 ds + \frac{2}{t} \int_{t/2}^t \int_{t/2}^t \|f(\tau)\|_{L^2} d\tau ds \\
+ \frac{2}{t} CM_0^2 \int_{t/2}^t \int_{t/2}^t (\|u\|_{BMO}^2 + \|f\|_{L^2}) d\tau ds
\]
\[ \leq \frac{2}{t} \int_{t/2}^{t} \|w(s)\|_{L^2}^2 ds + \int_{t/2}^{t} \|f(\tau)\|_{L^2} d\tau \]

\[ + C M_0^2 \int_{t/2}^{t} (\|u(\tau)\|_{BMO}^2 + \|f(\tau)\|_{L^2}) d\tau. \]  

(3.28)

Since by (3.26),

\[ \lim_{t \to \infty} \frac{2}{t} \int_{t/2}^{t} \|w(s)\|_{L^2}^2 ds \leq \lim_{t \to \infty} \sup_{t/2 \leq \tau \leq t} \int_{\tau}^{\tau+1} \|w(s)\|_{L^2}^2 ds = 0, \]

it follows from (3.28) that

\[ \lim_{t \to \infty} \|w(t)\|_{L^2} = 0. \]

This completes the proof of Theorem 1.2.

4 Final Remark

If we know the decay rate of the external disturbances \( f \), one can show the decay rate of \( \|\nabla v - \nabla u\|_{L^2} \). More precisely, we have

**Theorem 4.1** Under the same assumption as Theorem 1.2, if in addition, \( f \in L^1(0, \infty; L^2) \cap C(0, \infty; L^2) \) with

\[ \|f\|_{L^2} = O(t^{-1}), \text{ as } t \to \infty. \]

Then for every weak solution \( v \) of (1.4) satisfying the strong energy inequality (1.7), there holds

\[ \|\nabla v - \nabla u\|_{L^2} = O(t^{-1/2}), \text{ as } t \to \infty. \]

The proof follows from the estimates done in section 2 and a similar argument as that in section 4 of [12] directly. The detailed proof is omitted just for concise.

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