Twists and Quantizations of Cartan type $S$ Lie algebras

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Abstract. We construct explicit Drinfeld twists for the generalized Cartan type $S$ Lie algebras and obtain the corresponding quantizations. By the modular reduction and base changes, we obtain certain quantizations of the restricted universal enveloping algebra $u(S(n;1))$ in characteristic $p$. They are new Hopf algebras of truncated $p$-polynomial noncommutative and noncocommutative deformation of dimension $p^{1+(n-1)(p^n-1)}$, which contain the well-known Radford algebra $[24]$ as a Hopf subalgebra. As a by-product, we also get some Jordanian quantizations for $sl_n$.

In [15], the authors studied the quantizations both for the generalized-Witt algebra $W$ in characteristic 0 and for the Jacobson-Witt algebra $W(n;1)$ in characteristic $p$. In the present paper, we continue to treat with the same questions both for the generalized Cartan type $S$ Lie algebras in characteristic 0 (for the definition, see [4]) and for the restricted simple special algebras $S(n;1)$ in the modular case (for the definition, see [27], [28]). Our techniques are somewhat different from those in [15] and the method of constructing the twist can be generalized to other Cartan type Lie algebras.

We survey some previous related work. In [6], Drinfel’d raised the question about the existence of a universal quantization for Lie bialgebras. Etingof and Kazhdan gave a positive answer to this question in [8, 9], where the Lie bialgebras they considered including finite- and infinite-dimensional ones are the Lie algebras defined by generalized Cartan matrices. Enriquez and Halbout showed that any coboundary Lie bialgebra, in principle, can be quantized via a certain Etingof-Kazhdan quantization functor [7], and Geer extended Etingof and Kazhdan’s work from Lie bialgebra to the setting of Lie superbialgebras [11]. After the work [8, 9], it is natural to consider the quantizations of Cartan type Lie algebras which are defined by differential operators. In 2004, Grunspan [14] obtained the quantization of the (infinite-dimensional) Witt algebra $W$ in characteristic 0 using the twist found by Giaquinto-Zhang [12], but his way didn’t work for the quantum version of its simple modular Witt algebra $W(1;1)$ in characteristic $p$. The authors in [15]…

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obtained the quantizations of the generalized-Witt algebra $W$ in characteristic 0 and the Jacobson-Witt algebra $W(n; 1)$ in characteristic $p$, they are new families of noncommutative and noncocommutative Hopf algebras of dimension $p^{1+np^n}$ in characteristic $p$. While in the rank 1 case, the work recovered the Grunspan’s work in characteristic 0 and gave the required quantum version in characteristic $p$.

Although, in principle, the possibility to quantize an arbitrary Lie bialgebra has been proved, an explicit formulation of Hopf operations remains nontrivial. In particular, only a few types of twists were known in an explicit expression, see [12, 19, 23, 25] and the references therein. In this research, we start an explicit Drinfel’d twist due to [12, 14], which, we found recently, is essentially a variation of the Jordanian twist which first appeared, using different expression, in Coll-Gerstenhaber-Giaquinto’s paper [2], and recently used extensively by Kulish et al (see [19], etc.). For this fact, we provide a strict proof in Remark 2.5. Using the explicit Drinfel’d twist, we obtain vertical basic twists and horizontal basic twists for the generalized Cartan type $S$ Lie algebras and the corresponding quantizations in characteristic 0. These basic twists can afford many more Drinfel’d twists, see section 2.3. To study the modular case, what we discuss first is about the arithmetic property of quantizations to work out their quantization integral forms. To this end, we have to work over the so-called “positive” part subalgebra $S^+$ of the generalized Cartan type $S$ Lie (shifted) algebra $x^nS$ (where $\eta = -1$). This is one of the crucial technical points here. It is an infinite-dimensional simple Lie algebra defined over a field of characteristic 0, while, over a field of characteristic $p$, it contains a maximal ideal $J$ and the corresponding quotient is exactly the algebra $S'(n; 1)$. Its derived subalgebra $S(n; 1) = S'(n; 1)$(1) is a Cartan type restricted simple modular Lie algebra of type. Secondly, in order to yield the expected finite-dimensional quantizations of the restricted universal enveloping algebra of the special algebra $S(n; 1)$, we need to carry out the modular reduction process: modulo $p$ reduction and modulo “$p$-restrictedness” reduction, among which, we have to take the suitable base changes. These are the other two crucial technical points. Our work gets a new class of noncommutative and noncocommutative Hopf algebras of prime-power dimension in characteristic $p$ [29].

The paper is organized as follows. In Section 1, we recall some definitions and basic facts related to Cartan type $S$ Lie algebra $x^nS$ and Drinfeld twist, and also recall some notations. In Section 2, we construct Drinfeld twist, including vertical basic twists and horizontal basic twists, and show that the twisted structures given by these twists are nonisomorphic. In Section 3, we quantize explicitly Lie bialgebra structures of generalized Cartan type $S$ Lie algebra $x^nS$ by the vertical basic Drinfeld twists, and obtain $n(n-1)$ quantizations with integral forms for $S^+_c$ in characteristic 0. We use this fact to equip the restricted universal enveloping algebra of the special algebra $S(n; 1)$ with noncommutative and noncocommutative Hopf algebra structures by modulo $p$ reduction and “$p$-restrictedness” reduction, together with two steps of base changes. These process leads to new Hopf algebras of dimension $p^{1+(n-1)(p^n-1)}$ with indeterminate $t$ or of dimension $p^{(n-1)(p^n-1)}$ with specializing $t$ into a scalar in $K$ in characteristic $p$. Considering some products of pointwise different basic Drinfel’d twists, we can get new quantizations not only of generalized Cartan type $S$ Lie algebra but also of the restricted universal enveloping algebra of the special algebra $S(n; 1)$. In Section 4, using the horizontal twists, we
get some new quantizations of horizontal type of $\mathfrak{u}(\mathfrak{s}(v; \mathbb{L}))$, which contain some quantizations of the Lie algebra $\mathfrak{s}(n)$ derived by the Jordanian twists (cf. [2], [20]).

1. Preliminaries

1.1. Generalized Cartan type S Lie algebra and its subalgebra $S^+$. We recall the definition of Generalized Cartan type S Lie algebra from [4] and some basics about their structure.

Let $F$ be a field with $\text{char}(F) = 0$ and $n > 0$. Let $Q_n = F[x_{1}^{\pm 1}, \ldots, x_{n}^{\pm 1}]$ be a Laurent polynomial algebra and $\partial_i$ coincides with the degree operator $x_i \frac{\partial}{\partial x_i}$. Set $T = \bigoplus_{i=1}^{n} \mathbb{Z} \partial_i$, and $x^\alpha = x_{1}^{\alpha_1} \cdots x_{n}^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$.

Denote $W = Q_n \otimes F T = \text{Span}_F \{ x^\alpha \partial \mid \alpha \in \mathbb{Z}^n, \partial \in T \}$, where we set $x^\alpha \partial = x^\alpha \otimes \partial$ for short. Then $W = \text{Der}_F(Q_n)$ is a Lie algebra of generalized-Witt type (see [3]) under the following bracket

$$[x^\alpha \partial, x^\beta \partial'] = x^{\alpha+\beta} (\partial(\beta) \partial' - \partial'(\alpha) \partial), \quad \forall \alpha, \beta \in \mathbb{Z}^n; \partial, \partial' \in T,$$

where $\partial(\beta) = (\partial, \beta) = \langle \beta, \partial \rangle = \sum_{i=1}^{n} a_i \beta_i \in \mathbb{Z}$ for $\partial = \sum_{i=1}^{n} a_i \partial_i \in T$ and $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{Z}^n$. The bilinear map $\langle \cdot, \cdot \rangle : T \times \mathbb{Z}^n \rightarrow \mathbb{Z}$ is non-degenerate in the sense that

$$\partial(\alpha) = \langle \partial, \alpha \rangle = 0 \quad (\forall \partial \in T), \quad \Rightarrow \alpha = 0,$$

$$\partial(\alpha) = \langle \partial, \alpha \rangle = 0 \quad (\forall \alpha \in \mathbb{Z}^n), \quad \Rightarrow \partial = 0.$$

$W$ is an infinite dimensional simple Lie algebra over $F$ (see [3]).

We recall that the divergence (cf. [4]) $\text{div} : W \rightarrow Q_n$ is the $F$-linear map such that

$$(1) \quad \text{div}(x^\alpha \partial) = \partial(x^\alpha) = \partial(\alpha)x^\alpha, \quad \text{for } \alpha \in \mathbb{Z}^n, \partial \in T.$$

The divergence has the following two properties:

$$(2) \quad \text{div}([u, v]) = u \cdot \text{div}(v) - v \cdot \text{div}(u),$$

$$(3) \quad \text{div}(fw) = f \text{div}(w) + w \cdot f,$$

for $u, v, w \in W, f \in Q_n$. In view of (2), the subspace

$$\tilde{S} = \ker(\text{div})$$

is a subalgebra of $W$.

The Lie algebra $W$ is $\mathbb{Z}^n$-graded, whose homogeneous components are

$$W_\alpha := x^\alpha T, \quad \alpha \in \mathbb{Z}^n.$$

The divergence $\text{div} : W \rightarrow Q_n$ is a derivation of degree 0. Hence, its kernel is a homogeneous subalgebra of $W$. So we have

$$\tilde{S} = \bigoplus_{\alpha \in \mathbb{Z}^n} \tilde{S}_\alpha, \quad \tilde{S}_\alpha := \tilde{S} \cap W_\alpha.$$

For each $\alpha \in \mathbb{Z}^n$, let $\tilde{\alpha} : T \rightarrow F$ be the corresponding linear function defined by $\tilde{\alpha}(\partial) = \langle \partial, \alpha \rangle = \partial(\alpha)$. We have

$$\tilde{S}_\alpha = x^\alpha T_\alpha, \quad \text{and} \quad T_\alpha = \ker(\tilde{\alpha}).$$

The algebra $\tilde{S}$ is not simple, but its derived subalgebra $S = (\tilde{S})'$ is simple, assuming only that $\dim T \geq 2$. According to Proposition 3.1 [4], we have $S = \bigoplus_{\alpha \neq 0} \tilde{S}_\alpha$. 
More generally, it turns out that the shifted spaces \( x^\alpha S, \eta \in \mathbb{Z}^n \setminus \{0\} \), are simple subalgebras of \( W \) if \( \dim \mathcal{Q} \geq 3 \). We refer to the simple Lie algebras \( x^\alpha S \) as the Lie algebras of \textit{generalized Cartan type} \( S \) (see [4]). The Lie algebra \( x^\alpha S \) is \( \mathbb{Z}^n \)-graded with \( x^\alpha T_{\alpha - \eta} (\alpha \neq \eta) \), as its homogeneous component of degree \( \alpha \), while its homogeneous component of degree \( \eta \) is 0.

Denote \( D_i = \frac{\partial}{\partial x_i} \). Set \( W^+ := \text{Span}_K \{ x^\alpha D_i \mid \alpha \in \mathbb{Z}_+^n, 1 \leq i \leq n \} \), where \( \mathbb{Z}_+ \) is the set of non-negative integers. Then \( W^+ = \text{Der}_K (\mathcal{K}[x_1, \ldots, x_n]) \) is the derivation Lie algebra of polynomial ring \( \mathcal{K}[x_1, \ldots, x_n] \), which, via the identification \( x^\alpha \), with \( x^\alpha \partial_i \) (here \( \alpha - \epsilon_i \in \mathbb{Z}^n \), \( \epsilon_i = (\delta_{i1}, \ldots, \delta_{in}) \)), can be viewed as a Lie subalgebra (the “positive” part) of the generalized-Witt algebra \( W \) over a field \( K \).

For \( X = \sum_{i=1}^n a_i D_i \in W \), we define \( \text{Div}(X) = \sum_{i=1}^n D_i (a_i) \) as usual. Note that \( \text{Div}(X) = \sum_{i=1}^n x_i D_i (x_i^{-1} a_i) \) (since \( \partial_i = x_i D_i \)). Thus we have \( \text{Div}(x_1 \cdots x_n X) = x_1 \cdots x_n \text{Div}(X) \). This means that \( X \in \text{Ker}(\text{Div}) \) if and only if \( x_1 \cdots x_n X \in S \), and if and only if \( X \in x^{-1} S \), where \( 1 = \epsilon_1 + \cdots + \epsilon_n \).

Set \( S^+ := \ker(\text{Div}) \cap W^+ \), then we have \( S^+ = (x^{-1} S) \cap W^+ \) since \( S = \bigoplus_{\alpha \neq 0} S_\alpha \) and \( x^{-1} S_\alpha \cap W^+ = 0 \) (where \( S_0 = T \) ), which is a subalgebra of \( W^+ \). Note that \( \{ x_\alpha x^{-\epsilon_i} D_i - \alpha_i x^{-\epsilon_i} D_\alpha \mid \alpha \in \mathbb{Z}_+^n, 1 \leq i < n \} \) is a basis of \( S^+ \), where \( \alpha x^{-\epsilon_i} D_i - \alpha_i x^{-\epsilon_i} D_\alpha \in x^{-\epsilon_i} T_{\alpha - \epsilon_i} - \epsilon_i + 1 \) indicates once again that \( S^+ \) is indeed a subalgebra of \( x^{-1} S \) since \( \partial_i = x_i D_i \).

1.2. The special algebra \( S(n; 1) \). Assume now that \( \text{char}(K) = p \), then by definition, the Jacobson-Witt algebra \( W(n; 1) \) is a restricted simple Lie algebra over a field \( K \). Its structure of \( p \)-Lie algebra is given by \( D[p] = D^n, \forall D \in W(n; 1) \) with a basis \( \{ x^{(\alpha)} D_j \mid 1 \leq j \leq n, 0 \leq \alpha \leq \tau \} \), where \( \tau = (p-1, \ldots, p-1) \in \mathbb{N}^n \); \( \epsilon_i = (\delta_{i1}, \ldots, \delta_{in}) \) such that \( x^{(\alpha)} = x_\alpha \) and \( D_j (x_\alpha) = \delta_{ij} \); and \( O(n; 1) := \{ x^{(\alpha)} \mid 0 \leq \alpha \leq \tau \} \) is the restricted divided power algebra with \( x^{(\alpha)} x^{(\beta)} = x^{(\alpha + \beta)} \) and a convention: \( x^{(0)} = 0 \) if \( \alpha \) has a component \( \alpha_j < 0 \) or \( \geq p \), where \( x^{(\alpha)} = \prod_{i=1}^n (x_\alpha^{(i)})^{\epsilon_i} \).

Note that \( O(n; 1) \) is \( \mathbb{Z} \)-graded by \( O(n; 1)_\alpha := \text{Span}_K \{ x^{(\alpha)} \mid 0 \leq \alpha \leq \tau, |\alpha| = i \} \), where \( |\alpha| = \sum_{i=1}^n \alpha_j \). Moreover, \( W(n; 1) \) is isomorphic to \( \text{Der}_K (O(n; 1)) \) and inherits a gradation from \( O(n; 1) \) by means of \( W(n; 1)_\alpha = \sum_{\alpha = 1}^n O(n; 1)_{\alpha} + 1 D_j \). Then the subspace

\[ S'(n; 1) = \{ E \in W(n; 1) \mid \text{Div}(E) = 0 \} \]

is a subalgebra of \( W(n; 1) \).

Its derived subalgebra \( S(n; 1) = S'(n; 1)^{(1)} \) is called the \textit{special algebra}. Then \( S(n; 1) = \bigoplus_{\alpha = 1}^n S(n; 1) \cap W(n; 1) \) is graded with \( s = |\tau| - 2 \). Recall the mappings \( D_j : O(n; 1) \longrightarrow W(n; 1) \), \( D_j (f) = D_j (f) D_j - D_j (f) D_j \) \( \forall f \in O(n; 1) \). Note that \( D_\alpha = 0 \) and \( D_\alpha = -D_\alpha \), \( 1 \leq i, j \leq n \). Then by Lemma 4.2.2 [27],

\[ S(n; 1) = \text{Span}_K \{ D_j (f) \mid f \in O(n; 1)_\alpha, 1 \leq i < n \} \]

is a \( p \)-subalgebra of \( W(n; 1) \) with restricted gradation. Evidently, we have the following result (see the proof of Theorem 3.7, p.159 in [28])

**Lemma 1.1.** \( S'(n; 1) = S(n; 1) + \sum_{j=1}^n K x^{(\tau - (p-1)\epsilon_j)} D_j \). And \( \dim_K S'(n; 1) = (p^n - 1) + (n-1)p^n + 1 \).
Let us fix some notation. For any element $\pi$ of length $n$, the number of the first kind.

Since $\dim S(n; 1) = (n-1)(p^n - 1)$, we have $\dim u(S(n; 1)) = p^{(n-1)(p^n - 1)}$.

1.3. Quantization by Drinfel’d twists. The following result is well-known (see [2, 1, 5, 10, 25], etc.).

**Lemma 1.2.** Let $(A, m, \iota, \Delta_0, \varepsilon_0, S_0)$ be a Hopf algebra over a commutative ring. A Drinfel’d twist $F$ on $A$ is an invertible element of $A \otimes A$ such that

$$(F \otimes 1)((\Delta_0 \otimes \text{Id})(F) = (1 \otimes F)(\text{Id} \otimes \Delta_0)(F),$$

$$(\varepsilon_0 \otimes \text{Id})(F) = 1 = (\text{Id} \otimes \varepsilon_0)(F).$$

Then, $w = m(\text{Id} \otimes S_0)(F)$ is invertible in $A$ with $w^{-1} = m(S_0 \otimes \text{Id})(F^{-1})$.

Moreover, if we define $\Delta : A \longrightarrow A \otimes A$ and $S : A \longrightarrow A$ by

$$\Delta(a) = F\Delta_0(a)F^{-1}, ~ S = wS_0(a)w^{-1},$$

then $(A, m, \iota, \Delta, \varepsilon, S)$ is a new Hopf algebra, called the twisting of $A$ by the Drinfel’d twist $F$.

Let $F[[t]]$ be a ring of formal power series over a field $F$ with char$(F) = 0$. Assume that $L$ is a triangular Lie bialgebra over $F$ with a classical Yang-Baxter $r$-matrix $r$ (see [5, 10]). Let $U(L)$ denote the universal enveloping algebra of $L$, with the standard Hopf algebra structure $(U(L), m, \iota, \Delta_0, \varepsilon_0, S_0)$.

Let us consider the topologically free $F[[t]]$-algebra $U(L)[[t]]$ (for the definition, see p. 4, [10]), which can be viewed as an associative $F$-algebra of formal power series with coefficients in $U(L)$. Naturally, $U(L)[[t]]$ equips with an induced Hopf algebra structure arising from that on $U(L)$ (via the coefficient ring extension), by abuse of notation, denoted still by $(U(L)[[t]], m, \iota, \Delta_0, \varepsilon_0, S_0)$.

**Definition 1.3.** ([15]) For a triangular Lie bialgebra $L$ over $F$ with char$(F) = 0$, $U(L)[[t]]$ is called a quantization of $U(L)$ by a Drinfel’d twist $F$ over $U(L)[[t]]$ if $U(L)[[t]]/[U(L)][[t]] \cong U(L)$, and $F$ is determined by its $r$-matrix $r$ (namely, its Lie bialgebra structure).

1.4. Notations. We fix some notation. For any element $x$ of a unital $R$-algebra $(R$ a ring) and $a \in R$, we set (see [12])

$$(4) \quad x_0^{(n)} := (x + a)(x + a + 1) \cdots (x + a + n - 1),$$

then $x^{(n)} := x_0^{(n)} = \sum_{k=0}^{n} c(n, k)x^k$ where $c(n, k)$ is the number of $\pi \in S_n$ with exactly $k$ cycles (cf. [26]). Given a $\pi \in S_n$, let $c_i = c_i(\pi)$ be the number of cycles of $\pi$ of length $i$. Note that $n = \sum ic_i$. Define the type of $\pi$, denoted type $\pi$, to be the $n$-tuple $\underline{c} = (c_1, \ldots, c_n)$. The total number of cycles of $\pi$ is denoted $c(\pi)$, so $c(\pi) = |\underline{c}| = c_1 + \cdots + c_n$. Denote by $S_n(\underline{c})$ the set of all $\sigma \in S_n$ of type $\underline{c}$, then $|S_n(\underline{c})| = n!/1!^{c_1}2!^{c_2}3!^{c_3} \cdots n!^{c_n}$! (see Proposition 1.3.2 [26]).

We also set

$$(5) \quad x_0^{[n]} := (x + a)(x + a - 1) \cdots (x + a - n + 1),$$

then $x^{[n]} := x_0^{[n]} = \sum_{k=0}^{n} s(n, k)x^k$ where $s(n, k) = (-1)^{n-k}c(n, k)$ is the Stirling number of the first kind.
2. Drinfel’\’d twists in $U(x^n S)[[t]]$

2.1. Construction of Drinfel’\’d twists. Let $L$ be a Lie algebra containing linearly independent elements $h$ and $e$ satisfying $[h, e] = e$, then the classical Yang-Baxter $r$-matrix $r = h \otimes e - e \otimes h$ equips $L$ with the structure of triangular coboundary Lie bialgebra (see \cite{21}). To describe a quantization of $U(L)$ by a Drinfel’\’d twist $\mathcal{F}$ over $U(L)[[t]]$, we need an explicit construction for such a Drinfel’\’d twist. In what follows, we shall see that such a twist depends on the choice of two distinguished elements $h$, $e$, arising from its $r$-matrix $r$.

Recall the following results proved in \cite{14} and \cite{15}. Note that $h$ and $e$ satisfy the following equalities:

$$e^s \cdot h_a^{[m]} = h_a^{[m]} \cdot e^s,$$

$$e^s \cdot h_a^{[m]} = h_a^{(m)} \cdot e^s,$$

where $m$, $s$ are non-negative integers, $a \in F$.

For $a \in F$, we set $\mathcal{F}_a = \sum_{r=0}^{\infty} (-1)^r h_a^{[r]} \otimes e^r t^r$, $F_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{[r]} \otimes e^r t^r$, $u_a = m \cdot (S_0 \otimes \text{Id})(F_a)$, $v_a = m \cdot (\text{Id} \otimes S_0)(F_a)$. Write $\mathcal{F} = \mathcal{F}_0$, $F = F_0$, $u = u_0$, $v = v_0$. Since $S_0(h_a^{[r]}) = (-1)^r h_a^{[r]}$ and $S_0(e^r) = (-1)^r e^r$, one has $v_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{[r]} e^r t^r$, $u_b = \sum_{r=0}^{\infty} \frac{(-1)^r e^r}{r!} h_{-b}^{[r]} e^r t^r$.

**Lemma 2.1.** (\cite{14}) For $a, b \in F$, one has $\mathcal{F}_a F_b = 1 \otimes (1 - et)^{a-b}$ and $v_a u_b = (1 - et)^{-(a+b)}$.

**Corollary 2.2.** For $a \in F$, $\mathcal{F}_a$ and $u_a$ are invertible with $\mathcal{F}_a^{-1} = F_a$ and $u_a^{-1} = v_a$. In particular, $\mathcal{F}^{-1} = F$ and $u^{-1} = v$.

**Lemma 2.3.** (\cite{15}) For any positive integers $r$, we have

$$\Delta_0 (h_a^{[r]}) = \sum_{i=0}^{r} \binom{r}{i} h_a^{[i]} \otimes h_a^{[r-i]}.$$

Furthermore, $\Delta_0 (h_a^{[r]}) = \sum_{i=0}^{r} \binom{r}{i} h_a^{[i]} \otimes h_a^{[r-i]}$ for any $s \in F$.

**Proposition 2.4.** (\cite{14, 15}) If a Lie algebra $L$ contains a $2$-dimensional solvable Lie subalgebra with a basis $\{h, e\}$ satisfying $[h, e] = e$, then $\mathcal{F} = \sum_{r=0}^{\infty} (-1)^r h_a^{[r]} \otimes e^r t^r$ is a Drinfel’\’d twist on $U(L)[[t]]$.

**Remark 2.5.** Recently, we observed that Kulish et al extensively used (see \cite{19}) the so-called Jordanian twist, which first appeared in \cite{2} in a different way, with the two-dimensional carrier subalgebra $B(2)$ such that $[H, E] = E$, defined by the canonical twisting element

$$\mathcal{F}_J^t = \exp(H \otimes \sigma(t)), \quad \sigma(t) = \ln(1 + Et),$$

where $\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$ and $\ln(1 + X) = \sum_{n=1}^{\infty} \frac{(-1)^n+1}{n} X^n$. 
Expanding it, we get
\[
\exp(H \otimes \sigma(t)) = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} H \otimes (Et)^n \right) \\
= \prod_{n \geq 1} \left( \sum_{\ell \geq 0} \frac{(-1)^{\ell}}{\ell!} H^\ell \otimes (Et)^{n\ell} \right) \\
= \sum_{n \geq 0} \sum_{c_1, \ldots, c_n \geq 0} \frac{(-1)^{c_1 + 2c_2 + \cdots + nc_n - \frac{n}{2}}}{c_1! \cdots c_n!} \sum_{k=1}^n c_1 2c_2 \cdots n c_n H^{|c_1| + 2c_2 + \cdots + nc_n} \otimes (Et)^{c_1 + 2c_2 + \cdots + nc_n} \\
= \sum_{n \geq 0} \sum_{\ell = 0}^n \frac{(-1)^n c(n, k)}{n!} H^k \otimes (Et)^n \\
= \sum_{n = 0}^{\infty} \frac{1}{n!} H^{[n]} \otimes E^n t^n,
\]
where we set \( n = c_1 + 2c_2 + \cdots + nc_n, c(n, k) = \sum_{|c| = k} |\mathcal{S}_n(c)|. \) So

\[(F_\eta^-)^{-1} = \exp((-H) \otimes \sigma(t)) = \sum_{r=0}^{\infty} \frac{1}{r!} (-H)^{[r]} \otimes E^r t^r = \sum_{r=0}^{\infty} \frac{(-1)^r H^{[r]} \otimes E^r t^r}.\]

Consequently, we can rewrite the twist \( \mathcal{F} \) in Proposition 2.4 as

\[\mathcal{F} = \sum_{r=0}^{\infty} \frac{(-1)^r H^{[r]} \otimes E^r t^r} = \exp(H \otimes \sigma'(t)), \quad \sigma'(t) = \ln(1 - Et),\]

where \([H, -E] = -E. So there is no difference between the twists \( \mathcal{F} \) and \( F_\eta^- \). They are essentially the same up to an isomorphism on the carrier subalgebra \( \tilde{B}(2) \).

### 2.2. Basic Drinfel’d twists.

We assume that \( \eta \neq 0, \eta_k = \eta_{k'} \). Take two distinguished elements \( h = \partial_k - \partial_{k'}, e = x^\gamma \partial_0 \in x^a \mathfrak{S} \) such that \([h, e] = e\) where \( 1 \leq k \neq k' \leq n \). It is easy to see that \( \partial_0(\gamma - \eta) = 0, \) and \( \gamma_k - \gamma_{k'} = 1. \) Using a result of [21], we have the following

**Proposition 2.6.** There is a triangular Lie bialgebra structure on \( x^a \mathfrak{S} \) \( (\eta \neq 0, \eta_k = \eta_{k'}) \) given by the classical Yang-Baxter \( r \)-matrix

\[ r := (\partial_k - \partial_{k'}) \otimes x^\gamma \partial_0 + x^\gamma \partial_0 \otimes (\partial_k - \partial_{k'}), \quad \forall \partial_k, \partial_{k'} \in T, \gamma \in \mathbb{Z}^n, \]

where \( \gamma_k - \gamma_{k'} = 1, \partial_0(\gamma) = \partial_0(\eta) \) and \([\partial_k - \partial_{k'}, x^\gamma \partial_0] = x^\gamma \partial_0. \]

Fix two distinguished elements \( h := \partial_k - \partial_{k'}, e := x^\gamma \partial_0 \in x^a T_{\gamma - \eta} \) with \( \gamma_k - \gamma_{k'} = 1 \) for \( x^a \mathfrak{S} \), then \( \mathcal{F} = \sum_{r=0}^{\infty} \frac{(-1)^r h^{[r]} \otimes e^r t^r} = \text{a Drinfel’d twist on } U(x^a \mathfrak{S})[[t]]. \) But the coefficients of the quantizations of standard Hopf structure \( (U(x^a \mathfrak{S})[[t]], m, \iota, \Delta_0, S_0, \varepsilon_0) \) by \( \mathcal{F} \) may be not integral. In order to get integral forms, it suffices to consider what conditions are needed for those coefficients to be integers.

**Lemma 2.7.** ([14]) For any \( a, k, \ell \in \mathbb{Z}, \) \( a^\ell \prod_{j=0}^{\ell-1} (k+ja)/\ell! \) is an integer. \( \square \)
From this Lemma, we are interested in the following two cases:

(i) \( h = \partial_{q_n} - \partial_{q'_n}, \quad e = x^e (\partial_{q_n} - 2\partial_{q'_n}), \quad (1 \leq k \neq k' \leq n); \)

(ii) \( h = \partial_{q_{n_1}} - \partial_{q'_{n_1}}, \quad e = x^{e_{n_1}} - \epsilon_{n_1} \partial_{q_{n_1}}, \quad (1 \leq k \neq k' \neq n_1). \)

Let \( \mathcal{F}(k, k') \) be the corresponding Drinfel’d twist in case (i) and \( \mathcal{F}(k, k'; m) \) the corresponding Drinfel’d twist in case (ii).

**Definition 2.8.** \( \mathcal{F}(k, k')(1 \leq k \neq k' \leq n) \) are called vertical basic Drinfel’d twists; \( \mathcal{F}(k, k'; m)(1 \leq k \neq k' \neq m \leq n) \) are called horizontal basic Drinfel’d twists.

**Remark 2.9.** In case (i): we get \( n(n-1) \) vertical basic Drinfel’d twists \( \mathcal{F}(1, 2), \cdots, \mathcal{F}(1, n), \mathcal{F}(2, 1), \cdots, \mathcal{F}(n, n-1) \) over \( U(S^+_{Z})[[t]] \). It is interesting to consider the products of some basic Drinfel’d twists, one can get many more new Drinfel’d twists which will lead to many more complicated quantizations not only over the \( U(S^+_{Z})[[t]] \), but over the \( u(S(n; 1)) \) as well, via our modulo reduction approach developed in the next section.

In case (ii): according to the parametrization of twists \( \mathcal{F}(k, k'; m) \), we get \( n(n-1)(n-2) \) horizontal basic Drinfel’d twists over \( U(S^+_{Z})[[t]] \) and consider the products of some basic Drinfel’d twists. We will discuss these twists and corresponding quantizations in Section 4.

### 2.3. More Drinfel’d twists.

We consider the products of pairwise different and mutually commutative basic Drinfel’d twists and can get many more new complicated quantizations not only over the \( U(S^+_{Z})[[t]] \), but over the \( u(S(n; 1)) \) as well. Note that \( [\mathcal{F}(i, j), \mathcal{F}(k, m)] = 0 \) for \( i \neq k, m \) and \( j \neq k \). This fact, according to the definition of \( \mathcal{F}(k, m) \), implies the commutative relations in the case when \( i \neq k, m \) and \( j \neq k \):

\[
(F(k, m) \otimes 1)(\Delta_0 \otimes \text{Id})(F(i, j)) = (\Delta_0 \otimes \text{Id})(F(i, j))(F(k, m) \otimes 1),
\]

\[
(1 \otimes F(k, m))(\text{Id} \otimes \Delta_0)(F(i, j)) = (\text{Id} \otimes \Delta_0)(F(i, j))(1 \otimes F(k, m)),
\]

which give rise to the following property.

**Theorem 2.10.** \( \mathcal{F}(i, j)\mathcal{F}(k, m)(i \neq k, m; j \neq k) \) is still a Drinfel’d twist on \( U(S^+_{Z})[[t]] \).

**Proof.** Note that \( \Delta_0 \otimes \text{id}, \text{id} \otimes \Delta_0, e_0 \otimes \text{id} \) and \( \text{id} \otimes e_0 \) are algebraic homomorphisms. According to Lemma 1.2, it suffices to check that

\[
(F(i, j)F(k, m) \otimes 1)(\Delta_0 \otimes \text{Id})(F(i, j)F(k, m))
\]

\[
= (1 \otimes F(i, j)F(k, m))(\text{Id} \otimes \Delta_0)(F(i, j)F(k, m)).
\]

Using (8), we have

\[
\text{LHS} = (F(i, j) \otimes 1)(F(k, m) \otimes 1)(\Delta_0 \otimes \text{Id})(F(i, j))(\Delta_0 \otimes \text{Id})(F(k, m))
\]

\[
= (F(i, j) \otimes 1)(\Delta_0 \otimes \text{Id})(F(i, j))(F(k, m) \otimes 1)(\Delta_0 \otimes \text{Id})(F(k, m))
\]

\[
= (1 \otimes F(i, j))(\text{Id} \otimes \Delta_0)(F(i, j))(1 \otimes F(k, m))(\text{Id} \otimes \Delta_0)(F(k, m))
\]

\[
= (1 \otimes F(i, j))(1 \otimes F(k, m))(\text{Id} \otimes \Delta_0)(F(i, j))(\text{Id} \otimes \Delta_0)(F(k, m)) = \text{RHS}.
\]

This completes the proof.

More generally, we have the following

**Corollary 2.11.** Let \( F(i_1, j_1), \cdots, F(i_m, j_m) \) be \( m \) pairwise different basic Drinfel’d twists and \( [F(i_k, j_k), F(i_s, j_s)] = 0 \) for all \( 1 \leq k \neq s \leq m \). Then \( F(i_1, j_1) \cdots F(i_m, j_m) \) is still a Drinfel’d twist.
We denote $\mathcal{F}_m = \mathcal{F}(i_1, j_1) \cdots \mathcal{F}(i_m, j_m)$ and its length as $m$. We shall show that the twisted structures given by Drinfel’d twists with different product-length are nonisomorphic.

**Definition 2.12.** A Drinfel’d twist $\mathcal{F} \in A \otimes A$ on any Hopf algebra $A$ is called compatible if $\mathcal{F}$ commutes with the coproduct $\Delta_0$.

In other words, twisting a Hopf algebra $A$ with a compatible twist $\mathcal{F}$ gives exactly the same Hopf structure, that is, $\Delta_{\mathcal{F}} = \Delta_0$. The set of compatible twists on $A$ thus forms a group.

**Lemma 2.13.** ([13]) Let $\mathcal{F} \in A \otimes A$ be a Drinfel’d twist on a Hopf algebra $A$. Then the twisted structure induced by $\mathcal{F}$ coincides with the structure on $A$ if and only if $\mathcal{F}$ is a compatible twist.

Using the same proof as in Theorem 2.10, we obtain

**Lemma 2.14.** Let $\mathcal{F}, \mathcal{G} \in A \otimes A$ be Drinfel’d twists on a Hopf algebra $A$ with $\mathcal{F} \mathcal{G} = \mathcal{G} \mathcal{F}$ and $\mathcal{F} \neq \mathcal{G}$. Then $\mathcal{F}(\mathcal{G})$ is a Drinfel’d twist. Furthermore, $\mathcal{G}$ is a Drinfel’d twist on $A_{\mathcal{F}}$, $\mathcal{F}$ is a Drinfel’d twist on $A_{\mathcal{G}}$ and $\Delta_{\mathcal{F} \mathcal{G}} = (\Delta_{\mathcal{F}})_{\mathcal{G}} = (\Delta_{\mathcal{G}})_{\mathcal{F}}$.

**Proposition 2.15.** Drinfel’d twists $\mathcal{F}^{(i)} := \mathcal{F}(2, 1) \mathcal{F}(1) \mathcal{F}(n, 1)^{\zeta_{n-1}}$ (where $\zeta(i) = (\zeta_1, \cdots, \zeta_{n-1}) = (1, \cdots, 1, 0, \cdots, 0) \in \mathbb{Z}_2^{n-1}$) lead to $n-1$ different twisted Hopf algebra structures on $U(S^\pm_n)[[t]]$.

**Proof.** For $i = 1$, $\mathcal{F}(2, 1)$ gives one twisted structure with a twisted coproduct different from the original one. For $i = 2$, using Lemma 2.14, we know that $\mathcal{F}(3, 1)$ is a Drinfel’d twist and not a compatible twist on $U(S^\pm_n)[[t]][\mathcal{F}(2, 1)]$. So the twist $\mathcal{F}(2, 1)\mathcal{F}(3, 1)$ gives new Hopf algebra structure with the coproduct different from the previous one twisted by $\mathcal{F}(2, 1)$. Using the same discussion, we obtain that the Drinfel’d twists $\mathcal{F}^{(i)}$ for $\zeta(i) = (1, \cdots, 1, 0, \cdots, 0) \in \mathbb{Z}_2^{n-1}$ give $n-1$ different twisted structures on $U(S^\pm_n)[[t]]$. \qed

### 3. Quantizations of vertical type for Lie bialgebra of Cartan type $S$

In this section, we explicitly quantize the Lie bialgebras of $x^n S$ by the vertical basic Drinfel’d twists, and obtain certain quantizations of the restricted universal enveloping algebra $u(S(n; 1))$ by the modular reduction and base changes.

#### 3.1. Quantization integral forms of $S^+_n$ in characteristic 0.

For the universal enveloping algebra $U(x^n S)$ for the Lie algebra $x^n S$ over $F$, denote by $(U(x^n S), m, \mathcal{G}, S_0, \varepsilon_0)$ the standard Hopf algebra structure. We can perform the process of twisting the standard Hopf structure by the vertical Drinfel’d twist $\mathcal{F}$. We need to give some commutative relations, which are important to the quantizations of Lie bialgebra structure of $x^n S$ in the sequel.

**Lemma 3.1.** Fix two distinguished elements $h := \partial_k - \partial_k$, $e := x^\gamma \partial_0 \in x^\gamma T_{\gamma - \eta}$ with $\gamma_k - \gamma_k = 1$ for $x^n S$. For $a \in F$, $x^\partial \partial \in x^\partial T_{\partial - \eta}$, $x^\partial \partial \in x^\partial T_{\partial - \eta}$, $m$ is
non-negative integer, the following equalities hold in $U(x^nS)$:
\begin{align}
(9) & \quad x^\alpha \partial \cdot h^m_u = h^m_{u+(\alpha_k - \alpha_k)} \cdot x^\alpha \partial, \\
(10) & \quad x^\alpha \partial \cdot h^m_{a} = h^m_{a+(\alpha_k - \alpha_k)} \cdot x^\alpha \partial, \\
(11) & \quad x^\alpha \partial \cdot (x^\beta \partial')^m = \sum_{\ell=0}^{m} (-1)^{\ell} \left( \frac{m}{\ell} \right) (x^\beta \partial')^{m-\ell} \cdot x^{\alpha + \ell \beta} (a_\ell \partial - b_\ell \partial'),
\end{align}

where $a_\ell = \prod_{j=0}^{\ell-1} \partial' (\alpha + j \beta) = \prod_{j=0}^{\ell-1} \partial' (\alpha + j \eta)$, $b_\ell = \ell \partial (\beta) a_{\ell-1}$, and set $a_0 = 1$, $b_0 = 0$.

**Proof.** See [16]. \qed

To simplify formulas in the sequel, we introduce the operator $d^{(\ell)} (\ell \geq 0)$ on $U(x^nS)$ defined by $d^{(\ell)} := \frac{1}{\ell} (\text{ad} e)^\ell$. It is easy to get

**Lemma 3.2.** For $Z^n$-homogeneous elements $x^\alpha \partial$, $a_i$, the following equalities hold in $U(x^nS)$:
\begin{align}
(12) & \quad d^{(\ell)} (x^\alpha \partial) = x^{\alpha + \ell \gamma} (A_\ell \partial - B_\ell \partial_0), \\
(13) & \quad d^{(\ell)} (a_1 \cdots a_s) = \sum_{\ell_1 + \cdots + \ell_s = \ell} d^{(\ell_1)} (a_1) \cdots d^{(\ell_s)} (a_s),
\end{align}

where $A_\ell = \frac{\ell-1}{\pi} \prod_{j=0}^{\ell-1} \partial_0 (\alpha + j \gamma) = \frac{\ell-1}{\pi} \prod_{j=0}^{\ell-1} \partial_0 (\alpha + j \eta)$, $B_\ell = \partial (\gamma) A_{\ell-1}$, and set $A_0 = 1$, $A_{-1} = 0$.

The following Lemmas is very useful to our main result and the proof can be found in [16].

**Lemma 3.3.** For $a \in F$, $\alpha \in Z^n$, and $x^\alpha \partial \in x^nT_{n-\eta}$, one has
\begin{align}
(14) & \quad ((x^\alpha \partial)^s \otimes 1) \cdot F_a = F_a + s(\alpha_k - \alpha_k) \cdot ((x^\alpha \partial)^s \otimes 1), \\
(15) & \quad (x^\alpha \partial)^s \cdot u_a = u_a + s(\alpha_k - \alpha_k) \cdot \left( \sum_{\ell=0}^{\infty} d^{(\ell)} (x^\alpha \partial)^s \cdot h^{(\ell)}_{1-a} \cdot t^\ell \right), \\
(16) & \quad (1 \otimes (x^\alpha \partial)^s) \cdot F_a = \sum_{\ell=0}^{\infty} (-1)^\ell F_a + \ell \cdot \left( h^{(\ell)}_{a} \otimes d^{(\ell)} ((x^\alpha \partial)^s) \cdot t^\ell \right).
\end{align}

**Lemma 3.4.** For $s \geq 1$, one has
\begin{align}
(i) & \quad \Delta ((x^\alpha \partial)^s) = \sum_{\ell \geq 0} \sum_{0 \leq j \leq s} \binom{s}{j} (-1)^{\ell} (x^\alpha \partial)^j h^{(\ell)} \otimes (1-\text{et})^{s(\alpha_k - \alpha_k) - \ell} d^{(\ell)} ((x^\alpha \partial)^{s-j}) \cdot t^\ell, \\
(ii) & \quad S((x^\alpha \partial)^s) = (-1)^s (1-\text{et})^{-s(\alpha_k - \alpha_k)} \cdot \left( \sum_{\ell=0}^{\infty} d^{(\ell)} ((x^\alpha \partial)^s) \cdot h^{(\ell)}_{1-a} \cdot t^\ell \right).
\end{align}

The following theorem gives the quantization of $U(x^nS)$ by Drinfel’d twist $F(k, k')$, which is essentially determined by the Lie bialgebra triangular structure on $x^nS$. 
Theorem 3.5. Fix two distinguished elements \( h = \partial_k - \partial_{k'} \), \( e = x^i \partial_{\xi} \), where \( \gamma \) satisfies \( \gamma_k - \gamma_{k'} = 1 \) such that \([h, e] = e\) in the generalized Cartan type \( S \) Lie algebra \( x^S \) over \( F \), there exists a structure of noncommutative and noncommutative Hopf algebra \( (U(x^S)[[t]], m, \iota, \Delta, S, \varepsilon) \) on \( U(x^S)[[t]] \) over \( F[[t]] \) with \( U(x^S)[[t]]/U(x^S)[[t]] \cong U(x^S) \), which leaves the product of \( U(x^S)[[t]] \) undeformed but with the deformed coproduct, antipode and counit defined by

\[
\Delta(x^\alpha \partial) = x^\alpha \partial \otimes (1 - et)^{\alpha_k - \alpha_{k'}} + \sum_{\ell=0}^{\infty} (-1)^\ell t^{\ell} (1 - et)^{-\ell} \cdot d^{(\ell)}(x^\alpha \partial) t^\ell,
\]

\[
S(x^\alpha \partial) = -(1 - et)^{-(\alpha_k - \alpha_{k'})} \cdot \left( \sum_{\ell=0}^{\infty} d^{(\ell)}(x^\alpha \partial) \cdot h_1^{(\ell)} t^\ell \right),
\]

\[
\varepsilon(x^\alpha \partial) = 0,
\]

where \( x^\alpha \partial \in x^\alpha T_{\alpha - \eta} \).

Proof. By Lemmas 1.2 and 2.1, it follows from (14) and (16) that

\[
\Delta(x^\alpha \partial) = F \cdot \Delta_0(x^\alpha \partial) \cdot F^{-1}
\]

\[
= F \cdot (x^\alpha \partial \otimes 1) \cdot F + F \cdot (1 \otimes x^\alpha \partial) \cdot F
\]

\[
= \left( F \Delta_{\alpha_k - \alpha_{k'}} \right) \cdot (x^\alpha \partial \otimes 1) + \sum_{\ell=0}^{\infty} (-1)^\ell F \left( \sum_{\ell=0}^{\infty} d^{(\ell)}(x^\alpha \partial) t^\ell \right)
\]

\[
= \left( 1 \otimes (1 - et)^{\alpha_k - \alpha_{k'}} \right) \cdot (x^\alpha \partial \otimes 1)
\]

\[
+ \sum_{\ell=0}^{\infty} (-1)^\ell \left( 1 \otimes (1 - et)^{-\ell} \right) \cdot \left( h_1^{(\ell)} \cdot d^{(\ell)}(x^\alpha \partial) t^\ell \right)
\]

\[
= x^\alpha \partial \otimes (1 - et)^{\alpha_k - \alpha_{k'}} + \sum_{\ell=0}^{\infty} (-1)^\ell h_1^{(\ell)} \otimes (1 - et)^{-\ell} \cdot d^{(\ell)}(x^\alpha \partial) t^\ell.
\]

By (15) and Lemma 2.1, we obtain

\[
S(x^\alpha \partial) = u^{-1} S_0(x^\alpha \partial) u = -v \cdot x^\alpha \partial \cdot u
\]

\[
= -v \cdot u_{\alpha_k - \alpha_{k'}} \cdot \left( \sum_{\ell=0}^{\infty} d^{(\ell)}(x^\alpha \partial) \cdot h_1^{(\ell)} t^\ell \right)
\]

\[
= -(1 - et)^{-(\alpha_k - \alpha_{k'})} \cdot \left( \sum_{\ell=0}^{\infty} d^{(\ell)}(x^\alpha \partial) \cdot h_1^{(\ell)} t^\ell \right).
\]

Hence, we get the result. \( \square \)

As we known, \( \{ \alpha_n x^{a - \epsilon_n} D_i - \alpha_i x^{a - \epsilon_i} D_n \mid \alpha \in \mathbb{Z}_+^n, 1 \leq i < n \} \) is a \( \mathbb{Z} \)-basis of \( S^+_x \), as a subalgebra of both the simple Lie \( \mathbb{Z} \)-algebras \( x^2 \mathbb{Z} \) and \( W^+_x \). As a result of Theorem 3.5, we have

Corollary 3.6. Fix distinguished elements \( h := \partial_k - \partial_{k'} \), \( e := x^i (\partial_k - 2\partial_{k'}) \) \((1 \leq k \neq k' \leq n)\), the corresponding quantization of \( U(S^+_x) \) over \( U(S^+_x)[[t]] \) by
Drinfel’d twist \( \mathcal{F}(k, k') \) with the product undeformed is given by

\[
\Delta(x^\alpha \partial) = x^\alpha \partial \otimes (1-\varepsilon t)^{\alpha k - \alpha k'} + \sum_{\ell=0}^{\infty} (-1)^\ell h^{(\ell)} \otimes (1-\varepsilon t)^{-\ell} \\
\cdot x^{\alpha+\ell k} (A_t \partial - B_t (\partial k - 2\partial k')) t^\ell,
\]

(20)

\[
S(x^\alpha \partial) = -(1-\varepsilon t)^{-(\alpha k - \alpha k')} \cdot \left( \sum_{\ell=0}^{\infty} x^{\alpha+\ell k} (A_t \partial - B_t (\partial k - 2\partial k')) \cdot h^{(\ell)} t^\ell \right),
\]

(21)

\[
\varepsilon(x^\alpha \partial) = 0,
\]

(22)

where \( A_t = \prod_{j=0}^{\ell-1} (\alpha_k - 2\alpha_k + j), \quad B_t = \partial(\epsilon_k)A_{t-1} \) with \( A_0 = 1, \ A_{-1} = 0 \).

### 3.2. Quantizations of special algebra \( S(n; 1) \)

In this subsection, firstly, we make modulo \( p \)-reduction and base change with the \( \mathcal{K}[[t]] \) replaced by \( \mathcal{K}[t] \), for the quantization of \( U(S^+_n) \) in char 0 (Corollary 3.6) to yield the quantization of \( U(S(n; 1)) \), for the restricted simple modular Lie algebra \( S(n; 1) \) in char \( p \). Secondly, we shall further make “\( p \)-restrictedness” reduction as well as base change with the \( \mathcal{K}[t] \) replaced by \( \mathcal{K}[t]^{(p)} \), for the quantization of \( U(S(n; 1)) \), which will lead to the required quantization of \( \mathfrak{u}(S(n; 1)) \), the restricted universal enveloping algebra of \( S(n; 1) \).

Let \( \mathbb{Z}_p \) be the prime subfield of \( \mathcal{K} \) with \( \text{char}(\mathcal{K}) = p \). When considering \( W^+_n \) as a \( \mathbb{Z}_p \)-Lie algebra, namely, making modulo \( p \)-reduction for the defining relations of \( W^+_n \), denoted by \( W^+_n \), we see that \( \{ J_2 \}_{\mathbb{Z}_p} = \text{Span}_{\mathbb{Z}_p} \{ x^\alpha D_i \mid \exists j : \alpha_j \geq p \} \) is a maximal ideal of \( W^+_n \), and \( W^+_n / \{ J_2 \}_{\mathbb{Z}_p} \cong W(n; 1)_{\mathbb{Z}_p} = \text{Span}_{\mathbb{Z}_p} \{ x^{(\alpha)} D_i \mid 0 \leq \alpha \leq \tau, 1 \leq i \leq n \} \). For the subalgebra \( S^+_n \), we have \( S^+_n / (\{ J_2 \}_{\mathbb{Z}_p} \cap \{ J_1 \}_{\mathbb{Z}_p}) \cong S^+(n; 1)_{\mathbb{Z}_p} \). We denote simply \( S^+_n \cap \{ J_1 \}_{\mathbb{Z}_p} \) by \( \{ J^+_1 \}_{\mathbb{Z}_p} \).

Moreover, we have \( S^+(n; 1) = \mathcal{K} \otimes_{\mathbb{Z}_p} S^+(n; 1)_{\mathbb{Z}_p} = \mathcal{K} S^+(n; 1)_{\mathbb{Z}_p} \), and \( S^+_n = \mathcal{K} S^+_n \).

Observe that the ideal \( J^+_1 = \mathcal{K} \{ J^+_1 \}_{\mathbb{Z}_p} \) generates an ideal of \( U(S^+_n) \) over \( \mathcal{K} \), denoted by \( J := J^+_1 U(S^+_n) \), where \( S^+_n / J^+_1 \cong \mathcal{K} S^+(n; 1) \). Based on the formulæ (20) & (21), \( J \) is a Hopf ideal of \( U(S^+_n) \) satisfying \( U(S^+_n) / J \cong U(S^+(n; 1)) \). Note that elements \( \sum_{\alpha} a_\alpha x^\alpha D_i \) in \( S^+_n \) for \( 0 \leq \alpha \leq \tau \) will be identified with \( \sum_{\alpha} a_\alpha x^{(\alpha)} D_i \) in \( \mathcal{K} S^+(n; 1) \), and those in \( J_1 \) with 0. Hence, by Lemma 1.1 and Corollary 3.6, we get the quantization of \( U(S^+(n; 1)) \) over \( U_t(S^+(n; 1)) := U(S^+(n; 1)) \otimes \mathcal{K}[t] \) (not necessarily in \( U(S^+(n; 1))[[t]] \)), as seen in formulæ (23) & (24) as follows.

**Theorem 3.7.** Fix two distinguished elements \( h := D_{kk'}(x^{(\alpha_k + \alpha_{k'} - \varepsilon)}) \), \( e := 2D_{kk'}(x^{(\alpha_k + \alpha_{k'} - \varepsilon)}) \) (\( 1 \leq k \neq k' \leq n \)), the corresponding quantization of \( U(S^+(n; 1)) \) over \( U_t(S^+(n; 1)) \) with the product undeformed is given by

\[
\Delta(D_{ij}(x^{(\alpha)})) = D_{ij}(x^{(\alpha)} \otimes (1-\varepsilon t)^{\alpha k - \alpha k'} + \sum_{\ell=0}^{p-1} (-1)^\ell h^{(\ell)} \otimes (1-\varepsilon t)^{-\ell} (A_t D_{ij}(x^{(\alpha+\ell k})) \\
\cdot \partial (\delta_{ik} D_{k'j} + \delta_{jk'} D_{ik'})(x^{(\alpha+(\ell-1)\varepsilon)}) t^\ell),
\]

(23)
\[
(24) \quad S(D_{ij}(x^{(\alpha)})) = -(1-ct)^{-\alpha_k+\delta_{ij}+\epsilon_k} \cdot \sum_{\ell=0}^{p-1} \left( \tilde{A}_\ell D_{ij}(x^{(\alpha+\ell \epsilon_k)}) \right. \\
\left. + \tilde{B}_\ell (\delta_{ij} D_{ij} + \delta_{ij} D_{ij})(x^{(\alpha+(\ell-1)\epsilon_k+\epsilon_j)}) \right) \cdot h_\ell^{(q)(\ell)}, \]
\]

\[
(25) \quad \varepsilon(D_{ij}(x^{(\alpha)})) = 0, \]

\[
(26) \quad \Delta(x^{(\tau-(p-1)\epsilon_i)}) D_j = x^{(\tau-(p-1)\epsilon_i)} D_j \otimes (1-ct)^p(\delta_{ij}-\delta_{ik}) + x^{(\tau-(p-1)\epsilon_i)} D_j, \]

\[
(27) \quad S(x^{(\tau-(p-1)\epsilon_i)}) D_j = -(1-ct)^p(\delta_{ij}-\delta_{ik}) x^{(\tau-(p-1)\epsilon_i)} D_j, \]

\[
(28) \quad \varepsilon(x^{(\tau-(p-1)\epsilon_i)}) D_j = 0, \]

where \(0 \leq \alpha \leq \tau, 1 \leq j < i \leq n, \tilde{A}_\ell = \ell!\left(\alpha_k+\ell\right)(\ell \epsilon_i-\delta_{jk} \epsilon_i-\delta_{ik} \epsilon_k-2 \delta_{ik} \epsilon_k+m) \) and \(A_0 = 1, A_{-1} = 0.\)

Note that (23), (24) & (25) give the corresponding quantization of \(U(S(n;1))\) over \(U_t(S(n;1)) := U(S(n;1)) \otimes_K K[t]\) (also over \(U(S(n;1))[[t]]\)). It should be noticed that in this step — inducing from the quantization integral form of \(U(S^2)\) and making the modulo \(p\) reduction, we used the first base change with \(K[[t]]\) replaced by \(K[t]\), and the objects from \(U(S(n;1))[[t]]\) turning to \(U_t(S(n;1))\).

Denote by \(I\) the ideal of \(U(S(n;1))\) over \(K\) generated by \((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})^p\) and \((D_{ij}(x^{(\alpha)})^p - D_{ij}(x^{(\alpha)}))^p\) with \(\alpha \neq \epsilon_i + \epsilon_j\) for \(0 \leq \alpha \leq \tau\) and \(1 \leq j < i \leq n.\)

\(u(S(n;1)) = U(S(n;1))/I\) is of prime-power dimension \(p^{(n-1)(p^n-1)}\). In order to get a reasonable quantization of prime-power dimension for \(u(S(n;1))\) in char \(p\), at first, it is necessary to clarify in concept the underlying vector space in which the required \(t\)-deformed object exists. According to our modular reduction approach, it should start to be induced from the \(K[t]\)-algebra \(U_t(S(n;1))\) in Theorem 3.7.

Firstly, we observe the following fact

\textbf{Lemma 3.8.} (i) \((1 - ct)^p \equiv 1 \pmod{p, I}\).

(ii) \((1 - ct)^{-1} \equiv 1 + ct + \cdots + c^{p-1}t^{p-1} \pmod{p, I}\).

(iii) \(h_a^{(\alpha)} \equiv 0 \pmod{p, I}\) for \(\ell \geq p, a \in \mathbb{Z}_p.\)

\textbf{Proof.} (i), (ii) follow from \(e^p = 0 \in u(S(n;1)).\)

(iii) For \(\ell \in \mathbb{Z}_+,\) there is a unique decomposition \(\ell = \ell_0 + \ell_1 p\) with \(0 \leq \ell_0 < p\) and \(\ell_1 \geq 0.\) Using the formulae (1.4), we have

\[
h_a^{(\ell)} \equiv h_a^{(\ell_0)} \cdot h_a^{(\ell_1 p)} \equiv h_a^{(\ell_0)} \cdot (h_a^{(p)})^{\ell_1} \pmod{p}, \]

where we used the facts that \((x + a + \ell_0 p) \equiv x + a + \ell_0 \pmod{p}\) and \((x + a + \ell_0 p) \equiv x + a + \ell_0 \pmod{p}\). Hence, \(h_a^{(\ell)} \equiv 0 \pmod{p, I}\) for \(\ell \geq p.\)

The above Lemma, together with Theorem 3.7, indicates that the required \(t\)-deformation of \(u(S(n;1))\) (if it exists) in fact only happens in a \(p\)-truncated polynomial ring (with degrees of \(t\) less than \(p\)) with coefficients in \(u(S(n;1))\), i.e., \(u_{t,q}(S(n;1)) = u(S(n;1)) \otimes_K K[t]^{(q)}\) (rather than in \(u_t(S(n;1)) = u(S(n;1)) \otimes_K\)
\(K[t]\), where \(K[t]^{(q)}\) is conveniently taken to be a \(p\)-truncated polynomial ring which is a quotient of \(K[t]\) defined as

\[
K[t]^{(q)} = K[t]/(t^p - qt), \quad \text{for } q \in K.
\]

Thereby, we obtain the underlying ring for our required \(t\)-deformation of \(u(S(n; \frac{1}{2}))\) over \(K[t]^{(q)}\), and \(\dim_K u_{t,q}(S(n; \frac{1}{2})) = p \cdot \dim_K u(S(n; \frac{1}{2})) = p^{1+(n-1)(p^n-1)}\). Via modulo “restrictedness” reduction, it is necessary for us to work over the objects from \(U_t(S(n; \frac{1}{2}))\) passage to \(U_{t,q}(S(n; \frac{1}{2}))\) first, and then to \(u_{t,q}(S(n; \frac{1}{2}))\) (see the proof of Theorem 3.11 below), here we used the second base change with \(K[t]^{(q)}\) instead of \(K[t]\).

We are now in a position to describe the following

**Definition 3.9.** With notations as above. A Hopf algebra \((u_{t,q}(S(n; \frac{1}{2})), m, \iota, \Delta, S, \epsilon)\) over a ring \(K[t]^{(q)}\) of characteristic \(p\) is said to be a finite-dimensional quantization of \(u(S(n; \frac{1}{2}))\) if its Hopf algebra structure, via modular reduction and base changes, inherits from a twisting of the standard Hopf algebra \(U(S^+_n)[[t]]\) by a Drinfeld twist such that \(u_{t,q}(S(n; \frac{1}{2}))/u_{t,q}(S(n; \frac{1}{2})) \cong u(S(n; \frac{1}{2}))\).

To describe \(u_{t,q}(S(n; \frac{1}{2}))\) explicitly, we still need an auxiliary Lemma.

**Lemma 3.10.** Let \(e = 2D_{kk'}(x^{(2\alpha_k+\epsilon_{k'})})\) and \(d^{(\ell)} = \frac{1}{\ell!}(\text{ad } e)^{\ell}\). Then

(i) \(d^{(\ell)}(D_{ij}(x^{(\alpha)})) = A_\ell D_{ij}(x^{(\alpha+\ell \epsilon_k)}) + \bar{B}_\ell (\delta_{ik} D_{k'j} + \delta_{jk} D_{k'i})(x^{(\alpha+(\ell-1)\epsilon_k+\epsilon_{k'})})\),
where \(\bar{A}_\ell, \bar{B}_\ell\) as in Theorem 3.1.

(ii) \(d^{(\ell)}(D_{ij}(x^{(\epsilon_k+\epsilon_{k'})})) = \delta_{i,j} (D_{ij}(x^{(\epsilon_{k'}+\epsilon_k)})) - \delta_{i,j}(\delta_{ik} - \delta_{jk})\).

(iii) \(d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) = \delta_{i,j,\ell}(D_{ij}(x^{(\alpha)})^p) - \delta_{i,j}(\delta_{ik} - \delta_{jk})\delta_{\alpha,\epsilon_k+\epsilon_{k'}}\).

**Proof.** (i) Note that \(A_\ell = \frac{1}{\ell!} \prod_{m=0}^{\ell-1} (\alpha_k - \delta_{ik} - 2\alpha_{k'} + 2\delta_{k'k} + 2\delta_{kk'} + m)\), for \(0 \leq \alpha \leq \tau\). By (12) and Theorem 3.7, we get

\[
d^{(\ell)}(D_{ij}(x^{(\alpha)})) = \frac{1}{\ell!} d^{(\ell)}(x^{(\alpha-k-\epsilon_{k'})(\alpha_j, \partial_{i,k} - \alpha_i \partial_j)})
\]

\[
= \frac{1}{\ell!} x^{(\alpha-k-\epsilon_{k'}+\epsilon_{k'})} D_{ij}(\alpha_k \partial_{i,k} - \alpha_i \partial_j) - (\alpha_j, \delta_{ik} - \alpha \delta_{jk}) A_{\ell-1}\partial_{k} - 2\partial_{k'})
\]

\[
= \bar{A}_\ell D_{ij}(x^{(\alpha+k)}) + \bar{B}_\ell (\delta_{ik} D_{k'j} + \delta_{jk} D_{k'i})(x^{(\alpha+(\ell-1)\epsilon_k+\epsilon_{k'})}).
\]

(ii) Note that \(A_\ell = 1\) and \(A_\ell = 0\) for \(\ell \geq 1\),

\[
\bar{A}_\ell = \ell! \left( \alpha_k + \frac{\ell}{\ell} \right) (A_\ell - \delta_{ik} A_{\ell-1} - \delta_{ik} A_{\ell-1}) \quad \text{(mod } p)\)
\]

\[
\bar{B}_\ell = 2\ell! \left( \alpha_k + \frac{\ell-1}{\ell} \right) (A_{\ell+1} - \delta_{ik} A_{\ell-1}) \quad \text{(mod } p)\)
\]

We obtain \(\bar{A}_0 = 1\) and \(\bar{B}_0 = 0\). We also obtain \(\bar{A}_1 = -(\delta_{ik} + \delta_{jk})(\alpha_k + 1)\), \(\bar{B}_1 = 2(\alpha_{k'} + 1)\) and \(\bar{A}_\ell = \bar{B}_\ell = 0\) for \(\ell \geq 2\), namely, \(d^{(\ell)}(D_{ij}(x^{(\epsilon_k+\epsilon_{k'})})) = 0\) for \(\ell \geq 2\). So by (i), we have

\[
d^{(\ell)}(D_{ij}(x^{(\epsilon_k+\epsilon_{k'})})) = -(\delta_{ik} + \delta_{jk})(\alpha_k + 1) D_{ij}(x^{(\epsilon_k+\epsilon_{k'})}) + 2(\alpha_{k'} + 1)(\delta_{ik} D_{k'j} + \delta_{jk} D_{k'i})(x^{(\epsilon_k+\epsilon_{k'})})
\]

\[
= -(\delta_{ik} - \delta_{jk}) e.
\]
In any case, we arrive at the result as required.

(iii) From (11), we obtain that for $0 \leq \alpha \leq \tau$,

\[
d^{(1)}((D_{ij}(x^{(\alpha)}))^p) = \frac{1}{(\alpha!)^p} [e, (D_{ij}(x^{\alpha}))^p] = \frac{1}{(\alpha!)^p} [e, (x^{\alpha-\epsilon_i-\epsilon_j}(\alpha_j \delta_i - \alpha_i \delta_j))^p]
\]

\[
= \frac{1}{(\alpha!)^p} \sum_{\ell=1}^{p} (-1)^\ell \binom{p}{\ell} (x^{\alpha-\epsilon_i-\epsilon_j}(\alpha_j \delta_i - \alpha_i \delta_j))^{p-\ell}
\]

\[
\cdot x^{\epsilon_i+\epsilon_j}(\alpha_i \delta_i - \alpha_i \delta_j)
\]

\[
\equiv -\frac{a_p}{\alpha!} x^{2\epsilon_i+\epsilon_j}(\alpha_i \delta_i - \alpha_i \delta_j) (\partial_k - 2 \partial_{k'}) (\mod p)
\]

\[
\equiv \begin{cases}
-\alpha_p \epsilon_i, & \text{if } \alpha = \epsilon_i + \epsilon_j \\
0, & \text{if } \alpha \neq \epsilon_i + \epsilon_j
\end{cases} (\mod J),
\]

where the last "\(\equiv\)" by using the identification w.r.t. modulo the ideal \(J\) as before, and \(a_\ell = \prod_{m=0}^{\ell-1} (\delta m, \epsilon_i, \delta_j) (\epsilon_k + m(\alpha - \epsilon_i - \epsilon_j))\), \(b_\ell = \ell (\partial_k - 2 \partial_{k'}) (\alpha_i - \epsilon_i - \epsilon_j) \alpha_{\ell-1}\), and \(a_p = \delta h_k - \delta j k\) for \(\alpha = \epsilon_i + \epsilon_j\).

Consequently, by the definition of \(d^{(\ell)}\), we get \(d^{(\ell)}((x^{(\alpha)}D_i)^p) = 0\) in \(u(S(n; 1))\) for \(2 \leq \ell \leq p - 1\) and \(0 \leq \alpha \leq \tau\).

Based on Theorem 3.7, Definition 3.9 and Lemma 3.10, we arrive at

**Theorem 3.11.** Fix two distinguished elements \(h := D_{kk'}(x^{(\epsilon_k+\epsilon_{k'})})\), \(e := 2D_{kk'}(x^{(\epsilon_k+\epsilon_{k'})})\) \((1 \leq k \neq k' \leq n)\), there is a noncommutative and noncocommutative Hopf algebra \((u_{t,q}(S(n; 1)), m, \Delta, S, \epsilon)\) over \(K[t]_{1}^{(q)}\) with its algebra structure undeformed, whose coalgebra structure is given by

\[
\Delta(D_{ij}(x^{(\alpha)})) = D_{ij}(x^{(\alpha)}) \otimes (1-et)^{\alpha_k-\delta_{j_k}-\alpha_k'-\delta_{j_k'}}
\]

\[
+ \sum_{\ell=0}^{p-1} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-(\ell)D_{ij}(x^{(\alpha)}))} t^\ell,
\]

\[
S(D_{ij}(x^{(\alpha)})) = -(1-et)^{-\alpha_k+\delta_{j_k}+\delta_{j_k'}-\alpha_k'-\delta_{j_k'}} \cdot \left( \sum_{\ell=0}^{p-1} d^{(\ell)}(D_{ij}(x^{(\alpha)})) \cdot h^{(\ell)} t^\ell \right),
\]

\[
\epsilon(D_{ij}(x^{(\alpha)})) = 0,
\]

for \(0 \leq \alpha \leq \tau\), which is finite dimensional with \(\dim_K u_{t,q}(S(n; 1)) = p^{1+\binom{n-1}{2}+1}\).

**Proof.** Set \(U_{t,q}(S(n; 1)) := U(S(n; 1)) \otimes_K K[t]^{(q)}\). Note that the result of Theorem 3.7, via the base change with \(K[t]^{(q)}\) instead of \(K[t]^{(q)}\), is still valid over \(U_{t,q}(S(n; 1))\). Denote by \(I_{t,q}\) the ideal of \(U_{t,q}(S(n; 1))\) over the ring \(K[t]^{(q)}\) generated by the same generators of the ideal \(I\) in \(U(S(n; 1))\) via the base change with \(K\) replaced by \(K[t]^{(q)}\). We shall show that \(I_{t,q}\) is a Hopf ideal of \(U_{t,q}(S(n; 1))\). It suffices to verify that \(\Delta\) and \(S\) preserve the generators in \(I_{t,q}\) of \(U_{t,q}(S(n; 1))\).
(I) By Lemmas 3.4, 3.8 & 3.10 (iii), we obtain
\[ \Delta((D_{ij}(x^{(\alpha)}))^p) = (D_{ij}(x^{(\alpha)}))^p \otimes (1-et)^p(\alpha_k-\alpha_{k'}) \]
\[ + \sum_{\ell=0}^{p-1}(-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)t^\ell \]
\[ \equiv (D_{ij}(x^{(\alpha)}))^p \otimes 1 + \sum_{\ell=0}^{p-1}(-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)t^\ell \quad \text{(mod } p) \]
\[ = (D_{ij}(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (D_{ij}(x^{(\alpha)}))^p + h \otimes (1-et)^{-1}(\delta_{ik} - \delta_{jk})\delta_{\alpha_k,\epsilon_{k'}}et. \]
Hence, when \( \alpha \neq \epsilon_i + \epsilon_j \), we get
\[ \Delta((D_{ij}(x^{(\alpha)}))^p) = (D_{ij}(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (D_{ij}(x^{(\alpha)}))^p \]
\[ \in I_{t,q} \otimes U_{t,q}(S(n;\frac{1}{2})) + U_{t,q}(S(n;\frac{1}{2})) \otimes I_{t,q}. \]
and when \( \alpha = \epsilon_i + \epsilon_j \), by Lemma 3.10 (ii), (23) becomes
\[ \Delta(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = D_{ij}(x^{(\epsilon_i+\epsilon_j)}) \otimes 1 + 1 \otimes D_{ij}(x^{(\epsilon_i+\epsilon_j)}) + h \otimes (1-et)^{-1}(\delta_{ik} - \delta_{jk})et. \]
Combining with (33), we obtain
\[ \Delta(D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)}) \equiv ((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})) \otimes 1 \]
\[ + 1 \otimes ((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})) \]
\[ \in I_{t,q} \otimes U_{t,q}(S(n;\frac{1}{2})) + U_{t,q}(S(n;\frac{1}{2})) \otimes I_{t,q}. \]
Thereby, we prove that the ideal \( I_{t,q} \) is also a coideal of the Hopf algebra \( U_{t,q}(S(n;\frac{1}{2})) \).

(II) By Lemmas 3.4, 3.8 & 3.10 (iii), we have
\[ S((D_{ij}(x^{(\alpha)}))^p) = -(1-et)^{-p(\alpha_k-\alpha_{k'})} \sum_{\ell=0}^{\infty} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) \cdot h_1^{(\ell)} t^\ell \]
\[ \equiv -(D_{ij}(x^{(\alpha)}))^p - \sum_{\ell=1}^{p-1} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) \cdot h_1^{(\ell)} t^\ell \quad \text{(mod } p) \]
\[ = -(D_{ij}(x^{(\alpha)}))^p + (\delta_{ik} - \delta_{jk})\delta_{\alpha_k,\epsilon_{k'}} e \cdot h_1^{(1)} t. \]
Hence, when \( \alpha \neq \epsilon_i + \epsilon_j \), we get
\[ S((D_{ij}(x^{(\alpha)}))^p) = -(D_{ij}(x^{(\alpha)}))^p \in I_{t,q}. \]
When \( \alpha = \epsilon_i + \epsilon_j \), by Lemma 3.10 (ii), (24) reads as
\[ S(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = -D_{ij}(x^{(\epsilon_i+\epsilon_j)}) + (\delta_{ik} - \delta_{jk}) e \cdot h_1^{(1)} t. \]
Combining with (34), we obtain
\[ S((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = -(D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})) \in I_{t,q}. \]
Thereby, the ideal \( I_{t,q} \) is indeed preserved by the antipode \( S \) of the quantization \( U_{t,q}(S(n;\frac{1}{2})) \), the same as in Theorem 3.7.

(III) It is obvious to notice that \( \varepsilon((D_{ij}(x^{(\alpha)}))^p) = 0 \) for all \( \alpha \) with \( 0 \leq \alpha \leq \tau \).
In other words, we prove that \( I_{t,q} \) is a Hopf ideal in \( U_{t,q}(S(n;\frac{1}{2})) \). We thus obtain the required \( t \)-deformation on \( u_{t,q}(S(n;\frac{1}{2})) \), for the Cartan type simple modular restricted Lie algebra of \( S \) type — the special algebra \( S(n;\frac{1}{2}) \). □
Remark 3.12. (i) Set $f = (1 - et)^{-1}$. By Lemma 3.10 & Theorem 3.11, one gets

$$[h, f] = f^2 - f, \quad h^p = h, \quad f^p = 1, \quad \Delta(h) = h \otimes f + 1 \otimes h,$$

where $f$ is a group-like element, and $S(h) = -hf^{-1}$, $\varepsilon(h) = 0$. So the subalgebra generated by $h$ and $f$ is a Hopf subalgebra of $u_{\ell,q}(S(n; 1))$, which is isomorphic to the well-known Radford Hopf algebra over $K$ in char $p$ (see [26]).

(ii) According to our argument, given a parameter $q \in K$, one can specialize $t$ to any root of the $p$-polynomial $t^p - qt \in K[t]$ in a split field of $K$. For instance, if take $q = 1$, then one can specialize $t$ to any scalar in $\mathbb{Z}_p$. If set $t = 0$, then we get the original standard Hopf algebra structure of $u(S(n; 1))$. In this way, we indeed get a new Hopf algebra structure over the same restricted universal enveloping algebra $u(S(n; 1))$ over $K$ under the assumption that $K$ is algebraically closed, which has the new coalgebra structure induced by Theorem 3.11, but has dimension $p^{(n-1)(p^r-1)}$.

3.3. More quantizations. We consider the modular reduction process for the quantizations of $U(S^+)([[t]])$ arising from those products of some pairwise different and mutually commutative basic Drinfel’d twists. We will then get lots of new families of noncommutative and noncocommutative Hopf algebras of dimension $p^{(n-1)(p^r-1)}$ with indeterminate $t$ or of dimension $p^{(n-1)(p^r-1)}$ with specializing $t$ into a scalar in $K$.

Let $A(k, k')_t$, $B(k, k')_t$ and $A(m, m')_n$, $B(m, m')_n$ denote the coefficients of the corresponding quantizations of $U(S^+_n)$ over $U(S^+_n)([[t]])$ given by Drinfeld’s twists $\mathcal{F}(k, k')$ and $\mathcal{F}(m, m')$ as in Corollary 3.6, respectively. Note that $A(k, k')_0 = A(m, m')_0 = 1$, $A(k, k')^\ell = A(m, m')^\ell = 0$.

Set

$$\partial(m, m'; k, k')_{\ell, n} := A(m, m')_nA(k, k')_t\partial - A(m, m')_nB(k, k')_e(\partial_k - 2\partial_{k'})\partial - A(k, k')_tB(m, m')_n(\partial_m - 2\partial_{m'})\partial.$$

Lemma 3.13. Fix distinguished elements $h(k, k') = \partial_k - \partial_{k'}$, $e(k, k') = x^{x+k}(\partial_k - 2\partial_{k'})$ ($1 \leq k \neq k' \leq n$) and $h(m, m') = \partial_m - \partial_{m'}$, $e(m, m') = x^{p^r}(\partial_m - 2\partial_{m'})$ ($1 \leq m \neq m' \leq n$) with $k \neq m'$ and $k' \neq m$, the corresponding quantization of $U(S^+_n)$ over $U(S^+_n)([[t]])$ by Drinfeld’s twist $\mathcal{F} = \mathcal{F}(m, m')\mathcal{F}(k, k')$ with the product undeformed is given by

$$\Delta(x^\partial) = x^\partial \otimes (1 - e(k, k')t)^{\alpha_k - \alpha_{k'}}(1 - e(m, m')t)^{\alpha_m - \alpha_{m'}} + \sum_{n, \ell = 0}^{\infty} (-1)^{n + \ell} h(k, k')^{(\ell)} \cdot h(m, m')^{(n)} \otimes (1 - e(k, k')t)^{-\ell} \cdot (1 - e(m, m')t)^{-n_{x^\partial + e(k, k') + n_{x^\partial} = \partial(m, m'; k, k')_\ell, n_{x^{p^r}}}}$$

and

$$S(x^\partial) = - (1 - e(k, k')t)^{-\alpha_k + \alpha_{k'}}(1 - e(m, m')t)^{-\alpha_m + \alpha_{m'}} \cdot \sum_{n, \ell = 0}^{\infty} x^{\partial + e(k, k') + n_{x^\partial} = \partial(m, m'; k, k')_\ell, n_{x^{p^r}} - \partial(m, m')^{(n)} h(k, k')^{(\ell)} t_{x^{p^r}}},$$

for $x^\partial \in S^+_n$.

Proof. See [16].
Set \( \alpha(k,k') = \alpha_k - \delta_{kk'} - \alpha_{kk'} + \delta_{kk'} \). Write coefficients \( A_\ell, B_\ell, A_\ell \) in Theorem 3.7 as \( A(k,k')_\ell, B(k,k')_\ell, A(k,k')_\ell \). Set
\[
D_{ij}(m,m';k,k')_{\ell,n} := A(k,k')_\ell A(m,m')_n D_{ij}(x^{(\alpha+\ell_m+n_e m)}) + B(k,k')_\ell B(m,m')_n (\delta_{kk} D_{kj} + \delta_{jk} D_{ik}) (x^{(\alpha+(\ell-1)\ell_m+n_e m+\epsilon_{k'})}) + A(k,k')_\ell B(m,m')_n (\delta_{km} D_{k'j} + \delta_{jm} D_{k'i})(x^{(\alpha+\ell_m+(n-1)\ell_m+\epsilon_{k'})}) \]

Using Lemma 3.13, we get a new quantization of \( U(S(n;1)) \) over \( U_t(S(n;1)) \) by Drinfeld’s twist \( \mathcal{F} = \mathcal{F}(m,m') \mathcal{F}(k,k') \) as follows.

**Lemma 3.14.** Fix distinguished elements \( h(k,k') = D_{kk'}(x^{(\epsilon_k+\epsilon_{k'})}), e(k,k') = 2D_{kk'}(x^{(2\epsilon_k+\epsilon_{k'})}); h(m,m') = D_{mm'}(x^{(\epsilon_m+\epsilon_{m'})}), e(m,m') = 2D_{mm'}(x^{(2\epsilon_m+\epsilon_{m'})}) \) with \( k \neq m; m' \neq m \), the corresponding quantization of \( U(S(n;1)) \) on \( U_t(S(n;1)) \) (also on \( U(S(n;1))[[[S]]] \)) with the product undeformed is given by
\[
\Delta(D_{ij}(x^{(\alpha)})) = D_{ij}(x^{(\alpha)} \otimes (1-e(k,k') t)\alpha(k,k')(1-e(m,m') t)^{n(m,m')} + \sum_{n,\ell=0}^{p-1} (-1)^{n+\ell} h(k,k')^{(\ell)} h(m,m')^{(n)} \otimes (1-e(m,k') t)^{-\ell} \cdot (1-e(m,m') t)^{-n} D_{ij}(m,m';k,k')_{\ell,n} t^{n+\ell},
\]
\[
S(D_{ij}(x^{(\alpha)})) = -(1-e(k,k') t)^{-\alpha(k,k')} (1-e(m,m') t)^{-\alpha(m,m')} \cdot \left( \sum_{n,\ell=0}^{p-1} D_{ij}(m,m';k,k')_{\ell,n} (h(k,k')^{(\ell)} h(m,m')^{(n)} t^{n+\ell}) \right),
\]
and
\[
\varepsilon(D_{ij}(x^{(\alpha)})) = 0,
\]
where \( 0 \leq \alpha \leq \tau \).

**Lemma 3.15.** For \( s \geq 1 \), one has
\[
(i) \quad \Delta((D_{ij}(x^{(\alpha)}))^s) = \sum_{0 \leq j \leq \ell_0} \left( \begin{array}{c} s \\ j \end{array} \right) (-1)^{s+\ell} (D_{ij}(x^{(\alpha)}))^{s} h(k,k')^{(\ell)} h(m,m')^{(n)} \otimes (1-e(k,k') t)^{s-\alpha(k,k')} \cdot (1-e(m,m') t)^{s-\alpha(m,m')} \cdot d_{mm'} d_{kk'} ((D_{ij}(x^{(\alpha)}))^{s-j} t^{j+n}).
\]
\[
(ii) \quad S((D_{ij}(x^{(\alpha)}))^s) = (-1)^s (1-e(m,m') t)^{-s\alpha(m,m')} (1-e(k,k') t)^{-s\alpha(k,k')} \cdot \left( \sum_{n,\ell=0}^{s} d_{mm'} d_{kk'} ((D_{ij}(x^{(\alpha)}))^{s}) h(k,k')^{(\ell)} h(m,m')^{(n)} t^{n+\ell}. \right).
\]

**Lemma 3.16.** Set \( e(k,k') = 2D_{kk'}(x^{(2\epsilon_k+\epsilon_{k'})}), e(m,m') = 2D_{mm'}(x^{(2\epsilon_m+\epsilon_{m'})}) \), \( d_{kk'}^{(\ell)} = \frac{1}{\ell} (\mathrm{ad} e(k,k') t) \) and \( d_{mm'}^{(n)} = \frac{1}{n} (\mathrm{ad} e(m,m') n) \). Then
\[
(i) \quad d_{mm'}^{(n)} d_{kk'}^{(\ell)} ((D_{ij}(x^{(\alpha)}))^{s}) = D_{ij}(m,m';k,k')_{\ell,n},
\]
where \( D_{ij}(m,m';k,k')_{\ell,n} \) as in Lemma 3.14.
\[
(ii) \quad d_{mm'}^{(n)} d_{kk'}^{(\ell)} ((D_{ij}(x^{(\alpha+\epsilon_i)}))^{s}) = \delta_{\ell,0} d_{nn'}^{(0)} D_{ij}(x^{(\alpha+\epsilon_i)}) - \delta_{\ell,0} \delta_{1,n} (\delta_{kk} - \delta_{jk}) e(k,k') - \delta_{\ell,0} \delta_{1,n} (\delta_{km} - \delta_{jm}) e(m,m'),
\]
\[
(iii) \quad d_{mm'}^{(n)} d_{kk'}^{(\ell)} ((D_{ij}(x^{(\alpha)}))^{s}) = \delta_{\ell,0} d_{nn'}^{(0)} (D_{ij}(x^{(\alpha)}))^{s} - \delta_{\ell,0} \delta_{1,n} (\delta_{kk} - \delta_{jk}) \delta_{\alpha+\epsilon_j}.
\]
For two distinguished elements twist. Using the horizontal Drinfeld twists and the same discussion in Sections 2, ≤ of two different and commutative basic Drinfel’d twists.

\( e(k, k') - \delta_{\ell, 0}\delta_{1, n}(\delta_{im} - \delta_{jm})\delta_{\alpha, \epsilon} + \epsilon(e(m, m')) \).

**Proof.** (i) For 0 ≤ α ≤ τ, using (12), we obtain

\[
\begin{align*}
d_{mn}^{(n)}d_{kk'}^{(f)}(D_{ij}(x^{(\alpha)})) &= d_{mn}^{(n)}d_{kk'}^{(f)} \left( \frac{1}{\alpha!} x^{\alpha - \epsilon - \epsilon_j} (\alpha_j \partial_i - \alpha_i \partial_j) \right) \\
&= d_{mn}^{(n)} \left( \frac{1}{\alpha!} x^{\alpha - \epsilon_i - \epsilon_j + \epsilon_k} (A(k, k')\ell(\alpha_j \partial_i - \alpha_i \partial_j) - B(k, k')\ell(\partial_k - 2\partial_{k'})) \right) \\
&= \frac{1}{\alpha!} x^{\alpha - \epsilon_i - \epsilon_j + \epsilon_k} \alpha \cdot \epsilon_{m, m'}( A(k, k')_{\ell}(\alpha_j \partial_i - \alpha_i \partial_j) - A(m, m')_{\ell}(\partial_k - 2\partial_{k'})) \\
&= D_{ij}(m, m'; k, k')_{\ell, n}.
\end{align*}
\]

(ii), (iii) may be proved directly using Lemma 3.10.

Using Lemmas 3.8, 3.10, 3.15 & 3.16, we get a new Hopf algebra structure over the same restricted universal enveloping algebra \( u(S(n; 1)) \) over \( K \) by the products of two different and commutative basic Drinfel’d twists.

**Theorem 3.17.** Fix two distinguished elements \( h(k, k') := D_{kk'}(x^{(k + \epsilon_i)}), e(k, k') := D_{kk'}(x^{(k + \epsilon_i)})(1 \leq k \neq k' \leq n) \) and \( h(m, m') := D_{mm'}(x^{(m + \epsilon_i)}), e(m, m') := D_{mm'}(x^{(m + \epsilon_i)})(1 \leq m \neq m' \leq n) \) with \( k \neq m, m'; k' \neq m \), there is a noncommutative and noncocommutative Hopf algebra \( (u_{\ell}(S(n; 1)), m, \ell, A, S, \epsilon) \) over \( K'\mathcal{p}(\ell) \) with the product undeformed, whose coalgebra structure is given by

\[
\begin{align*}
\Delta(D_{ij}(x^{(\alpha)})) &= D_{ij}(x^{(\alpha)}) \otimes (1 - e(k, k')t)^{\alpha(k, k')} (1 - e(m, m')t)^{\alpha(m, m')} \\
&+ \sum_{n, \ell=0}^{p-1} (-1)^{\ell+n}h(k, k')^{(\ell)} h(m, m')^{(n)} \otimes (1 - e(k, k')t)^{-\ell}, \\
&\cdot (1 - e(m, m')t)^{-n} d_{kk'}^{(f)} d_{mm'}^{(n)}(D_{ij}(x^{(\alpha)}))^{\ell+n},
\end{align*}
\]

\[
\begin{align*}
S(D_{ij}(x^{(\alpha)})) &= -(1 - e(k, k')t)^{-\alpha(k, k')} (1 - e(m, m')t)^{-\alpha(m, m')} \\
&\cdot \left( \sum_{n, \ell=0}^{p-1} d_{kk'}^{(f)} d_{mm'}^{(n)}(D_{ij}(x^{(\alpha)})) h(k, k')^{(\ell)} h(m, m')^{(n)} t^{\ell+n}, \right),
\end{align*}
\]

\[
\epsilon(D_{ij}(x^{(\alpha)})) = 0,
\]

where 0 ≤ α ≤ τ, and \( \dim_K u_{\ell,q}(S(n; 1)) = p^{1+(n-1)(p^n-1)} \).

4. Quantizations of horizontal type for Lie bialgebra of Cartan type S

In this section, we assume that \( n \geq 3 \). Take \( h := \partial_k - \partial_{k'} \) and \( \epsilon := x^{\epsilon_k - \epsilon_m} \partial_m \)
(1 ≤ k ≠ k' ≠ m ≤ n) and denote by \( \mathcal{F}(k', m) \) the corresponding Drinfel’d twist. Using the horizontal Drinfel’d twists and the same discussion in Sections 2, 3, we obtain some new quantizations of horizontal type for the universal enveloping algebra of the special algebra \( S(n; 1) \). The twisted structures given by the twists \( \mathcal{F}(k', m) \) on subalgebra \( S(n; 1)_n \) are the same as those on the special linear Lie algebra \( sl_n \) over a field \( K \) with char(\( K \)) = p derived by the Jordanian twists \( \mathcal{F} = \text{exp}(h \otimes \sigma), \sigma = \ln(1 - \epsilon) \) for some two-dimensional carrier subalgebra \( B(2) = \text{Span}_K \{h, \epsilon\} \) discussed in [19, 20], etc.
4.1. Quantizations of horizontal type of \( u(S(n; 1)) \). From Lemma 3.2 and Theorem 3.5, we have

**Lemma 4.1.** Fix two distinguished elements \( h := \partial_k - \partial_{k'} \), \( e := x^{\epsilon_k - \epsilon_{k'}} \) (for \( 1 \leq k \neq k' \neq m \leq n \)), the corresponding horizontal quantization of \( U(S^+_k) \) over \( U(S^+_k)[[t]] \) by Drinfel’d twist \( \mathcal{F}(k, k'; m) \) with the product undeformed is given by

\[
\Delta(x^\alpha \partial) = x^\alpha \partial \otimes (1-et)^{\alpha_k - \alpha_{k'}} + \sum_{\ell=0}^{\infty} (-1)^{\ell} h^{(\ell)} \otimes (1-et)^{-\ell},
\]

\[
x^{\alpha+\ell(\epsilon_k - \epsilon_m)}(A_{\ell} \partial - B_{\ell} \partial_m),
\]

\[
S(x^\alpha \partial) = -(1-et)^{-\alpha_k - \alpha_{k'}} \cdot \left( \sum_{\ell=0}^{\infty} x^{\alpha+\ell(\epsilon_k - \epsilon_m)}(A_{\ell} \partial - B_{\ell} \partial_m) \cdot h^{(\ell)} \right),
\]

\[
\varepsilon(x^\alpha \partial) = 0,
\]

where \( \alpha - \eta \in \mathbb{Z}_n^+ \), \( \eta = -1 \), \( A_{\ell} = \frac{1}{n} \sum_{j=0}^{\ell-1} (\alpha_m - j) \), \( B_{\ell} = \partial(\epsilon_k - \epsilon_m)A_{\ell-1} \), with a convention \( A_0 = 1, A_{-1} = 0 \).

Note that \( A_{\ell} = 0 \) for \( \ell > \alpha_m \) and \( B_{\ell} = 0 \) for \( \ell > \alpha_m + 1 \) in Lemma 4.1.

We firstly make the modulo \( p \) reduction for the quantizations of \( U(S^+_k) \) in Lemma 4.1 to yield the horizontal quantizations of \( U(S(n; 1)) \) over \( U_\ell(S(n; 1)) \).

**Theorem 4.2.** Fix distinguished elements \( h = D_{kk'}(x^{(\epsilon_k + \epsilon_{k'})}) \), \( e = D_{mk}(x^{(\epsilon_k)}) \) (for \( 1 \leq k \neq k' \neq m \leq n \)), the corresponding horizontal quantization of \( U(S(n; 1)) \) over \( U_\ell(S(n; 1)) \) with the product undeformed is given by

\[
\Delta(D_{ij}(x^{(\alpha)})) = D_{ij}(x^{(\alpha)} \otimes (1-et)^{\alpha_k - \alpha_{k'}} + \sum_{\ell=0}^{p-1} (-1)^{\ell} h^{(\ell)} \otimes (1-et)^{-\ell},
\]

\[
\cdot \left( A_{\ell}D_{ij}(x^{(\alpha+\ell(\epsilon_k - \epsilon_m))}) + B_{\ell}(\delta_{jk}D_{jm} - \delta_{jk}D_{jm})(x^{(\alpha+(\ell-1)(\epsilon_k - \epsilon_m))}) \right),
\]

\[
S(D_{ij}(x^{(\alpha)})) = -(1-et)^{-\alpha_k - \alpha_{k'}} \cdot \sum_{\ell=0}^{p-1} \left( A_{\ell}D_{ij}(x^{(\alpha+\ell(\epsilon_k - \epsilon_m))}) + B_{\ell}(\delta_{jk}D_{jm} - \delta_{jk}D_{jm})(x^{(\alpha+(\ell-1)(\epsilon_k - \epsilon_m))}) \right) \cdot h^{(\ell)} \cdot \ell,
\]

\[
\varepsilon(D_{ij}(x^{(\alpha)})) = 0,
\]

where \( 0 \leq \alpha \leq \tau \), \( \alpha(k, k') = \alpha_k - \delta_{ik} - \delta_{j\ell} + \delta_{ik'} + \delta_{j\ell'} \), \( \bar{A}_{\ell} \equiv (\alpha_{\ell + e}) \pmod{p} \) for \( 0 \leq \ell \leq \alpha_m \), \( \bar{B}_{\ell} \equiv (\alpha_{\ell + e-1}) \pmod{p} \) for \( 1 \leq \ell \leq \alpha_m + 1 \), and otherwise, \( A_{\ell} = B_{\ell} = 0 \).

**Proof.** Note that the elements \( \sum_{i, \alpha} \frac{1}{\alpha} a_{i, \alpha} x^{\alpha} D_i \) in \( S^+_k \) for \( 0 \leq \alpha \leq \tau \) will be identified with \( \sum_{i, \alpha} a_{i, \alpha} x^{\alpha} D_i \) in \( S(n; 1) \) and those in \( J_{\frac{1}{2}} \) (given in Section 3.2)
with 0. Hence, by Lemma 4.1, we get
\[ \Delta(D_{ij}(x^{(\alpha)})) = \frac{1}{\alpha!} \Delta(x^{\alpha-\epsilon_i-\epsilon_j}(\alpha_j \partial_i - \alpha_i \partial_j)) \]

\[ = D_{ij}(x^{(\alpha)}) \otimes (1-e^{\ell})^{\alpha(k,k')} + \sum_{\ell=0}^{p-1} (-1)^\ell h(\ell) \otimes (1-e^{\ell})^{-\ell}. \]

\[ \cdot \frac{1}{\alpha!} x^{\alpha-\epsilon_i-\epsilon_j+\ell(\epsilon_k-\epsilon_m)} (A_\ell(\alpha_j \partial_i - \alpha_i \partial_j) - B_\ell \partial_m) t^\ell, \]

where \( A_\ell = \frac{\ell-1}{\eta} \prod_{j=0}^{\ell-1} (\alpha_m-\delta_{im}-\delta_{jm}-j) \), \( B_\ell = (\alpha_j \partial_i - \alpha_i \partial_j)(\epsilon_k-\epsilon_m)A_{\ell-1}. \)

Write
\[ (*) = \frac{1}{\alpha!} x^{\alpha-\epsilon_i-\epsilon_j+\ell(\epsilon_k-\epsilon_m)} (A_\ell(\alpha_j \partial_i - \alpha_i \partial_j) - B_\ell \partial_m), \]

\[ (**) = \tilde{A}_\ell D_{ij}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))}) + \tilde{B}_\ell (\delta_{ik} D_{jm} - \delta_{jk} D_{im})(x^{(\alpha+(\ell-1)(\epsilon_k-\epsilon_m))}). \]

We claim that \((*) \equiv (**)\).

The proof will be given in the following steps:

(i) For \( \delta_{im} + \delta_{jm} = 1 \), we have

\[ (*) = \left\{ \begin{array}{ll}
\frac{(\alpha_k+\ell)!}{\alpha_k!} \frac{(\alpha_m-\ell)!}{\alpha_m!} (A_\ell + A_{\ell-1}) D_{ij}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))}), & \text{for } 0 \leq \ell \leq \alpha_m, \\
0, & \text{for } \ell > \alpha_m.
\end{array} \right. \]

A simple calculation shows that \( \frac{(\alpha_k+\ell)!}{\alpha_k!} \frac{(\alpha_m-\ell)!}{\alpha_m!} (A_\ell + A_{\ell-1}) \equiv \binom{\alpha_k+\ell}{\ell} \pmod{p} \), for \( 0 \leq \ell \leq \alpha_m \). So, \((*) \equiv (**)\).

(ii) For \( \delta_{im} + \delta_{jm} = 0 \), we consider three subcases:

If \( \delta_{ij} = 1 \), we have

\[ (*) = \left\{ \begin{array}{ll}
\frac{(\alpha_k+\ell)!}{\alpha_k!} \frac{(\alpha_m-\ell)!}{\alpha_m!} A_\ell D_{kj}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))}) + \frac{(\alpha_k+\ell)!}{\alpha_k!} \frac{(\alpha_m-(\ell-1))!}{\alpha_m!} A_{\ell-1} D_{jm}(x^{(\alpha+(\ell-1)(\epsilon_k-\epsilon_m))}), & \text{for } 0 \leq \ell \leq \alpha_m+1, \\
0, & \text{for } \ell > \alpha_m+1.
\end{array} \right. \]

A simple calculation indicates that for \( 0 \leq \ell \leq \alpha_m+1, \)

\[ \frac{(\alpha_k+\ell)!}{\alpha_k!} \frac{(\alpha_m-\ell)!}{\alpha_m!} A_\ell \equiv \binom{\alpha_k+\ell}{\ell} \equiv \tilde{A}_\ell \pmod{p}, \]

\[ \frac{(\alpha_k+\ell-1)!}{\alpha_k!} \frac{(\alpha_m-(\ell-1))!}{\alpha_m!} A_{\ell-1} \equiv \binom{\alpha_k+\ell-1}{\ell-1} \equiv \tilde{B}_\ell \pmod{p}. \]

So, \((*) \equiv (**)\).

If \( \delta_{ij} = 1 \), we have

\[ (*) = \left\{ \begin{array}{ll}
\frac{(\alpha_k+\ell)!}{\alpha_k!} \frac{(\alpha_m-\ell)!}{\alpha_m!} A_\ell D_{ik}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))}) \frac{(\alpha_k+\ell)!}{\alpha_k!} \frac{(\alpha_m-(\ell-1))!}{\alpha_m!} A_{\ell-1} D_{jm}(x^{(\alpha+(\ell-1)(\epsilon_k-\epsilon_m))}), & \text{for } 0 \leq \ell \leq \alpha_m+1, \\
0, & \text{for } \ell > \alpha_m+1.
\end{array} \right. \]

A simple computation shows that

\[ \frac{(\alpha_k+\ell)!}{\alpha_k!} \frac{(\alpha_m-\ell)!}{\alpha_m!} A_\ell \equiv \binom{\alpha_k+\ell}{\ell} \equiv \tilde{A}_\ell \pmod{p}, \text{ for } 0 \leq \ell \leq \alpha_m, \]

\[ \frac{(\alpha_k+\ell-1)!}{\alpha_k!} \frac{(\alpha_m-(\ell-1))!}{\alpha_m!} A_{\ell-1} \equiv \binom{\alpha_k+\ell-1}{\ell-1} \equiv \tilde{B}_\ell \pmod{p}, \text{ for } 0 \leq \ell \leq \alpha_m+1. \]
So, \((*) = (**).\)

If \(\delta_{ik} = \delta_{jk} = 0\), we have \((*) = \frac{(\alpha_k + \ell)! (\alpha_m - \ell)!}{\alpha_k! \alpha_m!} A_{\ell} D_{ij}(x^{(\alpha + \ell (e_k - e_m))})\), and

\[
\frac{(\alpha_k + \ell)! (\alpha_m - \ell)!}{\alpha_k! \alpha_m!} A_{\ell} \equiv \left(\frac{\alpha_k + \ell}{\ell}\right) = \bar{A}_\ell \quad \text{(mod } p\text{)}, \quad \text{for } 0 \leq \ell \leq \alpha_m, \\
\bar{B}_\ell \equiv 0 \quad \text{(mod } p\text{)}, \quad \text{for } 0 \leq \ell \leq \alpha_m + 1.
\]

So, \((*) = (**).\)

Therefore, we verify the formula (47).

Applying a similar argument to the antipode, we can get the formula (48).

This completes the proof. □

To describe \(u_{e,0}(S(n; 1))\) explicitly, we still need an auxiliary Lemma.

**Lemma 4.3.** Denote \(e = D_{mk}(x^{(2e_k)}), d^{(\ell)} = \frac{1}{\ell!} (\text{ad } e)^\ell\). Then

(i) \(d^{(\ell)}(D_{ij}(x^{(\alpha)})) = \bar{A}_\ell D_{ij}(x^{(\alpha + \ell (e_k - e_m))}) + \bar{B}_\ell (\delta_{ik} D_{jm} - \delta_{jk} D_{im})(x^{(\alpha + (\ell - 1)(e_k - e_m))}),\)

where \(\bar{A}_\ell, \bar{B}_\ell\) as in Theorem 4.2.

(ii) \(d^{(\ell)}(D_{ij}(x^{(e_i + e_j)})) = \delta_{i,0} D_{ij}(x^{(e_i + e_j)}) - \delta_{1,\ell} (\delta_{ik} - \delta_{jk} - \delta_{im} + \delta_{jm}) e.\)

(iii) \(d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) = \delta_{i,0} (D_{ij}(x^{(\alpha)}))^p - \delta_{1,\ell} (\delta_{ik} - \delta_{jk} - \delta_{im} + \delta_{jm}) e^p p.\)

**Proof.** We can get (ii) from the proof of Theorem 4.2.

(ii) Note that \(\bar{A}_0 = 1, \bar{B}_0 = 0.\) Using Theorem 4.2, for \(\delta_{im} + \delta_{jm} = 1,\) we obtain \(\bar{A}_1 = 1\) and \(\bar{B}_1 = 0;\)

for \(\delta_{im} + \delta_{jm} = 0,\) we obtain \(\bar{A}_1 = 0\) and \(\bar{B}_1 = 1.\) We have \(\bar{A}_\ell = \bar{B}_\ell = 0\) for \(\ell > 1.\) Therefore, in any case, we arrive at the result as desired.

(iii) From (11), we obtain that for \(0 \leq \alpha \leq \tau,\)

\[
d^{(1)}((D_{ij}(x^{(\alpha)}))^p) = \frac{1}{(\alpha)!} \left[ e, (D_{ij}(x^{(\alpha)}))^p \right] = \frac{1}{(\alpha)!} \left[ e, (x^{\alpha - e_i - e_j} (\alpha_j \partial_1 - \alpha_i \partial_j))^p \right]
\]

\[
= \frac{1}{(\alpha)!} \sum_{\ell=1}^{p} (-1)^\ell \binom{p}{\ell} (x^{\alpha - e_i - e_j} (\alpha_j \partial_1 - \alpha_i \partial_j))^{p-\ell} \cdot x^{\ell (e_i - e_j)} (a_{\ell} \partial_m - b_{\ell} (\alpha_j \partial_1 - \alpha_i \partial_j))
\]

\[
\equiv \frac{a_p}{\alpha!} x^{\alpha - e_i - e_j + \ell (e_i - e_j)} \partial_m \quad \text{(mod } p\text{)}
\]

\[
\equiv \begin{cases} 
- a_p e, & \text{if } \alpha = e_i + e_j \\
0, & \text{if } \alpha \neq e_i + e_j
\end{cases} \quad \text{(mod } J),
\]

where the last “\(\equiv\)” by using the identification with respect to modulo the ideal \(J\) as before, and \(a_p = \prod_{m=0}^{p-1} (\alpha_j \partial_1 - \alpha_i \partial_j)(e_k - e_m + m (e_i - e_j)), \)

\(b_{\ell} = \ell \partial_m (e_i - e_j) a_{\ell-1},\) and \(a_p = \delta_{ik} - \delta_{jk} - \delta_{im} + \delta_{jm},\) for \(\alpha = e_i + e_j.\)

Consequently, by the definition of \(d^{(\ell)},\) we get \(d^{(\ell)}((x^{(\alpha)} D_{ij}^p) = 0\) in \(u(S(n; 1))\)

for \(2 \leq \ell \leq p - 1\) and \(0 \leq \alpha \leq \tau.\) □

Based on Theorem 4.2 and Lemma 4.3, we arrive at

**Theorem 4.4.** Fix distinguished elements \(h = D_{kk'}(x^{(e_k + e_{k'})}), e = D_{mk}(x^{(2e_k)})(1 \leq k \neq k' \neq m \leq n),\) there exists a noncommutative and nonco-commutative Hopf
algebra (of horizontal type) \((\mathfrak{u}_{t,q}(\mathbb{S}(n;\underline{1})), m, \iota, \Delta, S, \varepsilon)\) over \(K[t]^{(q)}\) with the product undeformed, whose coalgebra structure is given by

\[
\Delta(D_{ij}(x^{(\alpha)})) = D_{ij}(x^{(\alpha)}) \otimes (1-\epsilon t)^{\alpha(k,k')}
\]
\[
+ \sum_{\ell=0}^{p-1} (-1)^{\ell} h^{(\ell)} \otimes (1-\epsilon t)^{-\ell} d^{(\ell)}(D_{ij}(x^{(\alpha)})) t^{\ell},
\]

\[
S(D_{ij}(x^{(\alpha)})) = -(1-\epsilon t)^{-\alpha(k,k')} \sum_{\ell=0}^{p-1} d^{(\ell)}(D_{ij}(x^{(\alpha)})) \cdot h^{(\ell)} t^{\ell},
\]

\[
\varepsilon(D_{ij}(x^{(\alpha)})) = 0,
\]

where \(0 \leq \alpha \leq \tau\) and \(\alpha(k,k') = \alpha_k - \delta_{jk} - \delta_{jk'} + \alpha_{k'} + \delta_{k} + \delta_{j} - \delta_{j''}\), which is finite dimensional with \(\dim K \mathfrak{u}_{t,q}(\mathbb{S}(n;\underline{1})) = p^{1+(n-1)(p^n-1)}\).

**Proof.** Utilizing the same arguments as in the proofs of Theorems 3.11 & 3.17, we shall show that the ideal \(I_{t,q}\) is a Hopf ideal of the twisted Hopf algebra \(U_{t,q}(\mathbb{S}(n;\underline{1}))\) as in Theorem 4.2. To this end, it suffices to verify that \(\Delta\) and \(S\) preserve the generators in \(I_{t,q}\).

(I) By Lemmas 3.4, 4.2 & 4.3 (iii), we obtain

\[
\Delta((D_{ij}(x^{(\alpha)}))^p) = (D_{ij}(x^{(\alpha)}))^p \otimes (1-\epsilon t)^p(\alpha - \alpha_{k'})
\]
\[
+ \sum_{\ell=0}^{\infty} (-1)^{\ell} h^{(\ell)} \otimes (1-\epsilon t)^{-\ell} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) t^{\ell}
\]
\[
\equiv (D_{ij}(x^{(\alpha)}))^p \otimes 1 + \sum_{\ell=0}^{p-1} (-1)^{\ell} h^{(\ell)} \otimes (1-\epsilon t)^{-\ell} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) t^{\ell} \pmod p
\]
\[
= (D_{ij}(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (D_{ij}(x^{(\alpha)}))^p
\]
\[
+ h \otimes (1-\epsilon t)^{-1}(\delta_{ik} - \delta_{jk} - \delta_{jm} + \delta_{jm}) \delta_{\alpha_1 + \epsilon_j} \epsilon t.
\]

Hence, when \(\alpha \neq \epsilon_i + \epsilon_j\), we get

\[
\Delta((D_{ij}(x^{(\alpha)}))^p) \equiv (D_{ij}(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (D_{ij}(x^{(\alpha)}))^p
\]
\[
\in I_{t,q} \otimes U_{t,q}(\mathbb{S}(n;\underline{1})) + U_{t,q}(\mathbb{S}(n;\underline{1})) \otimes I_{t,q}.
\]

When \(\alpha = \epsilon_i + \epsilon_j\), by Lemma 4.3 (ii), (47) becomes

\[
\Delta(D_{ij}(x^{(\epsilon_i + \epsilon_j)})) = D_{ij}(x^{(\epsilon_i + \epsilon_j)}) \otimes 1 + 1 \otimes D_{ij}(x^{(\epsilon_i + \epsilon_j)})
\]
\[
+ h \otimes (1-\epsilon t)^{-1}(\delta_{ik} - \delta_{jk} - \delta_{jm} + \delta_{jm}) \epsilon t.
\]

Combining with (53), we obtain

\[
\Delta((D_{ij}(x^{(\epsilon_i + \epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i + \epsilon_j)})) \equiv ((D_{ij}(x^{(\epsilon_i + \epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i + \epsilon_j)})) \otimes 1
\]
\[
+ 1 \otimes ((D_{ij}(x^{(\epsilon_i + \epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i + \epsilon_j)}))
\]
\[
\in I_{t,q} \otimes U_{t,q}(\mathbb{S}(n;\underline{1})) + U_{t,q}(\mathbb{S}(n;\underline{1})) \otimes I_{t,q}.
\]

Thereby, we prove that \(I_{t,q}\) is a coideal of the Hopf algebra \(U_{t,q}(\mathbb{S}(n;\underline{1}))\).
(II) By Lemmas 3.4, 4.2 & 4.3(iii), we have

\[ S((D_{ij}(x^{(\alpha)}))^p) = -(1-\epsilon t)^{-p(\alpha_k - \alpha_{k'})} \sum_{\ell=0}^{\infty} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) \cdot h^{(\ell)}_1 t^\ell \]

(54)

\[ \equiv -(D_{ij}(x^{(\alpha)}))^p - \sum_{\ell=1}^{p-1} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) \cdot h^{(\ell)}_1 t^\ell \pmod{p} \]

\[ = -(D_{ij}(x^{(\alpha)}))^p + (\delta_{ik} - \delta_{jk} - \delta_{im} - \delta_{jm}) \delta_{\alpha, \epsilon_i + \epsilon_j} e \cdot h^{(1)}_1 t. \]

Hence, when \( \alpha \neq \epsilon_i + \epsilon_j \), we get

\[ S((D_{ij}(x^{(\alpha)}))^p) = -(D_{ij}(x^{(\alpha)}))^p \in I_{t,q}. \]

When \( \alpha = \epsilon_i + \epsilon_j \), by Lemma 4.3 (ii), (48) reads as

\[ S(D_{ij}(x^{(\epsilon_i + \epsilon_j)})) = -(D_{ij}(x^{(\epsilon_i + \epsilon_j)}))^p + (\delta_{ik} - \delta_{jk} - \delta_{im} - \delta_{jm}) \delta_{\alpha, \epsilon_i + \epsilon_j} e \cdot h^{(1)}_1 t. \]

Combining with (54), we obtain

\[ S((D_{ij}(x^{(\epsilon_i + \epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i + \epsilon_j)})) = -(D_{ij}(x^{(\epsilon_i + \epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i + \epsilon_j)}) \in I_{t,q}. \]

Thereby, we show that \( I_{t,q} \) is preserved by the antipode \( S \) of \( U_{t,q}(\mathfrak{S}(n; 1)) \) as in Theorem 4.2.

(III) It is obvious to notice that \( \epsilon((D_{ij}(x^{(\alpha)}))^p) = 0 \) for all \( 0 \leq \alpha \leq \tau \).

So, \( I_{t,q} \) is a Hopf ideal in \( U_{t,q}(\mathfrak{S}(n; 1)) \). We get a finite-dimensional horizontal quantization on \( \mathfrak{u}_{\epsilon,q}(\mathfrak{S}(n; 1)) \). \( \square \)

4.2. Jordanian modular quantizations of \( \mathfrak{u}(\mathfrak{s}(n)) \). Let \( \mathfrak{u}(\mathfrak{s}(n)) \) denote the restricted universal enveloping algebra of \( \mathfrak{s}(n) \). Since Drinfeld twists \( \mathcal{F}(k, k'; m) \) of horizontal type act stably on the subalgebra \( U((\mathfrak{S}_k^m))[t] \), consequently on \( \mathfrak{u}_{t,q}(\mathfrak{S}(n; 1)_{0}) \), these give rise to the Jordanian quantizations on \( \mathfrak{u}_{t,q}(\mathfrak{s}(n)) \).

By Lemma 4.3 (i), we have

\[ d^{(\ell)}(D_{ij}(x^{(2\epsilon_i)})) = \delta_{\ell,0}D_{ij}(x^{(2\epsilon_i)}) + \delta_{1,\ell}(\delta_{jm}D_{ik}(x^{(2\epsilon_k)})) \]

\[ - \delta_{ik}D_{mj}(x^{(2\epsilon_j)}) + \delta_{jm}\delta_{ik}D_{km}(x^{(\epsilon_k + \epsilon_m)}) - \delta_{2,\ell}\delta_{jm}\delta_{ik}e. \]

By Theorem 4.2, we have

**Theorem 4.5.** Fix distinguished elements \( h = D_{kk'}(x^{(\epsilon_k + \epsilon'_k)}) \) (1 \( \leq k \neq k' \neq m \leq n \), the corresponding Jordanian quantization of \( \mathfrak{u}(\mathfrak{S}(n; 1)_{0}) \cong \mathfrak{u}(\mathfrak{s}(n)) \) over \( \mathfrak{u}_{t,q}(\mathfrak{S}(n; 1)_{0}) \cong \mathfrak{u}_{t,q}(\mathfrak{s}(n)) \) with the product undeformed, whose coalgebra
structure is given by

\[(55) \quad \Delta(D_{ij}(x^{(c_i + s_j)})) = D_{ij}(x^{(c_i + s_j)}) \otimes 1 + 1 \otimes D_{ij}(x^{(c_i + s_j)})
+ (\delta_{ik} - \delta_{jk} - \delta_{im} + \delta_{jm}) h \otimes (1 - et)^{-1} et,
\]

\[(56) \quad \Delta(D_{ij}(x^{(2c_s)})) = D_{ij}(x^{(2c_s)}) \otimes (1 - et)^{\delta_{jk} - \delta_{ik} - \delta_{im} + \delta_{jm}} + 1 \otimes D_{ij}(x^{(2c_s)})
- h \otimes (1 - et)^{-1} (\delta_{jm} D_{ik}(x^{(2c_s)}) - \delta_{ik} D_{mj}(x^{(2c_s)}) + \delta_{jm} \delta_{ik} D_{km}(x^{(c_k + c_m)})) t
- \delta_{jm} \delta_{ik} h^{(2)} \otimes (1 - et)^{-2} et^2,
\]

\[(57) \quad S(D_{ij}(x^{(c_i + s_j)})) = -D_{ij}(x^{(c_i + s_j)}) + (\delta_{ik} - \delta_{jk} - \delta_{im} + \delta_{jm}) ch_1 t,
\]

\[(58) \quad S(D_{ij}(x^{(2c_s)})) = -(1 - et)^{-\delta_{jk} - \delta_{ik} - \delta_{im} + \delta_{jm}} \cdot (D_{ij}(x^{(2c_s)}))
+ (\delta_{jm} D_{ik}(x^{(2c_s)}) - \delta_{ik} D_{mj}(x^{(2c_s)}) + \delta_{jm} \delta_{ik} D_{km}(x^{(c_k + c_m)})) h t
- \delta_{jm} \delta_{ik} ch_1^{(2)} t^2,
\]

\[(59) \quad \varepsilon(D_{ij}(x^{(c_i + s_j)})) = \varepsilon(D_{ij}(x^{(2c_s)})) = 0.
\]

for \(1 \leq i \neq j \leq n\).

Remark 4.6. As \(S(n, 1)_0 \cong \mathfrak{sl}_n\), which via the identification \(D_{ij}(x^{(c_i + s_j)})\) with \(E_{ii} - E_{jj}\) and \(D_{ij}(x^{(2c_s)})\) with \(E_{ii}\) for \(1 \leq i \neq j \leq n\), we get a Jordanian quantization for \(\mathfrak{sl}_n\), which has been discussed by Kulish et al (cf. [19], [20] etc.).

Corollary 4.7. Fix distinguished elements \(h = E_{kk} - E_{k'k'}, \ e = E_{km} (1 \leq k \neq k' \neq m \leq n)\), the corresponding Jordanian quantization of \(\mathfrak{u}(\mathfrak{sl}_n)\) over \(\mathfrak{u}_{e,q}(\mathfrak{sl}_n)\) with the product undeformed, whose coalgebra structure is given by

\[(60) \quad \Delta(E_{ii} - E_{jj}) = (E_{ii} - E_{jj}) \otimes 1 + 1 \otimes (E_{ii} - E_{jj})
+ (\delta_{ik} - \delta_{jk} - \delta_{im} + \delta_{jm}) h \otimes (1 - et)^{-1} et,
\]

\[(61) \quad \Delta(E_{ji}) = E_{ji} \otimes (1 - et)^{\delta_{jk} - \delta_{ik} - \delta_{im} + \delta_{jm}} + 1 \otimes E_{ji}
- h \otimes (1 - et)^{-1} (\delta_{jm} E_{ki} - \delta_{ik} E_{jm}) t
- \delta_{jm} \delta_{ik} h^{(2)} \otimes (1 - et)^{-2} et^2,
\]

\[(62) \quad S(E_{ii} - E_{jj}) = -(E_{ii} - E_{jj}) + (\delta_{ik} - \delta_{jk} - \delta_{im} + \delta_{jm}) ch_1 t,
\]

\[(63) \quad S(E_{ji}) = -(1 - et)^{-\delta_{jk} - \delta_{ik} - \delta_{im} + \delta_{jm}} \cdot (E_{ji} + (\delta_{jm} E_{ki} - \delta_{ik} E_{jm}) h t
- \delta_{jm} \delta_{ik} ch_1^{(2)} t^2),
\]

\[(64) \quad \varepsilon(E_{ii} - E_{jj}) = \varepsilon(E_{ji}) = 0.
\]

for \(1 \leq i \neq j \leq n\).

Example 4.8. For \(n = 3\), take \(h = E_{11} - E_{22}, \ e = E_{13}, \) and set \(h' = E_{22} - E_{33}, \ f = (1 - et)^{-1}\). By Corollary 4.7, we get a Jordanian quantization on \(\mathfrak{u}_{e,q}(\mathfrak{sl}_3)\) with the coproduct as follows (here we omit the antipode formulae which can be directly
written down from (62) & (63)):
\[
\Delta(h) = h \otimes f + 1 \otimes h,
\]
\[
\Delta(h') = h \otimes f + (h' - h) \otimes 1 + 1 \otimes h',
\]
\[
\Delta(e) = e \otimes f^{-1} + 1 \otimes e,
\]
\[
\Delta(E_{12}) = E_{12} \otimes f^{-2} + 1 \otimes E_{12},
\]
\[
\Delta(E_{21}) = E_{21} \otimes f^2 + (1 + h) \otimes E_{21} - h \otimes f E_{21} f^{-1},
\]
\[
\Delta(E_{31}) = E_{31} \otimes f + (1 + h) \otimes E_{31} - h \otimes f E_{31} f^{-1} - h^{(2)} \otimes f^2 e t^2,
\]
\[
\Delta(E_{23}) = E_{23} \otimes f + 1 \otimes E_{23},
\]
\[
\Delta(E_{32}) = E_{32} \otimes f^{-1} + (1 + h) \otimes E_{32} - h \otimes f E_{32} f^{-1},
\]
where \( \{f, h\} \) satisfying the relations: \([h, f] = f^2 - f, h^p = h, f^p = 1 \) generates the (finite-dimensional) Radford Hopf subalgebra (with \( f \) as a group-like element) over a field of characteristic \( p \).

Before concluding the paper, we prefer to propose the following interesting questions for further considerations.

**Open Question 1.** Assume \( K \) is an algebraically closed field with \( t, q \in K \). How many non-isomorphic Hopf algebra structures can be equipped on the universal restricted enveloping algebra \( u(S(n, 1)) \) or \( u(sl_n) \)?

**Open Question 2.** What are the conditions for \( u_{t,q}(S(n, 1)) \) or \( u_{t,q}(sl_n) \) to be a ribbon Hopf algebra (see \([17, 18]\) and references therein)?

**Open Question 3.** It might be interesting to consider the tensor product structures of representations for \( u_{t,q}(S(n, 1)) \) or \( u_{t,q}(sl_n) \). How do their tensor categories behave?

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