COMPARISON OF TWO EQUIVARIANT $\eta$-FORMS

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Abstract. In this paper, we define first the equivariant infinitesimal $\eta$-form, then we compare it with the equivariant $\eta$-form, modulo exact forms, by a locally computable form. As a consequence, we obtain the singular behavior of the equivariant $\eta$-form, modulo exact forms, as a function on the acting Lie group. This result extends Goette’s previous result to the most general case, and it plays an important role in our recent work on the localization of $\eta$-invariants.

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In order to find a well-defined index for a first order elliptic differential operator over an even-dimensional compact manifold with nonempty boundary, Atiyah-Patodi-Singer [1] introduced a global boundary condition which is particularly significant for applications. In this final index formula, the contribution from the boundary is given by the Atiyah-Patodi-Singer (APS) $\eta$-invariant associated with the restriction of the operator on the boundary. Formally, the $\eta$-invariant is equal to the number of positive eigenvalues of the self-adjoint operator minus the number of its negative eigenvalues. If the manifold admits a compact Lie group action, in [19], extending the APS index theorem [1], Donnelly proved a Lefschetz type formula for manifolds with boundary. The contribution of the boundary is expressed as the equivariant $\eta$-invariant $\eta_g$.

Note that the $\eta$-invariant and the equivariant $\eta$-invariant are well-defined for any compact manifold. In [21] Theorem 0.5], Goette studied the singularity of $\eta_g$ at $g = 1$ when the group action is locally free. He defined the equivariant infinitesimal $\eta$-invariant as a formal power series and express the singularity of $\eta_g$ at $g = 1$ as a locally computable term through the comparison of the equivariant infinitesimal $\eta$-invariant and the equivariant $\eta$-invariant.

In [13] [14], Bismut and Goette established the general comparison formulas for holomorphic analytic torsions and de Rham torsions. They use the analytic localization techniques developed by Bismut and Lebeau in [15] and developed new techniques to overcome the difficulty that the operators do not have lower bounds. In the holomorphic case [13] Theorem 0.1], besides the predictable Bott-Chern current, in the final formula, there is an exotic additive characteristic class of the normal bundle, which is closely related to the Gillet-Soulé R-genus [20] and Bismut’s equivariant extension [6]. In the real case [14] Theorem 0.1], in the final formula, besides the predictable Chern-Simons current, they discovered an exotic locally computable diffeomorphism invariant of the fixed point set so-called $V$-invariant. The mysterious $V$-invariant should be understood as a finite dimensional analogue of the real analytic (de Rham) torsion.

On the other hand, extending the work of Bismut-Freed’s holonomy theorem [12] on the determinant line bundle of a family of Dirac operators, Bismut and Cheeger [8] studied the adiabatic limit for a fibration of compact Spin manifolds and found that under the invertible assumption of the fiberwise Dirac operator, the adiabatic limit of the $\eta$-invariant of the associated Dirac operators on the total space is expressible in terms of a canonically constructed
differential form, \( \tilde{\eta} \), so-called Bismut-Cheeger \( \eta \)-form, on the base space. Later, Dai extended this result to the case when the kernels of the fiberwise Dirac operators form a vector bundle over the base manifold. The Bismut-Cheeger \( \eta \)-form, \( \tilde{\eta} \), is the family version of the \( \eta \)-invariant and its 0-degree part is just the APS \( \eta \)-invariant. The Bismut-Cheeger \( \eta \)-form appears naturally as the boundary contribution of the family index theorem for manifolds with boundary (cf. [9, 10, 24]).

When the fibration admits a fiberwise compact Lie group action, the Bismut-Cheeger \( \eta \)-form could be naturally extended to the equivariant \( \eta \)-form \( \tilde{\eta}_g \). Recently, the functoriality of equivariant \( \eta \)-forms with respect to the composition of two submersions was established in [24], which extends the previous work of Bunke-Ma for usual \( \eta \)-forms for flat vector bundles with duality, cf. [3, 30, 31, 32] for related works on holomorphic torsions.

In this paper, we use the analytic techniques of Bismut-Goette in [13] to define the equivariant infinitesimal Bismut-Cheeger \( \eta \)-form and prove a general comparison formula for the equivariant infinitesimal Bismut-Cheeger \( \eta \)-form and the equivariant Bismut-Cheeger \( \eta \)-form which extend the work of Goette [21]. In particular, we express the singularity of \( \tilde{\eta}_g \) modulo exact forms, at any \( g \in G \) as a locally computable differential form.

Let \( G \) be a compact Lie group with Lie algebra \( \mathfrak{g} \). We assume that \( G \) acts isometrically on an odd-dimensional compact oriented Riemannian manifold \( X \) and the \( G \)-action lifts on a Clifford module \( E \) over \( X \). In general, the equivariant APS \( \eta \)-invariant \( \eta_g \) is not a continuous function on \( g \in G \). In [21], Goette studied the singularity of the equivariant \( \eta \)-invariant \( \eta_g \) at \( g = 1 \). He defines a formal power series \( \eta_K \in \mathbb{C}[[\mathfrak{g}^*]] \) for \( K \in \mathfrak{g} \), called the equivariant infinitesimal \( \eta \)-invariant and shows that if the Killing vector field \( K^X \) induced by \( K \) has no zeroes on \( X \), for any \( N \in \mathbb{N} \), as \( 0 \neq t \to 0 \),

\[
[\eta_K]_N - \eta_{e^tK} = \mathcal{M}_{tK} + \mathcal{O}(t^N),
\]

where \( [\eta_K]_N \) is the part of the formal power series \( \eta_K \) with degree \( \leq N \) and \( \mathcal{M}_{tK} \) could be expressed precisely as a locally computable term. Moreover, there exist \( c_j(K) \in \mathbb{C} \) such that when \( t \to 0 \),

\[
\mathcal{M}_{tK} = \sum_{j=1}^{(\dim X+1)/2} c_j(K)t^{-j} + \mathcal{O}(t^0).
\]

It means that if the Killing vector field \( K^X \) is nowhere vanishing, the singular behavior of \( \eta_{e^tK} \) when \( t \to 0 \) could be computed as the integral of the local terms explicitly.

In this paper, we show first that \( \eta_{tK} \) is an analytic function on \( t \) for \( t \) small enough and for any \( 0 \neq K \in \mathfrak{g} \),

\[
\eta_{tK} - \eta_{e^tK} = \mathcal{M}_{tK}, \quad \text{for } t \neq 0 \text{ small enough}.
\]

In Theorem 0.2, we establish the most general version of (0.3), in particular, its family version.

Let’s explain in detail our result here. Let \( \pi : W \to B \) be a smooth submersion of smooth compact manifolds with fibers \( X \). Note that \( n = \dim X \) can be even or odd. Let \( TX = TM/B \) be the relative tangent bundle to the fibers \( X \). We assume that \( TX \) is oriented and that the compact Lie group \( G \) acts fiberwisely on \( W \) and as identity on \( B \) and preserves the orientation of \( TX \).
Let $g^{TX}$ be a $G$-invariant metric on $TX$. Let $(\mathcal{E}, h^\mathcal{E})$ be a Clifford module of $TX$ to the fibers $X$ and we assume that the $G$-action lifts on $(\mathcal{E}, h^\mathcal{E})$ and is compatible with the Clifford action. Let $\nabla^\mathcal{E}$ be a $G$-invariant Clifford connection on $(\mathcal{E}, h^\mathcal{E})$, i.e., $\nabla^\mathcal{E}$ is a $G$-invariant Hermitian connection on $(\mathcal{E}, h^\mathcal{E})$ and compatible with the Clifford action (see (1.19)). Let $D$ be the fiberwise Dirac operator associated with $(g^{TX}, \nabla^\mathcal{E})$ (see (1.20)).

We assume that the kernels $\text{Ker}(D)$ form a vector bundle over $B$.

Then for any $g \in G$, the equivariant $\eta$-form $\tilde{\eta}_g$ is well-defined (see Definition 1.4).

In the whole paper, if $n = \dim X$ is even, $\mathcal{E}$ is naturally $\mathbb{Z}_2$-graded by the chiral operator $\Gamma$ defined in (1.15) and the supertrace for $A \in \text{End}(\mathcal{E})$ is defined by $\text{Tr}_s[A] := \text{Tr}[\Gamma A]$; if dim $X$ is odd, $\mathcal{E}$ is ungraded. For $\sigma = \alpha \otimes A$ with $\alpha \in \Lambda(T^* B)$, $A \in \text{End} \mathcal{E}$, we define $\text{Tr}[\sigma] := \alpha \cdot \text{Tr}[A]$. We denote by $\text{Tr}^{\text{odd}}[\sigma]$ the odd degree part of $\text{Tr}[\sigma]$. Set

$$\text{(0.4)} \quad \tilde{\text{Tr}}[\sigma] = \begin{cases} \text{Tr}_s[\sigma] & \text{if } n = \dim X \text{ is even;} \\ \text{Tr}^{\text{odd}}[\sigma] & \text{if } n = \dim X \text{ is odd.} \end{cases}$$

For $\alpha \in \Omega^j(\mathbb{R} \times B)$, the space of $j$-th differential forms on $\mathbb{R} \times B$, set

$$\text{(0.5)} \quad \psi_{\mathbb{R} \times B}(\alpha) = \begin{cases} (2i\pi)^{-\frac{j}{2}} \cdot \alpha & \text{if } j \text{ is even;} \\ \pi^{-\frac{1}{2}} (2i\pi)^{-\frac{j+1}{2}} \cdot \alpha & \text{if } j \text{ is odd.} \end{cases}$$

Let $t$ be the coordinate of $\mathbb{R}$ in $\mathbb{R} \times B$. If $\alpha = dt \wedge \alpha_0 + \alpha_1$, with $\alpha_0, \alpha_1 \in \Lambda(T^* B)$, we denote by

$$[\alpha]^d := \alpha_0.$$

Let $\mathcal{L}_K$ be the infinitesimal action on $\mathcal{E}^\infty(W, \mathcal{E})$ induced by $K \in \mathfrak{g}$ (see (2.3)).

For $g \in G$, we denote by $Z(g) \subset G$ the centralizer subgroup of $g$ with Lie algebra $\mathfrak{z}(g)$. Let $W^g = \{ x \in W : gx = x \}$ be the fixed point set of $g$. Then the restriction of $\pi$ on $W^g$, $\pi|_{W^g} : W^g \rightarrow B$ is a fibration with compact fiber $X^g$.

Let $\mathcal{B}_t$ be the rescaled Bismut superconnection defined in (1.23). Let $d$ be the exterior differential operator.

The following result extends the equivariant infinitesimal $\eta$-invariant to the family case at any $g \in G$ (see (2.31), (2.32), (2.36), (2.37) and Definition 2.3).

Theorem 0.1. For any $g \in G$, there exists $\beta > 0$ such that if $K \in \mathfrak{z}(g)$ with $|K| < \beta$, the integral

$$\text{(0.7)} \quad \tilde{\eta}_{g, K} = -\int_0^{+\infty} \left\{ \psi_{\mathbb{R} \times B} \tilde{\text{Tr}} \left[ g \exp \left( - \left( \mathcal{B}_t + \frac{c(K^X)}{4\sqrt{t}} + dt \wedge \frac{\partial}{\partial t} \right)^2 - \mathcal{L}_K \right) \right] \right\}^d dt$$

is a well-defined differential form on $B$, which is called the equivariant infinitesimal $\eta$-form, and

$$\text{(0.8)} \quad d\tilde{\eta}_{g, K} = \begin{cases} \int_{X^g} \hat{A}_{g, K}(TX, \nabla^{TX}) \text{ch}_{g, K}(\mathcal{E}/\mathcal{S}, \nabla^\mathcal{E}) & \text{if } n \text{ is even;} \\ -\text{ch}_{g, K}(\text{Ker}(D), \nabla^{\text{Ker}(D)}) & \text{if } n \text{ is odd;} \\ \int_{X^g} \hat{A}_{g, K}(TX, \nabla^{TX}) \text{ch}_{g, K}(\mathcal{E}/\mathcal{S}, \nabla^\mathcal{E}) & \text{if } n \text{ is odd.} \end{cases}$$
Here $\hat{\eta}_{g,K}(\cdot)$ and $ch_{g,K}(\cdot)$ are equivariant infinitesimal versions of the $\hat{\eta}$-form and the Chern character form (cf. (2.13) and (2.16)). Moreover, for fixed $K \in \mathfrak{z}(g)$, $\tilde{\eta}_{g,K}$ depends analytically on $z \in \mathbb{C}$ for $|z|$ small enough.

If $B$ is a point and $g = 1$, $\tilde{\eta}_{g,K}$ as formal power series, is just the equivariant infinitesimal $\eta$-invariant $\eta_K$ in [21, Definition 0.4].

Let $\vartheta_K \in T^*X$ be the 1-form which is dual to $K_X$ by the metric $g_{TX}$. Let $d_K = d - 2i\pi i_{K_X}$.

Now we state the main result of this paper.

**Theorem 0.2.** For $g \in G$ and $K \in \mathfrak{z}(g)$, with $K \neq 0$ and $|K|$ small enough, modulo exact forms on $B$, we have

\[
\tilde{\eta}_{g,K} = \tilde{\eta}_{geK} + \mathcal{M}_{g,K},
\]

where $\mathcal{M}_{g,K}$ is a well-defined integral defined by

\[
\mathcal{M}_{g,K} = - \int_0^{+\infty} \int_{X^g} \frac{\vartheta_K}{2i\pi} \exp\left(\frac{v d_K \vartheta_K}{2i\pi}\right) \hat{\eta}_{g,K}(TX, \nabla^{TX}) ch_{g,K}(E/S, \nabla^E) dv,
\]

and there exists $\beta > 0$ such that $t^{[(\dim W^g+1)/2]} \mathcal{M}_{g,tK}$ is real analytic on $t \in \mathbb{R}$, $|t| \leq \beta$.

By Theorem 0.1, $\tilde{\eta}_{g,tK}$ is smooth near $t = 0$. Thus when $t \to 0$, modulo exact forms, the singular terms of $\tilde{\eta}_{geK}$ is the same as that of $-\mathcal{M}_{g,tK}$.

Note that the general comparison formula for the two versions of equivariant holomorphic analytic torsions is established in [13, Theorem 5.1], which is the model of our paper. All analytical tools in this paper are inspired of them with necessary modifications. For this problem on de Rham torsion forms, a comparison formula is stated in [14, Theorem 5.13].

The main result of this paper is announced in [26] and plays an important role in our recent work [27].

This paper is organized as follows. In Section 1 we recall the definition of the equivariant Bismut-Cheeger $\eta$-form. In Section 2 we state the family Kirillov formula and define the equivariant infinitesimal $\eta$-form, in particular, we establish Theorem 0.1 modulo some technical details. In Section 3 we prove that $\mathcal{M}_{g,tK}$ in (0.10) is well-defined and state our main result, Theorem 0.2. In Section 4 we state some intermediate results and prove Theorem 0.2. In Section 5 we give an analytic proof of the family Kirillov formula and the technical details to establish Theorem 0.1 following the lines of [13, §7]. For the convenience to compare the arguments here with those in [13, §7], especially how the extra terms for the family version appear, the structure of this section is formulated almost the same as in [13, §7]. In Section 6 we prove the intermediate results in Section 4 using the analytical techniques in [13, §8, §9].

From Remark 1.3, to simplify the presentation, in Sections 5, 6, we will assume that $TX^g$ is oriented.

**Notation:** we use the Einstein summation convention in this paper: when an index variable appears twice in a single term and is not otherwise defined, it implies summation of that term over all the values of the index.

We denote by $\lfloor x \rfloor$ the maximal integer not larger than $x$.

We denote by $d$ the exterior differential operator and $d^B$ when we like to insist the base manifold $B$. Let $\Omega^{\text{even/odd}}(B, \mathbb{C})$ be the space of even/odd degree complex valued differential forms on $B$. For real vector bundle $E$, we denote by $\dim E$ the real rank of $E$. 

In the whole paper, if $A$, $A'$ are $\mathbb{Z}_2$-graded algebras we will note $A \hat{\otimes} A'$ as the $\mathbb{Z}_2$-graded tensor product as in [2, §1.3]. If one of $A, A'$ is ungraded, we understand it as $\mathbb{Z}_2$-graded by taking its odd part as zero.

For the fibre bundle $\pi : W \to B$, we will often use the integration of the differential forms along the oriented fibres $X$ in this paper. Since the fibres may be odd dimensional, we must make precisely our sign conventions: for $\alpha \in \Omega^\bullet(B)$ and $\beta \in \Omega^\bullet(W)$, then

\begin{equation}
\int_X (\pi^* \alpha) \land \beta = \alpha \land \int_X \beta.
\end{equation}

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1. Equivariant $\eta$-forms

In this section, we recall the definition of the equivariant $\eta$-form in the language of Clifford modules. In Section 1.1, we recall the definition of the Clifford algebra. In Section 1.2, we explain the Bismut superconnection. In Section 1.3, we define the equivariant $\eta$-form for Clifford module.

1.1. Clifford algebras. Let $(V, \langle \cdot, \cdot \rangle)$ be a Euclidean space, such that $\dim V = n$, with orthonormal basis $\{e_i\}_{i=1}^n$. Let $c(V)$ be the Clifford algebra of $V$ defined by the relations

\begin{equation}
e_i e_j + e_j e_i = -2\delta_{ij}.
\end{equation}

To avoid ambiguity, we denote by $c(e_i)$ the element of $c(V)$ corresponding to $e_i$.

If $e \in V$, let $e^* \in V^*$ correspond to $e$ by the scalar product $\langle \cdot, \cdot \rangle$ of $V$. The exterior algebra $\Lambda V^*$ is a module of $c(V)$ defined by

\begin{equation}
c(e) \alpha = e^* \land \alpha - i_e \alpha
\end{equation}

for any $\alpha \in \Lambda V^*$, where $\land$ is the exterior product and $i$ is the contraction operator. The map $a \mapsto c(a) \cdot 1$, $a \in c(V)$, induces an isomorphism of vector spaces

\begin{equation}
\sigma : c(V) \to \Lambda V^*.
\end{equation}

1.2. Bismut superconnection. Let $\pi : W \to B$ be a smooth submersion of smooth compact manifolds with $n$-dimensional fibres $X$. Let $TX = TW/B$ be the relative tangent bundle to the fibres $X$.

Let $G$ be a compact Lie group acting on $W$ along the fibres $X$, that is, if $g \in G$, $\pi \circ g = \pi$. Then $G$ acts on $TW$ and on $TX$. Let $T^HW \subset TW$ be a $G$-equivariant horizontal subbundle, so that

\begin{equation}
TW = T^HW \oplus TX.
\end{equation}
Since $G$ is compact, such $T^H W$ always exists. Let $P^{TX} : TW \to TX$ be the projection associated with the splitting (1.4). Note that

$$T^H W \cong \pi^*TB.$$  

(1.5)

Let $g^{TX}$ be a $G$-invariant metric on $TX$. Let $g^{TB}$ be a Riemannian metric on $TB$. We equip $TW$ with the $G$-invariant metric via (1.4) and (1.5),

$$g^{TW} = \pi^*g^{TB} \oplus g^{TX}.$$  

(1.6)

Let $\nabla^{TW,L}$ (resp. $\nabla^{TB}$) be the Levi-Civita connection on $(TW,g^{TW})$ (resp. $(TB,g^{TB})$). Let $\nabla^{TX}$ be the connection on $TX$ defined by

$$\nabla^{TX} = P^{TX}\nabla^{TW,L}P^{TX}.$$  

(1.7)

It is $G$-invariant. Let $\nabla^{TW}$ be the $G$-invariant connection on $TW$, via (1.4) and (1.5),

$$\nabla^{TW} = \pi^*\nabla^{TB} \oplus \nabla^{TX}.$$  

(1.8)

Put

$$S = \nabla^{TW,L} - \nabla^{TW}.$$  

(1.9)

Then $S$ is a 1-form on $W$ with values in antisymmetric elements of $\text{End}(TW)$. Let $T$ be the torsion of $\nabla^{TW}$. By [5, (1.28)], if $U,V,Z \in TW$,

$$S(U)V - S(V)U + T(U,V) = 0,$$

$$2\langle S(U)V, Z \rangle + \langle T(U,V), Z \rangle + \langle T(Z,U), V \rangle - \langle T(V,Z), U \rangle = 0.$$  

(1.10)

If $U$ is a vector field on $B$, let $U^H$ be its lift in $T^H W$ and let $\mathcal{L}_{U^H}$ be the Lie derivative operator associated with the vector field $U^H$. One verifies easily that $\mathcal{L}_{U^H}$ acts on the tensor algebra of $TX$. In particular, if $U \in TB, (g^{TX})^{-1}\mathcal{L}_{U^H}g^{TX}$ defines a self-adjoint endomorphisms of $TX$. If $U,V$ are vector fields on $B$, from [7, Theorem 1.1],

$$T(U^H, V^H) = -P^{TX}[U^H, V^H],$$

and if $U \in TB, Z, Z' \in TX$,

$$T(U^H, Z) = \frac{1}{2}(g^{TX})^{-1}\mathcal{L}_{U^H}g^{TX}Z, \quad T(Z, Z') = 0.$$  

(1.11)

From (1.10) and (1.12), if $U \in TB, Z, Z' \in TX$, we have

$$\langle S(Z)Z', U^H \rangle = -\langle T(U^H, Z), Z' \rangle = -\langle T(U^H, Z'), Z \rangle.$$  

(1.13)

We recall some properties in [7, §1.1].

**Proposition 1.1.** 1) The connection $\nabla^{TX}$ does not depend on $g^{TB}$ and on each fiber $X$, it restricts to the Levi-Civita connection of $(TX,g^{TX})$.

2) If $U \in TB$, then

$$\nabla^{TX}_{U^H} = \mathcal{L}_{U^H} + \frac{1}{2}(g^{TX})^{-1}\mathcal{L}_{U^H}g^{TX}.$$  

(1.14)

3) The tensors $T$ and $\langle S(\cdot), \cdot \rangle$ do not depend on $g^{TB}$.
Let $c(TX)$ be the Clifford algebra bundle of $(TX, g^{TX})$, whose fiber at $x \in W$ is the Clifford algebra $c(T_xX)$ of the Euclidean space $(T_xX, g^{T_xX})$. Let $\mathcal{E}$ be a Clifford module of $c(TX)$. It means that $\mathcal{E}$ is a vector bundle and restricted on a fiber, $\mathcal{E}_x$ is a representation of $c(T_xX)$. We assume that the $G$-action lifts on $\mathcal{E}$ and commutes with the Clifford action.

From now on, we assume that $TX$ is $G$-equivariant oriented.

In the whole paper, if $n$ is even, as in [2], Lemma 3.17], for a locally oriented orthonormal frame $e_1, \ldots, e_n$ of $TX$, we define the chirality operator by
\begin{equation}
\Gamma = x^{n/2}c(e_1) \cdots c(e_n).
\end{equation}
Then $\Gamma$ does not depend on the choice of the frame, commutes with the $G$-action and $\Gamma^2 = \text{Id}$. Thus $\mathcal{E}$ is naturally $\mathbb{Z}_2$-graded by the chiral operator $\Gamma$. The supertrace for $A \in \text{End}(\mathcal{E})$ is defined by
\begin{equation}
\text{Tr}_s[A] := \text{Tr}[\Gamma A].
\end{equation}
If $n$ is odd, $\mathcal{E}$ is ungraded.

Let $h^\mathcal{E}$ be a $G$-invariant Hermitian metric on $\mathcal{E}$. For $b \in B$, let $\mathbb{E}_b$ be the set of smooth sections over $X_b = \pi^{-1}(b)$ of $\mathcal{E}|_{X_b}$. As in [3], we will regard $\mathbb{E}$ as an infinite dimensional fiber bundle over $B$. Let $dv_X(x)$ be the Riemannian volume element of $X_b$. The bundle $\mathbb{E}_b$ is naturally endowed with the Hermitian product
\begin{equation}
\langle s, s' \rangle_0 = \int_{X_b} \langle s, s' \rangle(x) dv_X(x), \quad \text{for} \ s, s' \in \mathbb{E}.
\end{equation}
Then $G$ acts on $\mathbb{E}_b = \mathcal{C}^{\infty}(X_b, \mathcal{E}|_{X_b})$ as
\begin{equation}
(g.s)(x) = g(s(g^{-1}x)) \quad \text{for any} \ g \in G.
\end{equation}
Let $\nabla^\mathcal{E}$ be a $G$-invariant Clifford connection on $\mathcal{E}$ (cf. [2] §10.2]), that is, $\nabla^\mathcal{E}$ is $G$-invariant, preserves $h^\mathcal{E}$ and for any $U \in TW$, $Z \in \mathcal{C}^{\infty}(W, TX)$,
\begin{equation}
[\nabla^\mathcal{E}_U, c(Z)] = c(\nabla^{TX}_U Z).
\end{equation}
The fiberwise Dirac operator is defined by
\begin{equation}
D = \sum_{i=1}^n c(e_i)\nabla^\mathcal{E}_{e_i}.
\end{equation}
Let $k \in (T^H W)^*$ such that for any $U \in TB$, $\mathcal{L}_{U^H} dv_X(x)/dv_X(x) = 2k(U^H)(x)$. The connection $\nabla^{\mathcal{E},u}$ on $\mathcal{E}$ defined by (cf. [11], Definition 1.3])
\begin{equation}
\nabla^{\mathcal{E},u}_U s := \nabla^\mathcal{E}_{U^H} s + k(U^H)s \quad \text{for} \ s \in \mathcal{C}^{\infty}(B, \mathcal{E}) = \mathcal{C}^{\infty}(W, \mathcal{E}),
\end{equation}
is $G$-invariant and preserves the $G$-invariant $L^2$-product ([11] (see e.g., [11], Proposition 1.4]).

Let $\{f_p\}$ be a local frame of $TB$ and $\{f^p\}$ be its dual. Set
\begin{equation}
\nabla^{\mathcal{E},u} = f^p \wedge \nabla^{\mathcal{E},u}_{f_p}, \quad c(T^H) = \frac{1}{2} \left( T(f^p, f^q) \right) f^p \wedge f^q \wedge .
\end{equation}
Then $c(T^H)$ is a section of $\pi^* \Lambda^2 (T^* B) \otimes \text{End}(\mathcal{E})$.

By [3] (3.18)), the rescaled Bismut superconnection $\mathbb{B}_u, u > 0$, is defined by
\begin{equation}
\mathbb{B}_u = \sqrt{u}D + \nabla^{\mathcal{E},u} - \frac{1}{4\sqrt{u}} c(T^H) : \mathcal{C}^{\infty}(B, \Lambda(T^* B) \otimes \mathbb{E}) \to \mathcal{C}^{\infty}(B, \Lambda(T^* B) \otimes \mathbb{E}).
\end{equation}
Obviously, the Bismut superconnection $\mathcal{B}_u$ commutes with the $G$-action. Furthermore, $\mathbb{B}_u^2$ is a 2-order elliptic differential operator along the fibers $X$ (cf. [5 (3.4)]) acting on $\Lambda(T^*B) \otimes \mathcal{E}$. Let $\exp(-\mathbb{B}_u^2)$ be the family of heat operators associated with the fiberwise elliptic operator $\mathbb{B}_u^2$.

1.3. **Equivariant \( \eta \)-forms.** Take $g \in G$ fixed and set $W^g = \{ x \in W : gx = x \}$, the fixed point set of $g$. Then $W^g$ is a submanifold of $W$ and $\pi|_{W^g} : W^g \rightarrow B$ is a fibration with compact fibers $X^g$. Let $N_{W^g/W}$ denote the normal bundle of $W^g$ in $W$, then

\[
(1.24) \quad N_{W^g/W} := \frac{TW}{TW^g} = \frac{TX}{TX^g} =: N_{X^g/X}.
\]

Let $\{X^g_\alpha\}_{\alpha \in \mathbb{B}}$ be the connected components of $X^g$ with \n
\[
(1.25) \quad \dim X^g_\alpha = \ell_\alpha.
\]

By an abuse of notation, we will often simply denote by all $\ell_\alpha$ the same $\ell$.

**Assumption 1.2.** We assume that the kernels $\ker(D)$ form a vector bundle over $B$.

For $\sigma = \alpha \otimes A$ with $\alpha \in \Lambda(T^*B)$, $A \in \text{End}(\mathcal{E})$, we define

\[
(1.26) \quad Tr(\sigma) = \alpha \cdot Tr[A], \quad Tr^{\text{odd}}(\sigma) = \{\alpha\}^{\text{odd}} \cdot Tr[A], \quad Tr^{\text{even}}(\sigma) = \{\alpha\}^{\text{even}} \cdot Tr[A],
\]

where $\{\alpha\}^{\text{odd/even}}$ is the odd or even degree part of $\alpha$. Set

\[
(1.27) \quad \overline{Tr}[\sigma] = \begin{cases} 
\Tr_\alpha[\sigma] := \alpha \cdot \Tr[\Gamma A] & \text{if } n = \dim X \text{ is even}; \\
\Tr^{\text{odd}}[\sigma] & \text{if } n = \dim X \text{ is odd}.
\end{cases}
\]

Let $\text{End}_{\xi(TX)}(\mathcal{E})$ be the set of endomorphisms of $\mathcal{E}$ supercommuting with the Clifford action. It is a vector bundle over $W$. As in [2 Definition 3.28], we define the relative trace $\text{Tr}^{\mathcal{E}/S} : \text{End}_{\xi(TX)}(\mathcal{E}) \rightarrow \mathbb{C}$ by: for any $A \in \text{End}_{\xi(TX)}(\mathcal{E})$,

\[
(1.28) \quad \text{Tr}^{\mathcal{E}/S}[A] = \begin{cases} 
2^{-n/2} \text{Tr}_\alpha[\Gamma A] & \text{if } n = \dim X \text{ is even}; \\
2^{-(n-1)/2} \text{Tr}[A] & \text{if } n = \dim X \text{ is odd}.
\end{cases}
\]

Let $R^TX = (\nabla^TX)^2$, $R^\mathcal{E} = (\nabla^\mathcal{E})^2$ be the curvatures of $\nabla^TX$, $\nabla^\mathcal{E}$. Then

\[
(1.29) \quad R^{\mathcal{E}/S} := R^\mathcal{E} - \frac{1}{4} (R^TX_{e_i,e_j} c(e_i) c(e_j) \in \mathcal{E}^\infty(W, \Lambda^2(T^*W) \otimes \text{End}_{\xi(TX)}(\mathcal{E}))
\]

is the twisting curvature of the Clifford module $\mathcal{E}$ as in [2 Proposition 3.43].

Note that if $TX$ has a $G$-equivariant spin structure, then there exists a $G$-equivariant Hermitian vector bundle $E$ such that $\mathcal{E} = S_X \otimes E$, with $S_X$ the spinor bundle of $TX$, $\nabla^\mathcal{E}$ is induced by $\nabla^TX$ and a $G$-invariant Hermitian connection $\nabla^E$ on $E$ and

\[
(1.30) \quad R^{\mathcal{E}/S} = R^E.
\]

We denote the differential of $g$ by $dg$ which gives a bundle isometry $dg : N_{X^g/X} \rightarrow N_{X^g/X}$. As $G$ is compact, we know that there is an orthonormal decomposition of real vector bundles over $W^g$,

\[
(1.31) \quad TX|_{W^g} = TX^g \oplus N_{X^g/X} = TX^g \oplus \bigoplus_{0<\theta<\pi} N(\theta),
\]

where $dg|_{N(\pi)} = -\text{Id}$ and for each $\theta$, $0 < \theta < \pi$, $N(\theta)$ is the underline real vector bundle of a complex vector bundle $N_\theta$ over $W^g$ on which $dg$ acts by multiplication by $e^{i\theta}$. Since $g$ preserves
the metric and the orientation of $TX$, thus $\det(dg|_{N(\sigma)}) = 1$, this means $\dim N(\pi)$ is even. So the normal bundle $N_{X^g/X}$ is even dimensional.

Since $\nabla^{TX}$ commutes with the group action, its restriction on $W^g$, $\nabla^{TX}|_{W^g}$, preserves the decomposition (1.31). Let $\nabla^{TX^g}$ and $\nabla^{N(\theta)}$ be the corresponding induced connections on $TX^g$ and $N(\theta)$, with curvatures $R^{TX^g}$ and $R^{N(\theta)}$.

Set

$$
\widehat{A}_g(TX, \nabla^{TX}) = \det^{\frac{i}{2}} \left( \frac{1}{4\pi} R^{TX^g} \sinh \left( \frac{i}{4\pi} R^{TX^g} \right) \right) \prod_{0 < \theta \leq \pi} \left( i^{\frac{1}{2} \dim N(\theta)} \det^{\frac{1}{2}} \left( 1 - g \exp \left( \frac{i}{2\pi} R^{N(\theta)} \right) \right) \right)^{-1} \in \Omega^2(W^g, \mathbb{C}).
$$

The sign convention in (1.32) is that the degree 0 part in $\prod_{0 < \theta \leq \pi}$ is given by $\left( \frac{\exp(\theta/2)}{e^{-\theta/2} - 1} \right)^{\frac{1}{2} \dim N(\theta)}$.

By [2, Lemma 6.10], along $W^g$, the action of $g \in G$ on $E$ may be identified with a section $g^\xi$ of $c(N_{X^g/X}) \otimes \text{End}_c(TX)(E)$. Under the isomorphism (1.3), $\sigma(g^\xi) \in \mathcal{C}^\infty(W^g, \Lambda(N_{X^g/X}) \otimes \text{End}_c(TX)(E))$. Let $\sigma_{n-\ell}(g^\xi) \in \mathcal{C}^\infty(W^g, \Lambda^{n-\ell}(N_{X^g/X}) \otimes \text{End}_c(TX)(E))$ be the highest degree part of $\sigma(g^\xi)$ in $\Lambda(N_{X^g/X})$. Then we define the localized relative Chern character $\chi_g(E/S, \nabla^E)$ as in [2, Definition 6.13]:

$$
\chi_g(E/S, \nabla^E) := \frac{2(n-\ell)/2}{\det^{1/2}(1 - g|_{N_{X^g/X}})} \text{Tr}^E \left[ \sigma_{n-\ell}(g^\xi) \exp \left( -\frac{R^{E/S}|_{W^g}}{2i\pi} \right) \right] \in \Omega^* (W^g, \det(N_{X^g/X})).
$$

**Remark 1.3.** In general, $TX^g$ is not necessary oriented. By using the Berezin integral and the orientation of $TX$, the integral $\int_{X^g}$ of a form in $\Omega^*(W^g, \det(N_{X^g/X}))$ makes sense as in [2, Theorem 6.16]. Assume that $TX^g$ is oriented, then the orientations of $TX^g$ and $TX$ induce canonically the orientation on $N_{X^g/X}$. By paring with the volume form of $N_{X^g/X}$, we obtain

$$
\chi_g(E/S, \nabla^E) \in \Omega^*(W^g, \mathbb{C}).
$$

If $TX$ has a $G$-equivariant spin structure, then $TX^g$ is canonically oriented (cf. [2, Proposition 6.14], [29, Lemma 4.1]). If $TX$ has a $G$-equivariant spin structure, $\chi_g(E/S, \nabla^E)$ under the above convention is just the usual equivariant Chern character (cf. (1.30))

$$
\chi_g(E, \nabla^E) = \text{Tr}^E \left[ g \exp \left( -\frac{R^E}{2i\pi} \right) \right].
$$

As in (0.4), for $\alpha \in \Omega^j(B)$, set

$$
\psi_B(\alpha) = \begin{cases} (2i\pi)^{-\frac{j}{2}} \cdot \alpha & \text{if } j \text{ is even;} \\ \pi^{-1/2} (2i\pi)^{-\frac{j-1}{2}} \cdot \alpha & \text{if } j \text{ is odd.} \end{cases}
$$

Then from the equivariant family local index theorem (see e.g., [3, Theorem 4.17], [12, Theorem 2.10], [25, Theorem 1.2], [28, Theorem 1.3]), for any $u > 0$, the differential form $\psi_B \widetilde{\text{Tr}}[g \exp(-B^2_u)] \in \Omega^*(B, \mathbb{C})$ is closed, its cohomology class is independent of $u > 0$, and

$$
\lim_{u \to 0} \psi_B \widetilde{\text{Tr}}[g \exp(-B^2_u)] = \int_{X^g} \widehat{A}_g(TX, \nabla^{TX}) \chi_g(E/S, \nabla^E).
$$
Let \( P^{\operatorname{Ker}(D)} : E \to \operatorname{Ker}(D) \) be the orthogonal projection with respect to (1.17). Let
\[
\nabla^{\operatorname{Ker}(D)} = P^{\operatorname{Ker}(D)} \nabla_{E,u} P^{\operatorname{Ker}(D)}
\]
and \( R^{\operatorname{Ker}(D)} \) be the curvature of the connection \( \nabla^{\operatorname{Ker}(D)} \) on \( \operatorname{Ker}(D) \).

If \( n = \dim X \) is even, from the natural equivariant extension of [2] Theorem 9.19, we have
\[
\lim_{u \to +\infty} \psi_B \operatorname{Tr}_s \left[ g \exp(-B^2_u) \right] = \operatorname{Tr}_s \left[ g \exp \left( -\frac{R^{\operatorname{Ker}(D)}}{2i\pi} \right) \right] = \operatorname{ch}_g(\operatorname{Ker}(D), \nabla^{\operatorname{Ker}(D)}).
\]

Since \( B_u \) is \( G \)-invariant, the equivariant version of [2] Theorem 9.17 shows that
\[
\frac{\partial}{\partial u} \operatorname{Tr}_s \left[ g \exp(-B^2_u) \right] = -d^B \operatorname{Tr}_s \left[ g \frac{\partial B_u}{\partial u} \exp(-B^2_u) \right].
\]

Thus for \( 0 < \varepsilon < T < +\infty \),
\[
\operatorname{Tr}_s \left[ g \exp(-B^2_\varepsilon) \right] - \operatorname{Tr}_s \left[ g \exp(-B^2_T) \right] = d^B \int_\varepsilon^T \operatorname{Tr}_s \left[ g \frac{\partial B_u}{\partial u} \exp(-B^2_u) \right] du.
\]

The natural equivariant extension of [2] Theorems 9.23 and 10.32(1) (cf. e.g., [24] (2.72) and (2.77)) shows that
\[
\operatorname{Tr}_s \left[ g \exp(-B^2_u) \right] = \mathcal{O}(u^{-1/2}) \quad \text{as } u \to 0,
\]
\[
\operatorname{Tr}_s \left[ g \frac{\partial B_u}{\partial u} \exp(-B^2_u) \right] = \mathcal{O}(u^{-3/2}) \quad \text{as } u \to +\infty.
\]

In this case, by (1.36) and (1.42), the equivariant \( \eta \)-form is defined by
\[
\tilde{\eta}_g = \int_0^{+\infty} \frac{1}{2i\sqrt{\pi}} \psi_B \operatorname{Tr}_s \left[ g \frac{\partial B_u}{\partial u} \exp(-B^2_u) \right] du \in \Omega^{\text{odd}}(B, \mathbb{C}).
\]

By (1.37), (1.39), (1.41) and (1.43), we have
\[
d^B \tilde{\eta}_g = \int_{X^g} \tilde{\Lambda}_g(TX, \nabla^{TX}) \operatorname{ch}_g(E/S, \nabla^E) - \operatorname{ch}_g(\operatorname{Ker}(D), \nabla^{\operatorname{Ker}(D)}).
\]

If \( n \) is odd, since the equivariant extension of [2] Theorem 9.19 also holds, we have
\[
\lim_{u \to +\infty} \operatorname{Tr}^{\text{odd}}[g \exp(-B^2_u)] = \operatorname{Tr}^{\text{odd}}[g \exp(-R^{\operatorname{Ker}(D)})] = 0.
\]

As an analogue of (1.41), for \( 0 < \varepsilon < T < +\infty \), we have
\[
\operatorname{Tr}^{\text{odd}}[g \exp(-B^2_\varepsilon)] - \operatorname{Tr}^{\text{odd}}[g \exp(-B^2_T)] = d^B \int_\varepsilon^T \operatorname{Tr}^{\text{even}} \left[ g \frac{\partial B_u}{\partial u} \exp(-B^2_u) \right] du.
\]

Following the same arguments in the proof of (1.42), we have
\[
\operatorname{Tr}^{\text{even}} \left[ g \frac{\partial B_u}{\partial u} \exp(-B^2_u) \right] = \mathcal{O}(u^{-1/2}) \quad \text{as } u \to 0,
\]
\[
\operatorname{Tr}^{\text{even}} \left[ g \frac{\partial B_u}{\partial u} \exp(-B^2_u) \right] = \mathcal{O}(u^{-3/2}) \quad \text{as } u \to +\infty.
\]

In this case, by (1.36) and (1.47), the equivariant \( \eta \)-form is defined by
\[
\tilde{\eta}_g = \int_0^{+\infty} \frac{1}{\sqrt{\pi}} \psi_B \operatorname{Tr}^{\text{even}} \left[ g \frac{\partial B_u}{\partial u} \exp(-B^2_u) \right] du \in \Omega^{\text{even}}(B, \mathbb{C}).
\]
From (1.37), (1.45), (1.46) and (1.48), we get

\[ d B \tilde{\eta}_g = \int_{X^s} \tilde{\alpha}_g(TX, \nabla^{TX}) ch_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}). \]  

(1.49)

We write the definition of the equivariant \( \eta \)-form (1.43) and (1.48) in a uniform way using the notation \( \{ \cdot \}^{du} \) as in (0.6).

**Definition 1.4.** [24, Definition 2.3] For \( g \in G \) fixed, under Assumption 1.2, the equivariant Bismut-Cheeger \( \eta \)-form is defined by

\[ \tilde{\eta}_g := - \int_0^{+\infty} \left\{ \psi_{R \times B} \tilde{\text{Tr}} \left[ g \exp \left( - \left( B_u + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right] \right\}^{du} \in \Omega^*(B, \mathbb{C}). \]  

(1.50)

If \( g = 1 \), (1.50) is exactly the Bismut-Cheeger \( \eta \)-form defined in [8]. If \( B \) is noncompact, (1.42) and (1.47) hold uniformly on any compact subset of \( B \), thus Definition 1.4, (1.44) and (1.49) still hold.

2. **EQUIVARIANT INFINITESIMAL \( \eta \)-FORMS**

In this section, we state the family Kirillov formula and define the equivariant infinitesimal \( \eta \)-form. In Section 2.1, we state the family version of the Kirillov formula. In Section 2.2, we define the equivariant infinitesimal \( \eta \)-form, and establish Theorem 0.1 modulo some technical details.

In this section, we use the same notations and assumptions in Section 1. Especially, \( TX \) is \( G \)-equivariant oriented and Assumption 1.2 holds in this section.

2.1. **Moment maps and the family Kirillov formula.** Let \( | \cdot | \) be a \( G \)-invariant norm on the Lie algebra \( g \) of \( G \). For \( K \in g \), let

\[ K^X(x) = \frac{\partial}{\partial t} \bigg|_{t=0} e^{tK} \cdot x \quad \text{for} \quad x \in W \]  

(2.1)

be the vector field induced by \( K \) on \( W \). Since \( G \) acts fiberwisely on \( W \), \( K^X \in \mathcal{C}^{\infty}(W, TX) \) and

\[ [K^X, K'^X] = -[K, K'^X] \quad \text{for any} \quad K, K' \in g. \]  

(2.2)

For \( K \in g \), let \( \mathcal{L}_K \) be its Lie derivative acting on varies spaces: for \( s \in \mathcal{C}^{\infty}(W, \mathcal{E}) \) (cf. (1.18))

\[ \mathcal{L}_K s = \frac{\partial}{\partial t} \bigg|_{t=0} (e^{-tK} \cdot s). \]  

(2.3)

We define the moments \( m^{TX} (\cdot) \), \( m^{\mathcal{E}} (\cdot) \) by: for \( K \in g \),

\[ m^{TX}(K) := \nabla^{TX}_{K^X} - \mathcal{L}_K \in \mathcal{C}^{\infty}(W, \text{End}(TX)), \]

\[ m^{\mathcal{E}}(K) := \nabla^{\mathcal{E}}_{K^X} - \mathcal{L}_K \in \mathcal{C}^{\infty}(W, \text{End}(\mathcal{E})). \]  

(2.4)

Since the vector field \( K^X \) is Killing and \( \nabla^{TX} \), \( \nabla^{\mathcal{E}} \) preserve the corresponding metrics, \( m^{TX}(K) \) and \( m^{\mathcal{E}}(K) \) are skew-adjoint actions of \( \text{End}(TX) \) and \( \text{End}(\mathcal{E}) \) respectively. By Proposition 1.1, the connection \( \nabla^{TX} \) is the Levi-Civita connection of \( (TX, g^{TX}) \) when it is restricted on a fiber. Since the \( G \)-action is along the fiber, we have

\[ m^{TX}(K) = \nabla^{TX}_{K^X} K^X \in \mathcal{C}^{\infty}(W, \text{End}(TX)). \]  

(2.5)
Since the connection $\nabla^{TX}$ is $G$-invariant, from (2.4) (cf. [2, (7.4)]),

\[
\nabla^{TX} m^{TX}(K) + i_K R^{TX} = 0.
\]

We denote by $m^S(K)$ on $E$ by

\[
m^S(K) := \frac{1}{4} \langle m^{TX}(K)e_i, e_j \rangle c(e_i)c(e_j).
\]

If $TX$ is spin, $m^S(K)$ is just the moment of the spinor. Set

\[
m^{E/S}(K) := m^E(K) - m^S(K).
\]

From (1.29), we set

\[
R^{TX}_K = R^{TX} - 2i\pi m^{TX}(K), \quad R^{E/S}_K = R^{E/S} - 2i\pi m^{E/S}(K).
\]

Then $R^{TX}_K$ (resp. $R^{E/S}_K$) is called the equivariant curvature of $TX$ (resp. equivariant twisted curvature of $E$).

Let $Z(g) \subset G$ be the centralizer of $g \in G$ with Lie algebra $\mathfrak{z}(g)$. Then in the sense of the adjoint action,

\[
\mathfrak{z}(g) = \{ K \in g : g.K = K \}.
\]

We fix $g \in G$ from now on. In the sequel, we always take $K \in \mathfrak{z}(g)$. Put

\[
W^K = \{ x \in W : K^X(x) = 0 \}.
\]

Then $W^K$, which is the fixed point set of the group generated by $K$, is a totally geodesic submanifold along each fiber $X$. Set

\[
W^{g,K} = W^g \cap W^K.
\]

Then $W^{g,K}$ is also a totally geodesic submanifold along each fiber $X$. Moreover, if $K_0 \in \mathfrak{z}(g)$ and $z \in \mathbb{R}$, for $z$ small enough, we have

\[
W^{g,zK_0} = W^{ge^{zK_0}}.
\]

Since the $G$-action is trivial on $B$, $W^K \to B$ and $W^{g,K} \to B$ are fibrations with compact fibers $X^K$ and $X^{g,K}$. As in (1.25), by an abuse of notation, we will often simply denote by

\[
\dim X^{g,K} = \ell'.
\]

Observe that $m^{TX}(K)|_{X^g}$ acts on $TX^g$ and $N_{X^g/X}$. Also it preserves the splitting (1.31). Let $m^{TX^g}(K)$ and $m^{N(\theta)}(K)$ be the restrictions of $m^{TX}(K)|_{X^g}$ to $TX^g$ and $N(\theta)$ respectively. We define the corresponding equivariant curvatures $R^{TX^g}_K$, $R^{N(\theta)}_K$ as in (2.9).

For $K \in \mathfrak{z}(g)$ with $|K|$ small enough, comparing with (1.32), set

\[
\hat{A}_{g,K}(TX, \nabla^{TX}) = \det \left( \frac{1}{4\pi R^{TX^g}_K} \right) \left( \prod_{k>0} \left( i^{2\dim N(\theta)} \det \left( 1 - g \exp \left( \frac{i}{2\pi R^{N(\theta)}_K} \right) \right) \right) \right)^{-1} \in \Omega^{2\ast}(W^g, \mathbb{C}).
\]
Note that $W$ compact and $|K|$ small guarantee that the denominator in (2.15) is invertible. Comparing with (1.33), set

$$
\text{ch}_{g,K}(E/S, \nabla^E) := \frac{2^{(n-\ell)/2}}{\det^{1/2}(1 - g|_{X_g/X})} \text{Tr}^{E/S} \left[ \sigma_{n-\ell}(g^E) \exp \left( -\frac{R_E^g |W^g|}{2i\pi} \right) \right].
$$

As in (1.35), if $TX$ has a $G$-equivariant spin structure, $\text{ch}_{g,K}(E/S, \nabla^E)$ is just the equivariant infinitesimal Chern character in [13, Definition 2.7],

$$
\text{ch}_{g,K}(E, \nabla^E) = \text{Tr}^E \left[ g \exp \left( -\frac{R_E^g |W^g|}{2i\pi} \right) \right] \in \Omega^{2\bullet}(W^g, \mathbb{C}),
$$

where $m^E(K) = \nabla^E_{KX} - L_K$, $R_E^K := R_E - 2i\pi m^E(K)$ as in (2.4) and (2.9).

Set

$$
d_K = d - 2i\pi i_{KX}.
$$

Then by (2.6) (cf. [2, Theorem 7.7]),

$$
d_K \hat{A}_{g,K}(TX, \nabla^{TX}) = 0, \quad d_K \text{ch}_{g,K}(E/S, \nabla^E) = 0.
$$

Recall that $\mathbb{B}_t$ is the rescaled Bismut superconnection in (1.23). Set

$$
\mathbb{B}_{K,t} = \mathbb{B}_t + \frac{c(K^X)}{4\sqrt{t}},
$$

Then $\mathbb{B}_{K,t}^2$ is a 2-order elliptic differential operator along the fibers $X$ acting on $\Lambda(T^*B) \otimes \mathbb{E}$. Now we state the family version of the Kirillov formula and delayed a heat kernel proof of it to Section 5.

**Theorem 2.1.** For any $K \in \mathfrak{g}(g)$ and $|K|$ small,

- if $n$ is even, for $t > 0$, the differential form

$$
\psi_B \text{Tr}_s \left[ g \exp \left( -\mathbb{B}_{K,t}^2 - \mathcal{L}_K \right) \right] \in \Omega^{\text{even}}(B, \mathbb{C})
$$

is closed, independent of $t$ and

$$
\lim_{t \to 0} \psi_B \text{Tr}_s \left[ g \exp \left( -\mathbb{B}_{K,t}^2 - \mathcal{L}_K \right) \right] = \int_{X^g} \hat{A}_{g,K}(TX, \nabla^{TX}) \text{ch}_{g,K}(E/S, \nabla^E).
$$

- if $n$ is odd, for $t > 0$, the differential form

$$
\psi_B \text{Tr}^{\text{odd}} \left[ g \exp \left( -\mathbb{B}_{K,t}^2 - \mathcal{L}_K \right) \right] \in \Omega^{\text{odd}}(B, \mathbb{C})
$$

is closed, independent of $t$ and

$$
\lim_{t \to 0} \psi_B \text{Tr}^{\text{odd}} \left[ g \exp \left( -\mathbb{B}_{K,t}^2 - \mathcal{L}_K \right) \right] = \int_{X^g} \hat{A}_{g,K}(TX, \nabla^{TX}) \text{ch}_{g,K}(E/S, \nabla^E).
$$

If $B$ is a point and $g = 1$, this heat kernel proof of the Kirillov formula is given by Bismut in [41] (see also [2, Theorem 8.2]). If $B$ is a point, (2.21) is established in [13]. For $g = 1$, (2.21) is obtained in [36].
2.2. Equivariant infinitesimal $\eta$-forms: Theorem 0.1. For $t > 0$, set

\[(2.23) B_{K,t} = B_{K,t} + dt \wedge \frac{\partial}{\partial t}\]

Then by (2.20),

\[(2.24) B_{K,t}^2 = B_{K,t} + dt \wedge \frac{\partial B_{K,t}}{\partial t} = \left( B_t + \frac{c(K_X)}{4\sqrt{t}} \right)^2 + dt \wedge \frac{\partial}{\partial t} \left( B_t + \frac{c(K_X)}{4\sqrt{t}} \right).\]

**Theorem 2.2.** There exist $\beta > 0, \delta, \delta' > 0, C > 0$, such that if $K \in \mathfrak{g}(g), z \in C, |zK| \leq \beta$, (a) for any $t \geq 1$,

\[(2.25) \left| \tilde{\text{Tr}} \left[ g \exp \left( -B_{K,t}^2 - zL_K \right) \right] \right| dt \leq \frac{C}{t^{1+\delta}}; \]

(b) for any $0 < t \leq 1$,

\[(2.26) \left| \tilde{\text{Tr}} \left[ g \exp \left( -B_{K,t}^2 - zL_K \right) \right] \right| dt \leq Ct^{\delta' - 1}.\]

We delay the proof of Theorem 2.2 to Section 5.

- If $n = \dim X$ is even, then for $t > 0$, as $B_{K,t}$ commutes with $g, L_K$, by [2, Lemma 9.15],

\[(2.27) d^B \text{Tr}_s \left[ g \exp(-B_{K,t}^2 - L_K) \right] = \text{Tr}_s \left[ [B_{K,t}, g \exp(-B_{K,t}^2 - L_K)] \right] = 0.\]

As in (1.39) (cf. [2, Theorems 8.11, 9.19]), we have

\[(2.28) \lim_{t \to +\infty} \psi_B \text{Tr}_s \left[ g \exp(-B_{K,t}^2 - L_K) \right] = \text{ch}_{ge,k}(\text{Ker}(D), \nabla^{\text{Ker}(D)}).\]

As in (1.40),

\[(2.29) \frac{\partial}{\partial t} \text{Tr}_s \left[ g \exp(-B_{K,t}^2 - L_K) \right] = -d^B \text{Tr}_s \left[ g \frac{\partial B_{K,t}}{\partial t} \exp(-B_{K,t}^2 - L_K) \right]
\]

\[= d^B \left\{ \text{Tr}_s \left[ g \exp(-B_{K,t}^2 - L_K) \right] \right\} dt.\]

Thus from (2.29), for $0 < \varepsilon < T < +\infty$,

\[(2.30) \text{Tr}_s \left[ g \exp(-B_{K,T}^2 - L_K) \right] - \text{Tr}_s \left[ g \exp(-B_{K,\varepsilon}^2 - L_K) \right]
\]

\[= d^B \int_{\varepsilon}^{T} \{ \text{Tr}_s \left[ g \exp(-B_{K,t}^2 - L_K) \right] \} dt.\]

In this case, for $|K| \leq \beta$, by Theorem 2.2, the equivariant infinitesimal $\eta$-form is defined by

\[(2.31) \tilde{\eta}_{g,K} = -\int_{0}^{+\infty} \frac{1}{2i\sqrt{\pi}} \psi_B \left\{ \text{Tr}_s \left[ g \exp(-B_{K,t}^2 - L_K) \right] \right\} dt \]

\[= \int_{0}^{+\infty} \frac{1}{2i\sqrt{\pi}} \psi_B \text{Tr}_s \left[ g \frac{\partial B_{K,t}}{\partial t} \exp(-B_{K,t}^2 - L_K) \right] dt \in \Omega^{\text{odd}}(B, C).\]

By (2.21), (2.30) and (2.31), we have

\[(2.32) d^B \tilde{\eta}_{g,K} = \int_{X^g} \hat{A}_{g,K} (TX, \nabla^{TX}) \text{ch}_{g,K}(E/S, \nabla^\varepsilon) - \text{ch}_{ge,K}(\text{Ker}(D), \nabla^{\text{Ker}(D)}).\]
• If $n$ is odd, then for $t > 0$, as $\mathbb{B}_{K,t}$ commutes with $g$, $\mathcal{L}_K$, again by the argument in $[2$, Lemma 9.15],

\begin{equation}
\tag{2.33}
\frac{d}{dt} \text{Tr}^{\text{odd}} \left[ g \exp(-\mathbb{B}^2_{K,t} - \mathcal{L}_K) \right] = \text{Tr}^{\text{even}} \left[ \mathbb{B}_{K,t} g \exp(-\mathbb{B}^2_{K,t} - \mathcal{L}_K) \right] = 0.
\end{equation}

As the same argument in (1.45),

\begin{equation}
\tag{2.34}
\lim_{t \to +\infty} \text{Tr}^{\text{odd}} \left[ g \exp(-\mathbb{B}^2_{K,t} - \mathcal{L}_K) \right] = 0.
\end{equation}

Comparing with (1.26) and (2.29), we have

\begin{equation}
\tag{2.35}
\frac{\partial}{\partial t} \text{Tr}^{\text{odd}} \left[ g \exp(-\mathbb{B}^2_{K,t} - \mathcal{L}_K) \right] = -d^B \text{Tr}^{\text{even}} \left[ g \frac{\partial\mathbb{B}_{K,t}}{\partial t} \exp(-\mathbb{B}^2_{K,t} - \mathcal{L}_K) \right] = d^B \left\{ \text{Tr}^{\text{odd}} \left[ g \exp(-\mathbb{B}^2_{K,t} - \mathcal{L}_K) \right] \right\} dt.
\end{equation}

From Theorem 2.2 in this case, for $|K| \leq \beta$, the equivariant infinitesimal $\eta$-form is defined by

\begin{equation}
\tag{2.36}
\tilde{\eta}_{g,K} = -\int_0^{+\infty} \frac{1}{\sqrt{\pi}} \psi_B \left\{ \text{Tr}^{\text{odd}} \left[ g \exp(-\mathbb{B}^2_{K,t} - \mathcal{L}_K) \right] \right\} dt = \int_0^{+\infty} \frac{1}{\sqrt{\pi}} \psi_B \text{Tr}^{\text{even}} \left[ g \frac{\partial\mathbb{B}_{K,t}}{\partial t} \exp(-\mathbb{B}^2_{K,t} - \mathcal{L}_K) \right] dt \in \Omega^{\text{even}}(B, \mathbb{C}).
\end{equation}

As in (1.49), by (2.22), (2.34), (2.35) and (2.36), we get

\begin{equation}
\tag{2.37}
d^B \tilde{\eta}_{g,K} = \int_{X^g} \tilde{A}_{g,K} (T X, \nabla^{TX}) \text{ch}_{g,K}(E/S, \nabla^E).
\end{equation}

**Definition 2.3.** For $K \in \mathfrak{g}(g)$, $|K| \leq \beta$, determined in Theorem 2.2 under Assumption 1.2 the equivariant infinitesimal $\eta$-form is defined by

\begin{equation}
\tag{2.38}
\tilde{\eta}_{g,K} = -\int_0^{+\infty} \left\{ \psi_{R \times B} \text{Tr} \left[ g \exp \left( -\mathbb{B}^2_{zK,t} - z\mathcal{L}_K \right) \right] \right\} dt.
\end{equation}

By (0.5) and (1.36), Equation (2.38) is a reformulation of (2.31) and (2.36). From (2.32) and (2.37), we established the first part of Theorem 0.1.

Remark that the compactness of $B$ guarantees the existence of the constant $\beta > 0$.

From (2.31) and (2.36), it is obvious that if $K = 0$, $\tilde{\eta}_{g,K} = \tilde{\eta}_g$ in (1.50).

From the Duhamel’s formula (cf. e.g., [2, Theorem 2.48]), we have

\begin{equation}
\tag{2.39}
\frac{\partial}{\partial z} \text{Tr} \left[ g \exp \left( -\mathbb{B}^2_{zK,t} - z\mathcal{L}_K \right) \right] = -\text{Tr} \left[ g \frac{\partial(\mathbb{B}^2_{zK,t} + z\mathcal{L}_K)}{\partial z} \exp \left( -\mathbb{B}^2_{zK,t} - z\mathcal{L}_K \right) \right] = 0.
\end{equation}

Thus, $\text{Tr} \left[ g \exp \left( -\mathbb{B}^2_{zK,t} - z\mathcal{L}_K \right) \right]$ is $\mathcal{C}^\infty$ on $t > 0$ and holomorphic on $z \in \mathbb{C}$.

We fix $K \in \mathfrak{g}(g)$. Thus for $0 < \varepsilon < T < +\infty$, the function

\[ \int_\varepsilon^T \left\{ \psi_{R \times B} \text{Tr} \left[ g \exp \left( -\mathbb{B}^2_{zK,t} - z\mathcal{L}_K \right) \right] \right\} dt \]

is holomorphic on $z$. By Theorem 2.2 and the dominated convergence theorem, we have

\begin{equation}
\tag{2.40}
\tilde{\eta}_{g,z,K} := -\int_0^{+\infty} \left\{ \psi_{R \times B} \text{Tr} \left[ g \exp \left( -\mathbb{B}^2_{zK,t} - z\mathcal{L}_K \right) \right] \right\} dt
\end{equation}

is holomorphic on $z \in \mathbb{C}$, $|zK| < \beta$. Thus we get the last part of Theorem 0.1.

The proof of Theorem 0.1 is completed.
3. Comparison of two equivariant $\eta$-forms

In this section, we state our main result. We use the same notations and assumptions in Sections 1 and 2.

Let $\vartheta_K \in T^*X$ be the 1-form which is dual to $K^X$ by the metric $g^{TX}$, i.e., for any $U \in TX$, we have

$$\vartheta_K(U) = \langle K^X, U \rangle.$$  
(3.1)

We identify $\vartheta_K$ to a vertical 1-form on $W$, i.e., to a 1-form which vanishes on $T^H W$. Then by (2.18) and (3.1), we have

$$d_K \vartheta_K = d \vartheta_K - 2i\pi |K^X|^2.$$  
(3.2)

Let $d^X$ be the exterior differential operator along the fibers $X$. By (2.5) and (3.1) (cf. [2, Lemma 7.15 (1)]), for $U, U' \in TX$, we have

$$d^X \vartheta_K(U, U') = 2\langle \nabla^T_U K^X, U' \rangle = 2\langle m^X(K)U, U' \rangle.$$  
(3.3)

Set

$$\tilde{T} = 2T(f_p^H, e_i) f^p \wedge e^i + \frac{1}{2} T(f_p^H, f_q^H) f^p \wedge f^q.$$  
(3.4)

From [2, Proposition 10.1] or [14, (3.61) and (3.94)],

$$d \vartheta_K = d^X \vartheta_K + \langle \tilde{T}, K^X \rangle = d^X \vartheta_K + \vartheta_K(\tilde{T}).$$  
(3.5)

For $K \in \mathfrak{z}(g)$, $|K|$ small, $v > 0$, set

$$\alpha_K = \tilde{A}_{g,K}(TX, \nabla^{TX}) \operatorname{ch}_{g,K}(E/S, \nabla^E) \in \Omega^{2*}(W^g, \det(N_{X^g/X})).$$  
(3.6)

$$\bar{c}_v = -\sum_{j=0}^{[\dim W^g/2]} \frac{1}{j!} \left( \frac{1}{2i\pi} \right)^{j+1} \int_{X^g} \frac{\vartheta_K}{4v} \left( \frac{d \vartheta_K}{4v} \right)^j \exp \left( -\frac{|K^X|^2}{4v} \right) \cdot \alpha_K.$$  
(3.7)

Thus when $v \to +\infty$, $\bar{c}_v = O(v^{-1})$ as $v \to +\infty$ and $\bar{c}_v = O(v^{1/2})$ as $v \to 0$.

**Lemma 3.1.** If $W^{g,K} \not\subset W^g$. Then $\bar{c}_v = O(v^{-1})$ as $v \to +\infty$ and $\bar{c}_v = O(v^{1/2})$ as $v \to 0$.

**Proof.** By (3.2) and (3.6), we have

$$\bar{c}_v = -\sum_{j=0}^{[\dim W^g/2]} \frac{1}{j!} \left( \frac{1}{2i\pi} \right)^{j+1} \int_{X^g} \frac{\vartheta_K}{4v} \left( \frac{d \vartheta_K}{4v} \right)^j \exp \left( -\frac{|K^X|^2}{4v} \right) \cdot \alpha_K.$$  
(3.7)

Thus when $v \to +\infty$, $\bar{c}_v = O(v^{-1})$.

Since $K^X$ is a Killing vector field and $TX^g$ is oriented, thus $TX^{g,K} = TX^g|_{W^{g,K}} \cap TX^K|_{W^{g,K}}$ is oriented (cf. [2, Proposition 7.12]). For $x \in W^g$, if $K^X_x \neq 0$, when $v \to 0$, the integral term in (3.7) at $x$ is exponential decay. So the integral in (3.7) could be localized on a neighbourhood of $W^{g,K}_x$. Recall that for any $v \in B$, $X^g_{b,K}$ is totally geodesic in $X^g_b$. Given $\epsilon > 0$, let $U''_x$ be the $\epsilon$-neighborhood of $W^{g,K}$ in $N_{b,X^g/K/X^g}$, the normal bundle of $W^{g,K}$ in $W^g$. There exists $\varepsilon_0$ such that for $0 < \varepsilon \leq \varepsilon_0$, the fiberwise exponential map $(y, Z) \in N_{b,X^g/K/X^g} \to \exp_y^X(Z) \in X^g_b$ is a diffeomorphism from $U''_x$ onto the tubular neighborhood $V''_x$ of $W^{g,K}$ in $W^g$. We denote $V''_x$ the fiber of the fibration $V'' \to B$. With this identification, let $k(y, Z)$ be the function such that

$$dv_{X^g}(y, Z) = k(y, Z)dv_{X^g,K}(y)dv_{N_{X^g/K/X^g}}(Z).$$  
(3.8)
here $dv_{X^g}$, $dv_{X^g,K}$ are the Riemannian volume forms of $X^g$, $X^g,K$ and $dv_{N_{X^g,K}/X^g}$ is the Euclidean volume form on $N_{X^g,K}/X^g$.

Let $e^1, \cdots, e^n$ be a locally oriented orthonormal frame of $T^*X$. For $\beta \in \Omega^\text{odd}(W^g, \det(N_{X^g,X}))$, let $[\beta]_{\text{max}}$ be the coefficient of $e^1 \wedge \cdots \wedge e^n$ of $\beta$. Consider the dilation $\delta_v, v > 0$, of $N_{X^g,K}/X^g$ by $\delta_v(y, Z) = (y, \sqrt{v}Z)$. We have

$$
(3.10) \quad TN_{X^g,K}/X^g = T^HN_{X^g,K}/X^g \oplus \pi^*_N N_{X^g,K}/X^g.
$$

By (1.4) and (3.10), we have

$$
(3.11) \quad T^HN_{X^g,K}/X^g \simeq \pi^*_N TW^g,K \simeq \pi^*_N(T^HW \oplus TX^g,K).
$$

On $N_{X^g,K}/X^g$, we have

$$
(3.12) \quad \Lambda(T^*N_{X^g,K}/X^g) = \Lambda(T^{\ast\ast}N_{X^g,K}/X^g) \otimes \pi^*_N \Lambda(N_{X^g,K}/X^g).
$$

For $y \in W^g,K$ fixed, we take $Y_1, Y'_1 \in T_yW^g,K$, $Y^V, Y^W \in N_{X^g,K}/X^g$, then $Y = Y_1 + Y^V$, $Y' = Y'_1 + Y^W$ are sections of $N_{X^g,K}/X^g$ along $N_{X^g,K}/X^g$, under our identification (3.10), i.e.,

$$
(3.13) \quad Y_{(y,Z)} = Y_1^H(y, Z) + Y^V, \quad Y'_{(y,Z)} = Y'_1^H(y, Z) + Y^W,
$$

here $Y_1^H, Y'_1^H \in T^HN_{X^g,K}/X^g$ are the lifts of $Y_1, Y'_1$.

Let $\theta_0$ be the one form on total space $\mathcal{N}$ of $N_{X^g,K}/X^g = N_{W^g,K}/W^g$ given by

$$
(3.14) \quad \theta_0(Y)_{(y,Z)} = \langle m^{TX}(K)Z, Y^V \rangle_y \quad \text{for } Y = Y_1^H + Y^V \in T^HN_{X^g,K}/X^g \oplus (\pi^*_N N_{X^g,K}/X^g).
$$

By [2] Lemma 7.15 (2), we have

$$
(3.15) \quad \frac{1}{v}\delta_v^*\theta_K = \theta_0 + \mathcal{O}(v^{1/2}).
$$

From (3.15), we get

$$
(3.16) \quad \frac{1}{v}d\delta_v^*\theta_K = d\theta_0 + \mathcal{O}(v^{1/2}).
$$
As the same argument in Section 2.1, \( \nabla^{TX^g} \), \( m^{TX}(K) \) preserve the splitting
\[
TX^g = TX^g, K \oplus N_{X^g, k/X^n} \quad \text{on } W^g, K
\]
and \( m^{TX}(K) = 0 \) on \( TX^g, K \). As in [2, p218], we calculate that for \((y, Z) \in N_{X^g, k/X^n}, \)
\[
d\theta(y, Y^*(y, Z)) = 2 \langle m^{TX}(K)Y^*, Y^* \rangle_y - \langle R^{TX}(Y^H, Y^*H)(m^{TX}(K))Z, Z \rangle_y.
\]
By (2.5) and (2.12), for \( y \in W^g, K \),
\[
\frac{1}{v} |K^X(y, \sqrt{v} Z)|^2 = |m^{TX}(K)|^2 + O(v^{1/2}).
\]
From (3.15), (3.16) and (3.19), for any \( \alpha \in \Omega^*(W^g, \det(N_{X^g/X})) \), as \( v \to 0, \)
\[
\int_{\nabla^v} \frac{\partial K}{4v} \frac{d\partial K}{4v} j \exp \left(-\frac{|K^X|}{4v} \right) \alpha
= \int_{X^g, k} \int_{Z \in N_{X^g, k/X^n}} \frac{\theta_0}{4} \frac{d\theta}{4} j \exp \left(-\frac{|m^{TX}(K)Z|^2}{4} \right) \alpha_y dv_{X^g, k}(y) dv_{N_{X^g, k/X^n}}(Z) + O(v^{1/2}).
\]
From (3.18), \( d\theta_0 \) is an even polynomial on \( Z \). However from (3.14), \( \theta_0 \) is linear on \( Z \). Thus the last integral in (3.20) is zero. Therefore, as \( v \to 0 \), we have
\[
\int_{\nabla^v} \frac{\partial K}{4v} \frac{d\partial K}{4v} j \exp \left(-\frac{|K^X|}{4v} \right) \alpha_K = O(v^{1/2}).
\]
The proof of Lemma 3.1 is completed. \( \square \)

From Lemma 3.1 and (3.6), the following integral is well-defined,
\[
\mathcal{M}_{g, K} := \int_0^{+\infty} \frac{dv}{v} \int_{X^g} \frac{\partial K}{2\pi} \exp \left(\frac{v d\partial K}{2\pi} \right) \hat{A}_{g, K}(TX, \nabla^{TX}) ch_{g, K}(E/S, \nabla^{E}) dv.
\]
Remark that when \( B \) is a point, for \( g = 1 \), Lemma 3.1 is proved in [22, Proposition 2.2].

**Proposition 3.2.** For any \( K \in \mathfrak{g}(g) \) fixed, there exist \( \beta > 0 \) and \( c_j(K) \in \Omega^*(B, \mathbb{C}) \) for \( 1 \leq j \leq [(\dim W^g + 1)/2] \) such that \( \mathcal{M}_{g, tK} \) is smooth on \( |t| < \beta, t \neq 0 \) and as \( t \to 0 \), we have
\[
\mathcal{M}_{g, tK} = \sum_{j=1}^{[(\dim W^g + 1)/2]} c_j(K) t^{-j} + O(t^0).
\]
Moreover, \( t^{[(\dim W^g + 1)/2]} \mathcal{M}_{g, tK} \) is real analytic on \( t \) for \( |t| < \beta \).

**Proof.** By (3.22) and changing the variables \( v \mapsto v/t^2 \), we have
\[
\mathcal{M}_{g, tK} = -\int_0^{+\infty} \int_{X^g} \frac{vt \partial K}{2\pi} \exp \left(\frac{vt d\partial K}{2\pi} \right) \exp(-vt^2|K^X|^2) \alpha_K dv
= -\int_0^{+\infty} \int_{X^g} \frac{\partial K}{2\pi t} \sum_{k=0}^{[(\dim W^g - 1)/2]} \left(\frac{v^k (d\partial K)^k}{(2\pi t)^{k!}} \right) \exp(-vt^2|K^X|^2) \alpha_K dv.
\]
From the arguments in the proof of (3.21), we get (3.26). From (2.14), (2.16) and (3.6), we see that $\alpha_{tK}$ is real analytic on $t$ for $t$ small enough. Following the same arguments in the proof of (3.21),

$$
\int_0^{\infty} \int_{X_s} \partial_K v^k (d\partial_K)^k \exp(-v|K_X|^2) \alpha_{tK} dv
$$

is uniformly absolutely integrable on $v$ for $t$ small enough. Thus $t^{(\dim W_g + 1)/2} \mathcal{M}_{g,tK}$ is real analytic on $t$ for $t$ small enough.

The proof of Proposition 3.2 is completed.

Now we could state our main result of this paper, Theorem 0.2 as follows.

**Theorem 3.3.** For $g \in G$ fixed, there exists $\beta > 0$ such that for $K \in \mathfrak{g}(g)$, $K \neq 0$ and $|K| < \beta$, we have

$$
(3.25) \quad \tilde{\eta}_{g,K} = \tilde{\eta}_{ge^K} + \mathcal{M}_{g,K} \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}).
$$

Observe that by (2.40), $\tilde{\eta}_{g,tK}$ is smooth for $t$ small. By (3.25), as $t \to 0$, modulo exact forms, the singular terms of $\tilde{\eta}_{ge^K}$ is the same as that of $-\mathcal{M}_{g,tK}$ in (3.23).

**Remark 3.4.** For $K \in \mathfrak{g}(g)$, if $K^X$ has no zeros on $W$, for $M = [(\dim W_g - 1)/2]$, we have

$$
(3.26) \quad -\int_0^{\infty} \frac{\partial_K}{2i\pi} \exp\left(\frac{v d\partial_K}{2i\pi}\right) dv = -\sum_{j=0}^{M-1} \frac{1}{j!} \left(\frac{1}{2i\pi}\right)^j \int_0^{\infty} \partial_K (v d\partial_K)^j e^{-v K_X^2} dv
$$

$$
= -\sum_{j=0}^{M-1} \frac{1}{j!} \left(\frac{1}{2i\pi}\right)^j \left(\frac{d\partial_K}{d K_X} (d\partial_K)^j (d K_X^2)^j + \frac{d\partial_K}{d K_X} \right)^j e^{-v K_X^2} dv
$$

Therefore when $K_X$ has no zeros for $t \neq 0$ small enough, by (3.6), (3.22), (3.25) and (3.26), we have

$$
(3.27) \quad \tilde{\eta}_{g,K} = \tilde{\eta}_{ge^K} - \sum_{j=0}^{M-1} \int_{X_s} \frac{\partial_K}{2i\pi t |K_X|^2} \left(1 - \frac{d\partial_K}{2i\pi t |K_X|^2}\right)^{-1} \alpha_{tK} \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}).
$$

In particular, for $g = 1$ and $B = pt$, (3.27) as Taylor expansion near $t = 0$ is the the main result of [21].

4. A PROOF OF THEOREM 3.3

In this section, we state some intermediate results and prove Theorem 3.3. The proofs of the intermediate results are delayed to Section 6.

4.1. Some intermediate results. For $t > 0$, $v > 0$, set

$$
(4.1) \quad \mathcal{C}_{v,t} = \mathbb{B}_t + \sqrt{tc(K_X)} \left(\frac{1}{t} - \frac{1}{v}\right) + dt \wedge \frac{\partial}{\partial t} + dv \wedge \frac{\partial}{\partial v}.
$$

Then $\mathcal{C}_{v,t}$ is a superconnection associated with the fibration $(\mathbb{R}_+^*)^2 \times W \to (\mathbb{R}_+^*)^2 \times B$. From [2, Theorem 9.17], we have

$$
(4.2) \quad d^{\mathbb{R}^2 \times B} \text{Tr}[g \exp(-\mathcal{C}_{v,t} - \mathcal{L}_K)] = 0.
$$
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For $\alpha \in \Lambda(T^*(\mathbb{R}^2 \times B))$,

$$\alpha = \alpha_0 + dv \wedge \alpha_1 + dt \wedge \alpha_2 + dv \wedge dt \wedge \alpha_3, \quad \alpha_i \in \Lambda(T^*B), \ i = 0, 1, 2, 3,$$

as in (0.6), we denote by

$$[\alpha]^{dv} := \alpha_1, \quad [\alpha]^{dt} := \alpha_2, \quad [\alpha]^{dv \wedge dt} := \alpha_3.$$  \hfill (4.4)

**Definition 4.1.** We define $\beta_{g,K}$ to be the part of $-\psi_{\mathbb{R}^2 \times B} \tilde{\text{Tr}}[g \exp (-C^2_{v,t} - L_K)]$ of degree one with respect to the coordinates $(v, t)$. We denote by

$$\alpha_{g,K} = -\left\{ \psi_{\mathbb{R}^2 \times B} \tilde{\text{Tr}}[g \exp (-C^2_{v,t} - L_K)] \right\}^{dv \wedge dt}. \hfill (4.5)$$

Then from comparing the coefficient of $dv \wedge dt$ part of (4.2), we have

$$\left( dv \wedge \frac{\partial}{\partial v} + dt \wedge \frac{\partial}{\partial t} \right) \beta_{g,K} = -dv \wedge dt \wedge dB \alpha_{g,K}. \hfill (4.6)$$

Take $a, A$, $0 < a \leq 1 \leq A < +\infty$. Let $\Gamma = \Gamma_{a,A}$ be the oriented contour in $\mathbb{R}_{+t} \times \mathbb{R}_{+v}$:

The contour $\Gamma$ is made of three oriented pieces $\Gamma_1, \Gamma_2, \Gamma_3$ indicated in the above picture. For $1 \leq k \leq 3$, set $I_k^0 = \int_{\Gamma_k} \beta_{g,K}$. Also $\Gamma$ bounds an oriented rectangular domain $\Delta$.

By Stocks' formula and (4.6),

$$\sum_{k=1}^3 I_k^0 = \int_{\partial \Delta} \beta_{g,K} = \int_{\Delta} \left( dv \wedge \frac{\partial}{\partial v} + dt \wedge \frac{\partial}{\partial t} \right) \beta_{g,K} = -d^B \left( \int_{\Delta} \alpha_{g,K} dv \wedge dt \right). \hfill (4.7)$$

The proof of the following theorem is left to Section 5.11.

**Theorem 4.2.** For $K \in \mathfrak{k}(g)$, $|K|$ small enough, there exist $\delta > 0$, $C > 0$ such that for any $t \geq 1, v \geq t$, we have

$$\left| [\beta_{g,K}(v,t)]^{dt} \right| \leq \frac{C}{t^{1+\delta}}. \hfill (4.8)$$

For $\alpha \in \Omega^j(B, \mathbb{C})$, we define

$$\phi(\alpha) := \{ \psi_{\mathbb{R} \times B}(dv \wedge \alpha) \}^{dv} = \begin{cases} \pi^{-\frac{j}{2}} (2i\pi)^{-\frac{j}{2}} \cdot \alpha & \text{if } j \text{ is even;} \\ (2i\pi)^{-\frac{j+1}{2}} \cdot \alpha & \text{if } j \text{ is odd.} \end{cases} \hfill (4.9)$$
Comparing with (1.27), we set
\[
\tilde{T}'_r = \begin{cases} 
T_r & \text{if } n \text{ is even;} \\
T_{r, \text{even}} & \text{if } n \text{ is odd.}
\end{cases}
\quad (4.10)
\]

For \(0 < t \leq v\), set
\[
B_{K,t,v} = \left( B_t + \frac{\sqrt{t}c(K^X)}{4} \left( \frac{1}{t} - \frac{1}{v} \right) \right)^2 + \mathcal{L}_K.
\quad (4.11)
\]

Then by Definition 4.1, (4.1) and (4.11), we have
\[
\beta_{g,K}(v,t) = - \left\{ \psi_{B_t \times B} \tilde{T}_r \left[ g \exp \left( -B_{K,t,v} - dt \wedge \frac{\partial}{\partial t} \left( B_t + \frac{\sqrt{t}c(K^X)}{4} \left( \frac{1}{t} - \frac{1}{v} \right) \right) \right] \right\} dt
\]
\[
= \phi \tilde{T}_r \left[ g \frac{\partial}{\partial t} \left( B_t + \frac{\sqrt{t}c(K^X)}{4} \left( \frac{1}{t} - \frac{1}{v} \right) \right) \exp (-B_{K,t,v}) \right],
\quad (4.12)
\]
\[
\beta_{g,K}(v,t) = - \left\{ \psi_{B_t \times B} \tilde{T}_r \left[ g \exp \left( -B_{K,t,v} - dv \frac{\sqrt{t}c(K^X)}{4v^2} \right) \right] \right\} dv
\]
\[
= \phi \tilde{T}_r \left[ g \frac{\sqrt{t}c(K^X)}{4v^2} \exp (-B_{K,t,v}) \right].
\]

Thus as \(B_{K,t,t} = B_t^2 + \mathcal{L}_K\), by (4.12), on \(\Gamma_2\), we have
\[
\beta_{g,K}(v,t) = dt \wedge \phi \tilde{T}_r \left[ g \frac{\partial}{\partial t} \exp \left( -B_t^2 - \mathcal{L}_K \right) \right]
\]
\[
= -dt \wedge \left\{ \psi_{B_t \times B} \tilde{T}_r \left[ g \exp \left( - \left( B_t + dt \wedge \frac{\partial}{\partial t} \right)^2 - \mathcal{L}_K \right) \right] \right\} dt.
\quad (4.13)
\]

In the rest of this section, we use Theorem 4.2 and the following estimates to prove Theorem 3.3. The proofs of these estimates are delayed to Section 6.

**Theorem 4.3.** For \(K \in \mathfrak{g}(g)\), \(|K|\) small enough,

(a) when \(t \to 0\),
\[
\phi \tilde{T}_r \left[ g \frac{\sqrt{t}c(K^X)}{4v} \exp (-B_{K,t,v}) \right] \to -\tilde{e}_v;
\quad (4.14)
\]

(b) there exist \(C > 0, \delta \in (0, 1)\), such that for \(t \in (0, 1)\), \(v \in [t, 1]\),
\[
\phi \tilde{T}_r \left[ g \frac{\sqrt{t}c(K^X)}{4v} \exp (-B_{K,t,v}) \right] + \tilde{e}_v \leq C \left( \frac{t}{v} \right)^\delta;
\quad (4.15)
\]

(c) there exists \(C > 0\) such that for \(t \in (0, 1)\), \(v \geq 1\),
\[
\phi \tilde{T}_r \left[ g \frac{\sqrt{t}c(K^X)}{4v} \exp (-B_{K,t,v}) \right] \leq \frac{C}{\sqrt{v}};
\quad (4.16)
\]

(d) for \(v \geq 1\),
\[
\lim_{t \to 0} \phi \tilde{T}_r \left[ g \frac{c(K^X)}{4\sqrt{tv}} \exp (-B_{K,t,v}) \right] = 0.
\quad (4.17)
4.2. A proof of Theorem 3.3. We now finish the proof of Theorem 3.3 by using Theorems 4.2 and 4.3. By (4.7), we know that $I_0^1 + I_0^2 + I_0^3$ is an exact form on $B$. We take the limits $A \to +\infty$ and then $a \to 0$ in the indicated order. We claim that the limit of the part $I_0^j(a, A)$ as $A \to +\infty$ exists, denoted by $I_1^j(a)$, and the limit of $I_1^j(a)$ as $a \to 0$ exists, denoted by $I_2^j$ for $j = 1, 2, 3$.

i) From Theorem 4.2, (2.24), (4.12) and the dominated convergence theorem, we see that

\begin{equation}
I_1^1(a) = \lim_{A \to +\infty} \int_0^A \left[ \beta_{g, K}(A, t) \right] dt = -\int_a^{+\infty} \left\{ \psi_{R \times B} \tilde{\text{Tr}} \left[ g \exp \left( -B_{K, t}^2 - \mathcal{L}_K \right) \right] \right\} dt.
\end{equation}

Thus by Theorem 2.2 and Definition 2.3, we have

\begin{equation}
I_2^1 = -\int_0^{+\infty} \left\{ \psi_{R \times B} \tilde{\text{Tr}} \left[ g \exp \left( -B_{K, t}^2 - \mathcal{L}_K \right) \right] \right\} dt = \tilde{\eta}_{g, K}.
\end{equation}

ii) From Definition 1.4 and (4.13), we have

\begin{equation}
I_2^2 = \int_0^{+\infty} \left\{ \psi_{R \times B} \tilde{\text{Tr}} \left[ g \exp \left( - \left( \mathbb{B}_t + dt \wedge \frac{\partial}{\partial t} \right)^2 - \mathcal{L}_K \right) \right] \right\} dt = \tilde{\eta}_{ge, K}.
\end{equation}

iii) For the term $I_3(a, A)$, set

\begin{align*}
J_1 &= -\int_1^a \tilde{e}_v \frac{dv}{v}, \\
J_2 &= \int_1^{+\infty} \phi \tilde{\text{Tr}} \left[ g \frac{\sqrt{ac(KX)}}{4v} \exp \left( -B_{K, av} \right) \right] \frac{dv}{v}, \\
J_3 &= \int_1^{1/a} \left( \phi \tilde{\text{Tr}} \left[ g \frac{c(KX)}{4\sqrt{av}} \exp \left( -B_{K, av} \right) \right] + \tilde{e}_av \right) \frac{dv}{v}.
\end{align*}

Clearly, by Theorem 4.3 c) and (4.12), we have

\begin{equation}
I_3^1(a) = J_1 + J_2 + J_3.
\end{equation}

By (4.14), (4.16) and (4.21), from the dominated convergence theorem, we find that as $a \to 0$,

\begin{equation}
J_2 \to J_2^1 = -\int_1^{+\infty} \tilde{e}_v \frac{dv}{v}.
\end{equation}

By (4.15), there exist $C > 0$, $\delta \in (0, 1]$ such that for $a \in (0, 1]$, $1 \leq v \leq 1/a$,

\begin{equation}
\left| \phi \tilde{\text{Tr}} \left[ g \frac{c(KX)}{4\sqrt{av}} \exp \left( -B_{K, av} \right) \right] + \tilde{e}_av \right| \leq \frac{C}{v^\delta}.
\end{equation}

Using Lemma 3.1, (4.14), (4.17), (4.21), (4.24), and the dominated convergence theorem, as $a \to 0$,

\begin{equation}
J_3 \to J_3^1 = 0.
\end{equation}

By (4.22) and (4.21)-(4.25), we have

\begin{equation}
I_3^2 = -\int_0^{+\infty} \tilde{e}_v \frac{dv}{v} = -\mathcal{M}_{g, K}.
\end{equation}
By [18, §22 Theorem 17], $d\Omega^\bullet(B, \mathbb{C})$ is closed under the uniformly convergence. Thus, by (4.7),
\begin{equation}
\sum_{j=1}^{3} I_j^2 \equiv 0 \mod d\Omega^\bullet(B, \mathbb{C}).
\end{equation}
By (4.19), (4.20), (4.26) and (4.27), the proof of Theorem 3.3 is completed.

5. CONSTRUCTION OF THE EQUIVARIANT INFINITESIMAL $\eta$-FORMS

In this section, we prove Theorems 2.2 and 4.2 following the lines of [13, §7] and give a heat kernel proof of the family Kirillov formula Theorem 2.1. For the convenience to compare the arguments in this section with those in [13], especially how the extra terms for the family version appear, the structure of this section is formulated almost the same as in [13, §7].

This section is organized as follows. In Section 5.1, we prove Theorem 2.2 a). In Sections 5.2-5.10, we give proofs of Theorems 2.1 and 2.2 b). In Section 5.11, we prove Theorem 4.2.

5.1. The behaviour of the trace as $t \to +\infty$. Set
\begin{equation}
C_{K,t} = \mathbb{B}_t + \frac{c(K^X)}{4\sqrt{t}} + t \cdot dt \wedge \frac{\partial}{\partial t}.
\end{equation}
For $z \in \mathbb{C}$, we denote by
\begin{equation}
A_{zK,t} := C_{zK,t}^2 + z\mathcal{L}_K.
\end{equation}
Then Theorem 2.2 a) is implied by the following estimate.

**Theorem 5.1.** For $\beta > 0$ fixed, there exist $C > 0$, $\delta > 0$ such that if $K \in \mathfrak{z}(g)$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $t \geq 1$,
\begin{equation}
|\tilde{\text{Tr}}[g \exp(-A_{zK,t})]| \leq \frac{C}{t^\delta}.
\end{equation}

**Proof.** This subsection is devoted to the proof of Theorem 5.1. \hfill \Box

In this section, we fix $\beta > 0$. The constants in this subsection may depend on $\beta$.

For $b \in B$, recall that $\mathbb{E}_b$ is the vector space of the smooth sections of $\mathcal{E}$ on $X_b$. For $\mu \in \mathbb{R}$, let $\mathbb{E}_b^\mu$ be the Sobolev spaces of the order $\mu$ of sections of $\mathcal{E}$ on $X_b$. We equip $\mathbb{E}_b^0$ by the Hermitian product $(\cdot, \cdot)_0$ in (1.17). Let $\| \cdot \|_0$ be the corresponding norm of $\mathbb{E}_b^0$. For $\mu \in \mathbb{Z}$, let $\| \cdot \|_\mu$ be the Sobolev norm of $\mathbb{E}_b^\mu$ induced by $\nabla^{TX}$ and $\nabla^\mathcal{E}$.

Recall that we assume that the kernels $\text{Ker}(D)$ form a vector bundle over $B$. We denote by $P$ the orthogonal projection from $\mathbb{E}_b^0$ to $\text{Ker}(D)$ and let $P^\perp = 1 - P$.

Recall that $e_1, \cdots, e_n$ is a local orthonormal frame of $TX$. For $s, s' \in \mathbb{E}$, $t \geq 1$, we set
\begin{equation}
|s|_{t,0}^2 := \|s\|_0^2;
\end{equation}
\begin{equation}
|s|_{t,1}^2 := \|Ps\|_0^2 + t\|P^\perp s\|_0^2 + t \sum_i \|\nabla^\mathcal{E}_{e_i} P^\perp s\|_0^2.
\end{equation}
Set
\begin{equation}
|s|_{t,-1} = \sup_{0 \neq s' \in \mathbb{E}_1} \frac{|\langle s, s' \rangle_0|}{|s'|_{t,1}}.
\end{equation}
Lemma 5.2. There exist $c_1, c_2, c_3, c_4 > 0$, such that for any $t \geq 1$, $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $s, s' \in \mathbb{E}$,
\[
\begin{align*}
\text{Re} \left\langle A_{zK,t}^{(0)}(s, s) \right\rangle_0 & \geq c_1 |s|_{l,1}^2 - c_2 |s|_{l,0}^2, \\
\text{Im} \left\langle A_{zK,t}^{(0)}(s, s) \right\rangle_0 & \leq c_3 |s|_{l,1} |s|_{l,0}, \\
\left\langle A_{zK,t}^{(0)}(s, s') \right\rangle_0 & \leq c_4 |s|_{l,1} |s'|_{l,1}.
\end{align*}
\]  
(5.6)

Proof. From (1.23) and (5.1), we have
\[
A_{zK,t}^{(0)} = tD^2 + \frac{z}{4} \left[ D, c(K^X) \right] - z^2 \frac{|K^X|^2}{16t} + z \mathcal{L}_K.
\]  
So we have
\[
\begin{align*}
\text{Re} & \left\langle A_{zK,t}^{(0)}(s, s) \right\rangle_0 = \left\langle \left(tD^2 + \text{Im}(z)i \left( \frac{1}{4} \left[ D, c(K^X) \right] + \mathcal{L}_K \right) - \text{Re}(z^2) \frac{|K^X|^2}{16t} \right), s, s \right\rangle_0, \\
\text{Im} & \left\langle A_{zK,t}^{(0)}(s, s) \right\rangle_0 = \left\langle \left(-\text{Re}(z)i \left( \frac{1}{4} \left[ D, c(K^X) \right] + \mathcal{L}_K \right) - \text{Im}(z^2) \frac{|K^X|^2}{16t} \right), s, s \right\rangle_0.
\end{align*}
\]  
(5.8)

From (5.4), there exists $c'_1, c'_2, c'_3, c'_4 > 0$ such that for any $t \geq 1$, $|zK| \leq \beta$, $\epsilon > 0$,
\[
\begin{align*}
\left\langle \left(tD^2 - \text{Re}(z^2) \frac{|K^X|^2}{16t} \right), s, s \right\rangle_0 & \geq c'_1 |s|_{l,1}^2 - c'_2 |s|_{l,0}^2, \\
\left\langle \frac{\text{Im}(z)}{4} \left[ D, c(K^X) \right], s, s \right\rangle_0 & \leq c'_3 |s|_{l,1} |s|_{l,0} \leq c'_4 \epsilon |s|_{l,1}^2 + \frac{c'_4}{4\epsilon} |s|_{l,0}^2, \\
|\langle z, \mathcal{L}_K, s \rangle_0 | & \leq c'_4 |s|_{l,1} |s|_{l,0} \leq c'_4 \epsilon |s|_{l,1}^2 + \frac{c'_4}{4\epsilon} |s|_{l,0}^2.
\end{align*}
\]  
(5.9)

By taking $\epsilon = \min\{c'_1/(4c'_3), c'_2/(4c'_4)\}$, from (5.8), we get the first estimate of (5.6).

The other estimates in (5.10) follow directly from (5.6), (5.4) and (5.8).

The proof of Lemma 5.2 is completed. \hfill \Box

By using Lemma 5.2 and exactly the same argument in [15 Theorem 11.27], we get

Lemma 5.3. There exist $c, C > 0$, such that if $t \geq 1$, $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \leq \beta$,
\[
\begin{align*}
\lambda \in U_{\epsilon} := \left\{ \lambda \in \mathbb{C} : \text{Re}(\lambda) \leq \frac{\text{Im}(\lambda)^2}{4c^2} - c^2 \right\},
\end{align*}
\]  
(5.10)

the resolvent $(\lambda - A_{zK,t}^{(0)})^{-1}$ exists, and moreover for any $t \geq 1$, $s \in \mathbb{E}$,
\[
\begin{align*}
|\langle \lambda - A_{zK,t}^{(0)} \rangle^{-1}s |_{l,0} & \leq C |s|_{l,0}, \\
|\langle \lambda - A_{zK,t}^{(0)} \rangle^{-1}s |_{l,1} & \leq C(1 + |\lambda|)^2 |s|_{l,-1}.
\end{align*}
\]  
(5.11)

The following lemma is the analogue of [7 Theorem 9.15].
Lemma 5.4. There exist $C > 0, k \in \mathbb{N}$, such that for $t \geq 1$, $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $\lambda \in U_c$, with $c$ in Lemma 5.3, the resolvent $(\lambda - A_{zK,t})^{-1}$ exists, extends to a continuous linear operator from $\Lambda(T^*_0B) \otimes \mathbb{E}^{-1}$ into $\Lambda(T^*_0B) \otimes \mathbb{E}^1$, and moreover for $s \in \mathbb{E}$,
\begin{equation}
\| (\lambda - A_{zK,t})^{-1}s \|_0 \leq C(1 + |\lambda|)^k \| s \|_0.
\end{equation}
\begin{equation}
\| (\lambda - A_{zK,t})^{-1}s \|_{t,1} \leq C(1 + |\lambda|)^k \| s \|_{t,-1}.
\end{equation}

Proof. From (1.23) and (5.1),
\begin{equation}
A_{zK,t} - A_{zK,t}^{(0)} = \sqrt{t} \left( [D, \nabla^E.u] + \frac{1}{2} dt \land D \right) + (\nabla^E.u)^2 - \frac{1}{4} [D, c(T^H)]
+ \frac{1}{8} \sqrt{t} \left( 2[\nabla^E.u, zc(K^X) - c(T^H)] - dt \land (zc(K^X) - c(T^H)) \right)
+ \frac{1}{16} \left( 2z(K^X, T^H) + c(T^H)^2 \right).
\end{equation}
By [5, Theorem 2.5], we see that $[D, \nabla^E.u]$ and $(\nabla^E.u)^2$ are first order differential operators along the fibers. Since
\begin{equation}
\left| \langle \sqrt{t}[D, \nabla^E.u]s, s \rangle \right| \leq C(|s|_{t,0}|s'|_{t,1} + |s|_{t,1}|s'|_{t,0}),
\end{equation}
by (5.13), there exists $C' > 0$ such that for any $t \geq 1$, we have
\begin{equation}
\| (A_{zK,t} - A_{zK,t}^{(0)})s \|_{t,-1} \leq C'|s|_{t,1}.
\end{equation}
Take $\lambda \in U_c$. Then since $A_{zK,t} - A_{zK,t}^{(0)}$ has positive degree in $\Lambda^*(T^*(\mathbb{R} \times B))$, we have
\begin{equation}
(\lambda - A_{zK,t})^{-1} = \sum_{m=0}^{1 + \dim B} (\lambda - A_{zK,t}^{(0)})^{-1} \left( (A_{zK,t} - A_{zK,t}^{(0)})(\lambda - A_{zK,t}^{(0)})^{-1} \right)^m.
\end{equation}
Therefore, by (5.11), (5.15) and (5.16), we obtain (5.12).

The proof of Lemma 5.4 is completed.

Proposition 5.5. There exists $C > 0$, such that for $t \geq 1$, $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $s \in \mathbb{E}$,
\begin{equation}
\| P^\perp \exp(-A_{zK,t})P^\perp s \| \leq \frac{C}{\sqrt{t}} \| s \|_0.
\end{equation}

Proof. For $m \geq 0$, by (5.4), (5.11) and (5.15), for $\lambda \in U_c$,
\begin{equation}
\| P^\perp (\lambda - A_{zK,t}^{(0)})^{-1} \left( (A_{zK,t} - A_{zK,t}^{(0)})(\lambda - A_{zK,t}^{(0)})^{-1} \right)^m P^\perp s \|_0 \leq \frac{1}{\sqrt{t}} \left| (\lambda - A_{zK,t}^{(0)})^{-1} \left( (A_{zK,t} - A_{zK,t}^{(0)})(\lambda - A_{zK,t}^{(0)})^{-1} \right)^m P^\perp s \right|_{t,1}
\leq \frac{C(1 + |\lambda|)^2}{\sqrt{t}} \left| \left( (A_{zK,t} - A_{zK,t}^{(0)})(\lambda - A_{zK,t}^{(0)})^{-1} \right)^m P^\perp s \right|_{t,-1} \leq \frac{C(1 + |\lambda|)^{km+2}}{\sqrt{t}} \| s \|_0.
\end{equation}
Therefore, by (5.16) and (5.18), there exists $k' > 0$, such that for $\lambda \in U_c$,
\begin{equation}
\| P^\perp (\lambda - A_{zK,t})^{-1}P^\perp s \| \leq \frac{C}{\sqrt{t}}(1 + |\lambda|)^{k'} \| s \|_0.
\end{equation}
From Lemma 5.6 we have
\begin{equation}
\exp(-A_{zK,t}) = \frac{1}{2i\pi} \int_{\partial U_c} e^{-\lambda} (\lambda - A_{zK,t})^{-1} d\lambda.
\end{equation}

From (5.19) and (5.20), we get (5.17).

The proof of Proposition 5.5 is completed. \qed

Since $B$ is compact, there exists a family of smooth sections of $TX$, $U_1, \cdots, U_m$ such that for any $x \in W$, $U_1(x), \cdots, U_m(x)$ spans $T_xX$.

Let $D$ be a family of operators on $E$,
\begin{equation}
D = \{ P^\perp \nabla_{\xi}^i P^\perp \}.
\end{equation}

From (5.7), by the same argument as the proof of [7, Theorem 9.17] (see also e.g., [24, Lemma 5.17]), we get the following lemma.

**Lemma 5.6.** For any $k \in \mathbb{N}$ fixed, there exists $C_k > 0$ such that for $t \geq 1$, $K \in g$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $Q_1, \cdots, Q_k \in D$ and $s, s' \in E$, we have
\begin{equation}
|\langle (Q_1, [Q_2, \cdots [Q_k, A_{zK,t}], \cdots]s, s' \rangle_0 | \leq C_k |s|_{t,1} |s'|_{t,1}.
\end{equation}

For $k \in \mathbb{N}$, let $D^k$ be the family of operators $Q$ which can be written in the form
\begin{equation}
Q = Q_1 \cdots Q_k, \quad Q_i \in D.
\end{equation}

If $k \in \mathbb{N}$, we define the Hilbert norm $|| \cdot ||_k'$ by
\begin{equation}
||s||^2_k = \sum^k_{\ell=0} \sum_{Q \in D^\ell} ||Qs||^2_0.
\end{equation}

Since $P^\perp \nabla_{\xi}^i$ and $\nabla_{\xi}^i P$ are operators along the fibers with smooth kernels, the Sobolev norm $|| \cdot ||_k'$ is equivalent to the Sobolev norm $|| \cdot ||_k$. Thus, we also denote the Sobolev space with respect to $|| \cdot ||_k'$ by $E^k$.

By using Lemma 5.6 as the proof of [7, Theorem 9.18], we get

**Lemma 5.7.** For any $m \in \mathbb{N}$, there exist $p_m \in \mathbb{N}$ and $C_m > 0$ such that for $t \geq 1$, $\lambda \in U_c$, $s \in E^m$,
\begin{equation}
|| P^\perp (\lambda - A_{zK,t})^{-1} P^\perp s ||_{m+1} \leq C_m (1 + |\lambda|)^p_m ||s||_m^2.
\end{equation}

By using Lemma 5.7 following the same progress as in the proof of [7, Theorem 9.20], we could get

**Proposition 5.8.** For $m \in \mathbb{N}$, there exists $C > 0$, such that for $b \in B$, $x, x' \in X_b$, $t \geq 1$, $K \in g$, $z \in \mathbb{C}$, $|zK| \leq \beta$,
\begin{equation}
\sup_{|\alpha|, |\alpha'| \leq m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial x^\alpha \partial x'^{\alpha'}} P^\perp \exp(-A_{zK,t}) P^\perp \exp(-A_{zK,t}) P^\perp (x, x') \right| \leq C,
\end{equation}

where $P^\perp \exp(-A_{zK,t}) P^\perp (x, x')$ is the smooth kernel of the operator $P^\perp \exp(-A_{zK,t}) P^\perp$ associated with $d\nu_X(x')$. 
From Propositions 5.5 and 5.8 by the arguments in [15 §11 p], there exist $C > 0$, $\delta > 0$, such that for $t \geq 1$, $|zK| \leq \beta$,

\begin{equation}
|P^\perp \exp (-A_{zK,t}) P^\perp (x, x')| \leq \frac{C}{t^\delta}.
\end{equation}

From (5.27), for $K \in \mathfrak{g}(g)$, $t \geq 1$, $|zK| \leq \beta$, we have

\begin{equation}
|\hat{\text{Tr}} [g P^\perp \exp (-A_{zK,t}) P^\perp]| \leq \frac{C}{t^\delta}.
\end{equation}

Forgetting the $dt \wedge$ part and following the proof of Proposition 5.5 the operator

\[
\exp \left( -(B_t + zc(K^X)/4\sqrt{t})^2 - zL_K \right)
\]

is uniformly bounded for $t \geq 1$, $|zK| \leq \beta$. Therefore there exists $C > 0$, such that for $K \in \mathfrak{g}(g)$, $|zK| < \beta$, $t \geq 1$,

\begin{equation}
|\hat{\text{Tr}} [g P \exp (-A_{zK,t}) P]|^dt = -\frac{1}{8\sqrt{t}} \left| \hat{\text{Tr}} \left[ g P (c(T^H) - zc(K^X)) \exp \left( -\left( B_t + zc(K^X)/4\sqrt{t} \right)^2 - zL_K \right) P \right] \right| \leq \frac{C}{\sqrt{t}}.
\end{equation}

Since

\begin{equation}
\hat{\text{Tr}} [g \exp (-A_{zK,t})]^dt = \hat{\text{Tr}} [g P \exp (-A_{zK,t}) P]^dt + \hat{\text{Tr}} [g P^\perp \exp (-A_{zK,t}) P^\perp]^dt,
\end{equation}

we obtain Theorem 5.1 from (5.28) and (5.29).

The proof of Theorem 5.1 is completed.

5.2. A proof of Theorems 2.1 and 2.2 b). Sections 5.2 and 5.3 are devoted to the proof of the following theorem.

**Theorem 5.9.** There exist $\beta > 0$, $C > 0$, $0 < \delta \leq 1$ such that if $K \in \mathfrak{g}(g)$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $0 < t \leq 1$,

\begin{equation}
|\psi_B \hat{\text{Tr}} [g \exp (-A_{zK,t})] - \int_{X^g} \tilde{A}_{g,zK}(TX, \nabla TX) \text{ch}_{g,zK}(\mathcal{E}/\mathcal{S}, \nabla \mathcal{E})| \leq C t^\delta.
\end{equation}

Since $\int_{X^g} \tilde{A}_{g,zK}(TX, \nabla TX) \text{ch}_{g,zK}(\mathcal{E}/\mathcal{S}, \nabla \mathcal{E})$ does not have the $dt$ term, we get the second part of Theorem 2.2 from Theorem 5.9 which we reformulate as follows.

**Theorem 5.10.** There exist $\beta > 0$, $C > 0$, $\delta > 0$, such that if $K \in \mathfrak{g}(g)$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $0 < t \leq 1$,

\begin{equation}
|\hat{\text{Tr}} [g \exp (-A_{zK,t})]^dt| \leq C t^\delta.
\end{equation}

**Proof of Theorem 2.1.** If we omit the $dt$ term in (5.31) and take $z = 1$, it follows that

\begin{equation}
|\psi_B \hat{\text{Tr}} \left[ g \exp \left( - \left( B_t + \frac{c(K^X)}{4\sqrt{t}} \right)^2 - L_K \right) \right] - \int_{X^g} \tilde{A}_{g,K}(TX, \nabla TX) \text{ch}_{g,K}(\mathcal{E}/\mathcal{S}, \nabla \mathcal{E})| \leq C t^\delta.
\end{equation}

Therefore, we get (2.21) and (2.22).
From (5.32), (5.33), (5.34), and (5.35), we get other parts of Theorem 2.1. The proof of Theorem 2.1 is completed. □

For simplicity, we will assume in the rest of this section that \( n = \text{dim} X \) is even. The functional analysis part is exactly the same for even and odd dimensional. We only explain in Remark 5.22 how to use the argument in the proof [12, Theorem 2.10] to compute the local index in odd dimension case.

5.3. Finite propagation speed and localization. The proof of the following lemma is the same as Lemma 5.12.

**Lemma 5.11.** Given \( \beta > 0 \), there exist \( C_1, C_2, C_2'(\beta), C_3(\beta), C_3'(\beta), C_4, C_5(\beta) > 0 \) such that if \( K \in g, z \in \mathbb{C}, |zK| \leq \beta, s, s' \in E, 0 < t \leq 1, \)

\[
\text{Re}(tA_z^{(0)}(0)s, s)_0 \geq C_1t^2\|s\|^2_1 - (C_2t^2 + C_2'(\beta))\|s\|_0^2,
\]

\[
\text{Im}(tA_z^{(0)}(0)s, s)_0 \leq C_3(\beta)t\|s\|_1\|s\|_0 + C_3'(\beta)\|s\|_0^2,
\]

\[
|\langle tA_z^{(0)}(0)s, s' \rangle_0| \leq C_4(t\|s\|_1 + C_5(\beta))\|s\|_0(t\|s'\|_1 + C_5(\beta))\|s'\|_0).
\]

Moreover, as \( \beta \to 0, C_2'(\beta), C_3(\beta), C_3'(\beta), C_5(\beta) \to 0. \)

In the sequel, we take \( \beta > 0 \) and always assume that \( K \in g, |zK| \leq \beta. \)

For \( c > 0 \), put

\[
V_c = \left\{ \lambda \in \mathbb{C} : \text{Re}\lambda \geq \frac{\text{Im}(\lambda)^2}{4c^2} - c^2 \right\},
\]

\[
\Gamma_c = \left\{ \lambda \in \mathbb{C} : \text{Re}\lambda = \frac{\text{Im}(\lambda)^2}{4c^2} - c^2 \right\}.
\]

Note that \( U_c, V_c, \Gamma_c \) are the images of \( \{ \lambda \in \mathbb{C} : |\text{Im}\lambda| \geq c \}, \{ \lambda \in \mathbb{C} : |\text{Im}\lambda| \leq c \}, \{ \lambda \in \mathbb{C} : |\text{Im}\lambda| = c \} \) by the map \( \lambda \mapsto \lambda^2. \)

The following lemma is an analogue of [13, Theorem 7.12].

**Lemma 5.12.** There exists \( C > 0 \) such that given \( c \in (0, 1) \), for \( \beta > 0 \) and \( t \in (0, 1] \) small enough, if \( \lambda \in U_c, |zK| \leq \beta \), the resolvent \( (\lambda - A_z^{(0)})^{-1} \) exists, extends to a continuous operator from \( E^{-1} \) into \( E^1 \), and moreover, for \( s \in E, \)

\[
\| (\lambda - tA_z^{(0)}(0)s)_0 \|_1 \leq \frac{2}{C^2} \| s \|_0,
\]

\[
\| (\lambda - tA_z^{(0)}(0)s)_1 \|_1 \leq \frac{C}{C^2 t^4} (1 + |\lambda|)^2 \| s \|_1.
\]

**Proof.** From the same arguments in [13, (7.47)-(7.49)], by Lemma 5.11, if \( \lambda \in \mathbb{R}, \lambda \leq -(C_2t^2 + \beta'), \) the resolvent \( (\lambda - tA_z^{(0)}(0))^{-1} \) exists.

Now we take \( \lambda = a + ib, a, b \in \mathbb{R}. \) By (5.33),

\[
|\langle (tA_z^{(0)}(0) - \lambda)s, s \rangle_0 | \geq \text{sup} \{ C_1t^2\|s\|_1^2 - (C_2t^2 + C_2'(\beta) + a)\|s\|_0^2 - C_3(\beta)t\|s\|_1\|s\|_0 + (|b| - C_3'(\beta))\|s\|_0^2 \}.
\]

Set

\[
C(\lambda, t) = \inf_{u \geq 1} \text{sup} \{ C_1(tu)^2 - (C_2t^2 + C_2'(\beta) + a), -C_3(\beta)tu - C_3'(\beta) + |b| \}.
\]
Since \( \|s\|_0 \leq \|s\|_1 \), using (5.37), (5.38), we get
\[ (5.39) \quad |\langle (tA_{eK,t}^{(0)} - \lambda)s, s \rangle| \geq C(\lambda, t)|s|_0^2. \]

Now we fix \( c \in (0, 1] \). Suppose that \( \lambda \in U_c \), i.e.,
\[ (5.40) \quad a \leq \frac{b^2}{4c^2} - c^2. \]

Assume that \( u \in \mathbb{R} \) is such that
\[ (5.41) \quad |b| - C_3(\beta)tu - C''_3(\beta) \leq c^2. \]

Then by (5.40) and (5.41),
\[ (5.42) \quad C_1(tu)^2 - (C_2t^4 + C'_2(\beta) + a) \geq C_1(tu)^2 - \frac{b^2}{4c^2} + c^2 - C_2t^2 - C'_2(\beta) \]
\[ \geq \left( C_1 - \frac{C_2^2(\beta)}{4c^2} \right) (tu)^2 - \frac{(c^2 + C'_3(\beta))C_3(\beta)}{2c^2}tu + c^2 - \frac{(c^2 + C'_3(\beta))^2}{4c^2} - C_2t^2 - C'_2(\beta). \]

The discriminant \( \Delta \) of the polynomial in the variable \( tu \) in the right-hand side of (5.42) is given by
\[ (5.43) \quad \Delta = -3c^2C_1 + 2C_1(C'_3(\beta) + 2C_2t^2 + 2C'_2(\beta)) + C_3(\beta)^2 \]
\[ + \frac{1}{c^2}(C_1C'_3(\beta)^2 - C_2C_3(\beta)^2t^2 - C'_2(\beta)C_3(\beta)^2). \]

Clearly, for \( \beta, t \) small enough,
\[ (5.44) \quad \Delta \leq -2c^2C_1, \quad C_1 - \frac{C_2^2(\beta)}{4c^2} > 0. \]

From (5.42)-(5.44), we get
\[ (5.45) \quad C_1(t^2u)^2 - (C_2t^4 + C'_2(\beta) + a) \geq -\frac{\Delta}{4(C_1 - \frac{C_2^2(\beta)}{4c^2})} \geq \frac{c^2}{2}. \]

Ultimately, by (5.38)-(5.45), we find that for \( \beta > 0, t \in (0, 1] \) small enough, if \( \lambda \in U_c \),
\[ (5.46) \quad C(\lambda, t) \geq \frac{c^2}{2}. \]

The other part of the proof of Lemma 5.12 is the same as that of [13, Theorem 7.12]. The proof of Lemma 5.12 is completed. \( \square \)

As in (5.15), since \( t \leq 1 \), from (5.13), there exists \( C > 0 \), such that for any \( s \in \mathbb{H}^1 \),
\[ (5.47) \quad \|(tA_{eK,t} - tA_{eK,t}^{(0)})s\|_{-1} \leq C\|s\|_1. \]

From Lemma 5.12 following the same process as the proof of (5.12), we get the following lemma.
Lemma 5.13. There exist $k, m \in \mathbb{N}$, $C > 0$, such that given $c \in (0, 1]$, for $\beta > 0$ and $t \in (0, 1]$ small enough, if $\lambda \in U_c$, $|zK| \leq \beta$, the resolvent $(\lambda - tA_{zK,t})^{-1}$ exists, extends to a continuous operator from $\mathbb{E}^{-1}$ into $\mathbb{E}^1$, and moreover, for $s \in \mathbb{E}$,

$$
\|(\lambda - tA_{zK,t})^{-1}s\|_0 \leq \frac{C}{c^{2k}t^k}((1 + |\lambda|)^m s\|_0,
$$

(5.48)

$$
\|(\lambda - tA_{zK,t})^{-1}s\|_1 \leq \frac{C}{c^{2k}t^k}((1 + |\lambda|)^m s\|_1.
$$

Definition 5.14. If $H, H'$ are separable Hilbert spaces, if $1 \leq p < +\infty$, set

$$
\mathcal{L}_p(H, H') = \{ A \in \mathcal{L}(H, H') : \text{Tr}[(A^*A)^{p/2}] < +\infty \}.
$$

If $A \in \mathcal{L}_p(H, H')$, set

$$
\|A\|_{(p)} := \left(\text{Tr}[(A^*A)^{p/2}]\right)^{1/p}.
$$

Then by [35] Chapter IX Proposition 6], $\| \cdot \|_{(p)}$ is a norm on $\mathcal{L}_p(H, H')$. Similarly, if $A \in \mathcal{L}_p(H, H')$, let $\|A\|_\infty$ be the usual operator norm of $A$.

In the sequel, the norms $\| \cdot \|_{(p)}$, $\| \cdot \|_{\infty}$ will always be calculated with respect to the Sobolev spaces $\mathbb{E}^0$.

From Lemma 5.13 we get the following lemma with the same proof as in [13] Theorem 7.13.

Lemma 5.15. Given $q \geq 2\dim X + 1$, there exist $C, C' > 0$, $k, m \in \mathbb{N}$, such that given $c \in (0, 1]$, for $\beta > 0$ and $t \in (0, 1]$ small enough, if $\lambda \in U_c$, $|zK| \leq \beta$,

$$
\|(\lambda - tA_{zK,t})^{-1}\|_{(q)} \leq \frac{C}{c^{2k}t^k}((1 + |\lambda|)^m,
$$

(5.51)

$$
\|(\lambda - tA_{zK,t})^{-q}\|_{(1)} \leq \frac{C^q}{(c^{2k}t^k)^q((1 + |\lambda|)^mq}.
$$

Let $a_X$ be the inf of the injectivity radius of the fibers $X$. Let $\alpha \in (0, a_X/8]$. The precise value of $\alpha$ will be fixed later. The constants $C > 0$, $C' > 0$ may depend on the choice of $\alpha$.

Let $f : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$
f(u) = \begin{cases} 
1 & \text{for } |u| \leq \alpha/2; \\
0 & \text{for } |u| \geq \alpha.
\end{cases}
$$

(5.52)

Set

$$
g(u) = 1 - f(u).
$$

(5.53)

For $t > 0$, $a \in \mathbb{C}$, put

$$
\begin{cases} 
F_t(a) = \int_{-\infty}^{+\infty} \exp(\sqrt{2}iu a) \exp\left(-\frac{u^2}{2}\right) f(\sqrt{t}u) \frac{du}{\sqrt{2\pi}}, \\
G_t(a) = \int_{-\infty}^{+\infty} \exp(\sqrt{2}iu a) \exp\left(-\frac{u^2}{2}\right) g(\sqrt{t}u) \frac{du}{\sqrt{2\pi}}.
\end{cases}
$$

(5.54)

Then $F_t(a), G_t(a)$ are even holomorphic functions of $a$ such that

$$
\exp(-a^2) = F_t(a) + G_t(a).
$$

(5.55)
Moreover when restricted on $\mathbb{R}$, $F_t$ and $G_t$ both lie in the Schwartz space $\mathcal{S}(\mathbb{R})$. Put

$$I_t(a) = G_t(a/\sqrt{t}).$$

Clearly, there exist uniquely defined holomorphic functions $\tilde{F}_t(a)$, $\tilde{G}_t(a)$, $\tilde{I}_t(a)$ such that

$$F_t(a) = \tilde{F}_t(a^2), \quad G_t(a) = \tilde{G}_t(a^2), \quad I_t(a) = \tilde{I}_t(a^2).$$

By (5.56) and (5.55), we have

$$\beta > \frac{5.56}{5.55}.$$ 

By (5.61) and (5.60), we find that to establish (5.32), we may as well replace $\exp(-\mathcal{A}_{zK,\ell})$ by $\tilde{F}_t(\mathcal{A}_{zK,\ell})$.

Let $F_t(\mathcal{A}_{zK,\ell})(x, x')$, $(x, x' \in X_b, b \in B)$ be the smooth kernel associated with the operator $\mathcal{F}_t(\mathcal{A}_{zK,\ell})$ with respect to $dv_X(x')$. Clearly the kernel of $g\mathcal{F}_t(\mathcal{A}_{zK,\ell})$ is given by $g\mathcal{F}_t(\mathcal{A}_{zK,\ell})(g^{-1}x, x')$. Then,

$$\text{Tr}_x[g\mathcal{F}_t(\mathcal{A}_{zK,\ell})](g^{-1}x, x)]dv_X(x).$$

By the above, it follows that $g\mathcal{F}_t(\mathcal{A}_{zK,\ell})(g^{-1}x, x) \in X_b$ vanishes if $d^{X_b}(g^{-1}x, x) \geq \alpha$. Here $d^{X_b}$ is the distance in $(X_b, g^{TX})$.

Now we explain our choice of $\alpha$. Recall that $N_{X_b/X}$ is identified with the orthogonal bundle to $TX^g$ in $TX|_{X_b}$. Given $\varepsilon > 0$, let $U_\varepsilon$ be the $\varepsilon$-neighborhood of $X_b^g$ in $N_{X_b/X}$. There exists $\varepsilon_0 \in (0, a_X/32]$ such that if $\varepsilon \in (0, 16\varepsilon_0]$, the fiberwise exponential map $(x, Z) \in N_{X_b/X} \to \exp_x(Z)$ is a diffeomorphism of $U_\varepsilon$ on the tubular neighborhood $V_\varepsilon$ of $X^g$ in $X$. In the sequel, we identify $U_\varepsilon$ and $V_\varepsilon$. This identification is $g$-equivariant. We will assume that $\alpha \in (0, \varepsilon_0]$ is small enough so that for any $b \in B$, if $x \in X_b$, $d^{X_b}(g^{-1}x, x) \leq \alpha$, then $x \in V_\varepsilon$.

By (5.60), (5.61) and the above considerations, it follows that for $\beta > 0$ small enough, the problem is localized on the $\varepsilon_0$-neighborhood $V_{\varepsilon_0}$ of $X^g$. 

By (5.59) and (5.60), we have

$$\exp(-\mathcal{A}_{zK,\ell}) = \tilde{F}_t(\mathcal{A}_{zK,\ell}) + \tilde{I}_t(t\mathcal{A}_{zK,\ell}).$$

From Lemma 5.15, the proof of the following lemma is the same as that of [13, Theorem 7.15].

**Lemma 5.16.** There exist $\beta > 0$, $C > 0$, $C' > 0$ such that if $t \in (0, 1]$, $|zK| \leq \beta$,

$$\|\tilde{I}_t(t\mathcal{A}_{zK,\ell})\|_{(1)} \leq C \exp(-C'/t).$$

By (5.59) and (5.60), we find that to establish (5.32), we may as well replace $\exp(-\mathcal{A}_{zK,\ell})$ by $\tilde{F}_t(\mathcal{A}_{zK,\ell})$.

From Lemma 5.15, the proof of the following lemma is the same as that of [13, Theorem 7.15].

**Lemma 5.16.** There exist $\beta > 0$, $C > 0$, $C' > 0$ such that if $t \in (0, 1]$, $|zK| \leq \beta$,

$$\|\tilde{I}_t(t\mathcal{A}_{zK,\ell})\|_{(1)} \leq C \exp(-C'/t).$$

By the above, it follows that $g\mathcal{F}_t(\mathcal{A}_{zK,\ell})(g^{-1}x, x) \in X_b$ vanishes if $d^{X_b}(g^{-1}x, x) \geq \alpha$. Here $d^{X_b}$ is the distance in $(X_b, g^{TX})$.
As in (3.3), let \( k(x, Z) \) be the smooth function on \( U_0 \) such that
\[
dv_X(x, Z) = k(x, Z)dv_{X^*}(x)dv_N(Z).
\]
In particular, \( k|_{X^s} = 1 \).

For \( \alpha \in \Lambda(T^*W^g) \), via (1.4) and (1.5), we will write
\[
\alpha = \sum_{1 \leq i_1 < \cdots < i_p \leq \ell} \alpha_{i_1, \ldots, i_p} \wedge e^{i_1} \wedge \cdots \wedge e^{i_p}, \quad \text{for } \alpha_{i_1, \ldots, i_p} \in \pi^*\Lambda(T^sB).
\]
We denote by
\[
\alpha_{\text{max}} := \alpha_{1, \ldots, \ell} \in \pi^*\Lambda(T^sB).
\]
Note that if the fiber is odd dimensional, our sign convention here is compatible with that in (0.11).

**Theorem 5.17.** For \( K \in \mathfrak{g}(g) \), there exist \( \beta > 0, \delta \in (0, 1] \) such that if \( z \in \mathbb{C}, |zK| \leq \beta, \ t \in (0, 1], \ x \in X^g, \)
\[
\left| t^{\frac{1}{2}\dim \mathcal{N}_s / X} \int_{Z \in \mathcal{N}_s |Z| \leq \delta / \sqrt{t}} \psi_{\mathbb{R} \times B} \Tr_s [g \tilde{F}_t(A_{zK,t})(g^{-1}(x, \sqrt{t}Z), (x, \sqrt{t}Z))] \times k(x, \sqrt{t}Z)dv_N(Z) - \left\{ \hat{A}_{g,zK}(TX, \nabla^{TX}) \ch_{g,zK}(E, \nabla^E) \right\}_{\text{max}} \right| \leq Ct^\delta.
\]

**Proof.** Sections 5.4–5.10 are devoted to the proof of this theorem. \( \square \)

**Proof of Theorem 5.9.** By (5.61) and the finite propagation speed argument above, we have
\[
\int_X \Tr_s [g \tilde{F}_t(A_{zK,t})(g^{-1}(x, x))]dv_X(x) = \int_{U_0} \Tr_s [g \tilde{F}_t(A_{zK,t})(g^{-1}(x, x))]dv_X(x) = \int_{(x, z) \in U_0 / \sqrt{t}} t^{\frac{1}{2}\dim \mathcal{N}_s / X} \Tr_s [g \tilde{F}_t(A_{zK,t})(g^{-1}(x, \sqrt{t}Z), (x, \sqrt{t}Z))] \times k(x, \sqrt{t}Z)dv_{X^*}(x)dv_N(Z).
\]
By Lemma 5.16 and Theorem 5.17 there exist \( \beta > 0, \gamma \in (0, 1] \) such that for \( |zK| \leq \beta, \ t \in (0, 1], \)
\[
\left| \psi_{\mathbb{R} \times B} \Tr_s [\exp(-A_{zK,t})] - \int_X \hat{A}_{g,zK}(TX, \nabla^{TX}) \ch_{g,zK}(E / \mathcal{S}, \nabla^E) \right| \leq Ct^\delta.
\]
So we obtain Theorem 5.9. \( \square \)

### 5.4. A Lichnerowicz formula

Let \( e_1, \ldots, e_n \) be a locally defined smooth orthonormal frame of \( TX \). Let \((F, \nabla^F)\) be a vector bundle with connection on \( X \). We use the notation
\[
(\nabla^F_{e_i})^2 = \sum_{i=1}^n (\nabla^F_{e_i})^2 - \nabla^F_{\sum_{i=1}^n \nabla^{TX}_{e_i} e_i}.
\]
Let \( H \) be the scalar curvature of \( X \).
**Proposition 5.18.** The following identity holds,

\[
A_{x,K,t} = -t \left( \nabla_{e_i}^X - \frac{z(K^X, e_i)}{4t} \right)^2 + dt \wedge \frac{\sqrt{7} D}{2} - dt \wedge \frac{c(K^X)}{8t} + \frac{t}{4} H + \frac{t}{2} R^{E/S}(e_i, e_j) c(e_i) c(e_j) - zm^{E/S}(K)
\]

\[
= -t \left( \nabla_{e_i}^X - \frac{z(K^X, e_i)}{4t} - dt \wedge \frac{c(e_i)}{4t} \right)^2 + \frac{t}{4} H + \frac{t}{2} R^{E/S}(e_i, e_j) c(e_i) c(e_j) - zm^{E/S}(K).
\]

Proof. We first assume that \(B\) is a point. From [2 Proposition 8.12], (2.8) and (5.1), we have

\[
A_{x,K,t} = -t \left( \nabla_{e_i}^X - \frac{z(K^X, e_i)}{4t} \right)^2 + dt \wedge \frac{\sqrt{7} D}{2} - dt \wedge \frac{c(K^X)}{8t} + \frac{t}{4} H + \frac{t}{2} R^{E/S}(e_i, e_j) c(e_i) c(e_j) - zm^{E/S}(K).
\]

If \(B\) is not a point, since \(K^X \in TX\), (5.69) follows from (5.70) and the adiabatic limit process in the proof of [3, Theorem 3.5].

The proof of Proposition 5.18 is completed. \(\square\)

5.5. A local coordinate system near \(X^q\). Take \(x \in W^q\). Then the fiberwise exponential map \(Z \in T_x X \rightarrow \exp_x X(Z) \in X\) identifies \(B^{T_x X}(0, 16\varepsilon_0)\) with \(B^X(x, 16\varepsilon_0)\). With this identification, there exists a smooth function \(k'_x(Z), Z \in B^{T_x X}(0, a_X/2)\) such that

\[
dv_X(Z) = k'_x(Z) dv_{TX}(Z), \quad \text{with } k'_x(0) = 1.
\]

We may and we will assume that \(\varepsilon_0\) is small enough so that if \(Z \in T_x X, |Z| \leq 4\varepsilon_0\),

\[
\frac{1}{2} g^T_x \leq g^T_x \leq \frac{3}{2} g^T_x.
\]

Recall that \(\vartheta_K\) is the one form dual to \(K^X\) defined in (5.1).

**Definition 5.19.** Let \(1\nabla^{E,t}\) be the connection on \(\Lambda(T^*\mathbb{R}) \otimes \pi^* \Lambda(T^*B) \otimes \mathcal{E}\) along the fibres,

\[
1\nabla^{E,t} := \nabla^E + \frac{1}{2\sqrt{t}} \langle S(\cdot)e_j, f^H_p \rangle c(e_j) f^p \wedge + \frac{1}{4t} \langle S(\cdot)f^H_p, f^H_q \rangle f^p \wedge f^q \wedge - \frac{z\vartheta_K(\cdot)}{4t} - dt \wedge \frac{c(\cdot)}{4\sqrt{t}}.
\]

In the sequel, we will trivialize \(\Lambda(T^*\mathbb{R}) \otimes \pi^* \Lambda(T^*B) \otimes \mathcal{E}\) by parallel transport along \(u \in [0, 1] \rightarrow uZ\) with respect to the connection \(1\nabla^{E,t}\). Observe that the above connection is \(g\)-invariant.

From (1.10) and (1.13), we have \(S(e_i)e_j = S(e_j)e_i\). Let \(L\) be a trivial line bundle over \(W\). We equip a connection on \(L\) by

\[
\nabla^L = d - \frac{z\vartheta_K}{4}.
\]
Thus
\[(5.75)\quad R^L = (\nabla^L)^2 = -\frac{zd\vartheta_K}{4}.\]

From (1.30), (3.3), (3.5), (5.74) and (5.75), we could calculate that
\[(5.76)\quad \left(1\nabla^{\xi,1}\right)^2(e_i, e_j) = \frac{1}{4} (R^X e_k, e_i, e_j) c(e_k) c(e_i) + \frac{1}{2} (R^X f_p H e_i, e_j) c(e_k) f_p \wedge
\quad + \frac{1}{4} (R^X (f_p^H f_q) e_i, e_j) f_p \wedge f_q \wedge + R^K e_i, e_j - \frac{z}{2} \langle m^X (K) e_i, e_j \rangle.
\]

Note that from (5.73), \( \left(1\nabla^{\xi,1}\right)^2 \) could be obtained from \( \left(1\nabla^{\xi,1}\right)^2 \) by replacing \( f_p \wedge, f_q \wedge \) and \( K \) by \( \frac{L^p}{\sqrt{t}} \wedge, \frac{L^p}{\sqrt{t}} \wedge \) and \( \frac{K}{t} \).

Let \( \Gamma^{\xi,t} \) be the connection form of \( 1\nabla^{\xi,t} \) in the trivialization of \( \mathcal{E} \) associated with the connection \( 1\nabla^{\xi,t} \) on \( \mathcal{E} \). Then by [2 Proposition 1.18], we have
\[(5.77)\quad \Gamma^{\xi,t}_Z(e_i) = \frac{1}{2} \left(1\nabla^{\xi,t}\right)_Z^2 (Z, e_i) + f_{ijk}(Z) c(e_j) c(e_k) + \frac{1}{\sqrt{t}} f_{ijp}(Z) c(e_j) f_p \wedge
\quad + \frac{1}{t} f_{ipq}(Z) f_p \wedge f_q \wedge + \frac{1}{t} h_i(zK, Z) + g_i(Z),
\]
where \( f_{iab}(Z) = O(|Z|^2), g_i(Z) = O(|Z|^2) \in \text{End}_{c(TX)}(\mathcal{E}), h_i(zK, Z) \) is a function depending linearly on \( zK \) such that \( h_i(zK, Z) = O(|Z|^2) \) for \( |zK| < \beta \).

5.6. Replacing \( X \) by \( T_x X \). Let \( \gamma(u) \) be a smooth even function from \( \mathbb{R} \) into \([0, 1]\) such that
\[(5.78)\quad \gamma(u) = \begin{cases} 1 & \text{if } |u| \leq 1/2; \\ 0 & \text{if } |u| \geq 1. \end{cases}
\]
If \( Z \in T_x X \), put
\[(5.79)\quad \rho(Z) = \gamma \left( \frac{|Z|}{4\varepsilon_0} \right).\]
Then
\[(5.80)\quad \rho(Z) = \begin{cases} 1 & \text{if } |Z| \leq 2\varepsilon_0; \\ 0 & \text{if } |Z| \geq 4\varepsilon_0. \end{cases}
\]

For \( x \in W^\mu \), let \( H_x \) be the vector space of smooth sections of \( \Lambda(T^*\mathbb{R}) \otimes \pi^* (\Lambda(T^*B)) \otimes \mathcal{E}_x \) over \( T_x X \). Let \( \Delta^{TX} \) be the (negative) standard Laplacian on the fibers of \( TX \).

Let \( L^t_{x,zK} \) be the differential operator acting on \( H_x \),
\[(5.81)\quad L^t_{x,zK} = (1 - \rho^2(Z))(-t\Delta^{TX}) + \rho^2(Z)(\mathcal{A}_{xK,t}).
\]

Let \( \widetilde{F}_t(L^t_{x,zK})(Z, Z') \) be the smooth kernel of \( \widetilde{F}_t(L^t_{x,zK}) \) with respect to \( dv_{TX}(Z') \). Using the finite propagation speed for solutions of hyperbolic equations [33 Appendix D.2] and (5.71), we find that if \( Z \in N_{X^s/x,x}, |Z| \leq \varepsilon_0 \), then
\[(5.82)\quad \widetilde{F}_t(\mathcal{A}_{xK,t})(g^{-1}Z, Z)k^t_x(Z) = \widetilde{F}_t(L^t_{x,zK})(g^{-1}Z, Z).
\]
In our proof of Theorem 5.17 we can then replace \( \mathcal{A}_{xK,t} \) by \( L^t_{x,zK} \).
5.7. The Getzler rescaling. Let $\text{Op}_x$ be the set of scalar differential operators on $T_xX$ acting on $H_x$. Then by (5.62),

\begin{equation}
L_{x,zK}^{1,t} \in (\Lambda(T^*\mathbb{R}) \otimes \pi^*(\Lambda(T^*B)) \otimes c(TX) \otimes \text{End}(E))_x \otimes \text{Op}_x.
\end{equation}

For $t > 0$, let $H_t : H_x \to H_x$ be the linear map

\begin{equation}
H_t h(Z) = h(Z/\sqrt{t}).
\end{equation}

Let $L_{x,zK}^{2,t}$ be the differential operator acting on $H_x$ defined by

\begin{equation}
L_{x,zK}^{2,t} = H_t^{-1}L_{x,zK}^{1,t}H_t.
\end{equation}

By (5.83),

\begin{equation}
L_{x,zK}^{2,t} \in (\Lambda(T^*\mathbb{R}) \otimes \pi^*(\Lambda(T^*B)) \otimes c(TX) \otimes \text{End}(E))_x \otimes \text{Op}_x.
\end{equation}

Recall that $\dim X^g = \ell$ and $\dim N_{X^g/X} = n - \ell$. Let $(e_1, \cdots, e_\ell)$ be an orthonormal oriented basis of $T_xX^g$, let $(e_{\ell+1}, \cdots, e_n)$ be an orthonormal oriented basis of $N_{X^g/X}$, so that $(e_1, \cdots, e_n)$ is an orthonormal oriented basis of $T_xX$. We denote with an superscript the corresponding dual basis.

For $1 \leq j \leq \ell$, the operators $e^j \wedge$ and $i_{e_j}$ act as odd operators on $\Lambda(T^*X^g)$.

**Definition 5.20.** For $t > 0$, put

\begin{equation}
c_t(e_j) = \frac{1}{\sqrt{t}} e^j - \sqrt{t} i_{e_j}, \quad 1 \leq j \leq \ell.
\end{equation}

Let $L_{x,zK}^{3,t}$ be the differential operator acting on $H_x$ obtained from $L_{x,zK}^{2,t}$ by replacing $c(e_j)$ by $c_t(e_j)$ for $1 \leq j \leq \ell$.

Let $\nabla_{e_i}$ be the usual derivative along $e_i$ on $T_xX$. From (5.69), (5.76), (5.77), (5.81), (5.85) and (5.87), we calculate that

\begin{equation}
L_{x,zK}^{3,t} = (1 - \rho^2(\sqrt{t}Z))(-\Delta^X) + \rho^2(\sqrt{t}Z) \left\{ -g^{ij}(\sqrt{t}Z) \left( \nabla'_{e_i} \nabla'_{e_j} - \sqrt{t} \Gamma^k_{ij}(\sqrt{t}Z) \nabla'_{e_k} \right) 
+ \frac{t}{4} H(\sqrt{t}Z) + \sum_{i,j} \frac{1}{2} R_{\sqrt{t}Z}^E(e_i, e_j) (e^i \wedge -t i_{e_j}) (e^j \wedge -t i_{e_j}) 
+ \sum_{j \leq \ell, i > \ell} \sqrt{t} R_{\sqrt{t}Z}^E(e_i, e_j) c(e_i) (e^j \wedge -t i_{e_j}) 
+ \sum_{i,j > \ell} \frac{1}{2} R_{\sqrt{t}Z}^E(e_i, e_j) c(e_i) c(e_j) 
+ \sum_{j \leq \ell} R_{\sqrt{t}Z}^E(e_j, f^p_H) (e^j \wedge -t i_{e_j}) f^p \wedge + \sum_{j > \ell} \sqrt{t} R_{\sqrt{t}Z}^E(e_j, f^H_p) c(e_j) f^p \wedge + \frac{1}{2} R_{\sqrt{t}Z}^E(f^H_p, f^H_q) f^p \wedge f^q \wedge - m_{\sqrt{t}Z}(zK) \right\}.
\end{equation}
where

\begin{align}
(5.89) \quad \nabla'_{e_i} = \nabla_{e_i} + \sum_{k=1, j \leq \ell} \frac{1}{8} \left( \langle R^{TX}_x(e_k, e_l)Z, e_i \rangle + O(\sqrt{t}|Z|^2) \right) (e^k \wedge -ti_{e_k}) (e^l \wedge -ti_{e_l}) \\
+ \sum_{k \leq \ell, l > \ell} \sqrt{t} \left( \langle R^{TX}_x(e_k, e_l)Z, e_i \rangle + O(\sqrt{t}|Z|^2) \right) (e^k \wedge -ti_{e_k}) c(e_l) \\
+ \sum_{k \leq \ell} \frac{t}{8} \left( \langle R^{TX}_x(e_k, e_l)Z, e_i \rangle + O(\sqrt{t}|Z|^2) \right) c(e_k) (e^l \wedge -ti_{e_l}) f^p \wedge \\
+ \sum_{k > \ell} \frac{\sqrt{t}}{4} \left( \langle R^{TX}_x(e_k, f^p_H)Z, e_i \rangle + O(\sqrt{t}|Z|^2) \right) c(e_k) f^p \wedge \\
+ \frac{1}{8} \left( \langle R^{TX}_x(f^p_H, f^q_H)Z, e_i \rangle + O(\sqrt{t}|Z|^2) \right) f^p \wedge f^q \wedge \\
- \frac{1}{4} \langle m^{TX}_x(zK)Z, e_i \rangle + \frac{1}{\sqrt{t}} h_i(zK, \sqrt{t}Z). \nonumber
\end{align}

Here $h_i(zK, Z)$ is a function depending linearly on $zK$ and $h_i(zK, Z) = O(|z|^2)$ for $|z| < \beta$.

Let $\tilde{F}_t(L^{3,t}_{x,k})(Z, Z')$ be the smooth kernel associated with $\tilde{F}_t(L^{3,t}_{x,k})$ with respect to $d\nu_{TX}(Z')$.

From the finite propagation speed argument explained before (5.62), we could also assume that $TX^\sigma$ and $\tilde{N}_{TX^\sigma}$ are spin. Let $S_X^\sigma$ and $S_N$ be the spinors of $TX^\sigma$ and $\tilde{N}_{TX^\sigma}$ respectively. Then $S_X = S_X^\sigma \otimes S_N$. Recall that $g$ acts on $(S_N \otimes E)_x$.

We may write $\tilde{F}_t(L^{3,t}_{x,k})(Z, Z')$ in the form

\begin{align}
(5.90) \quad \tilde{F}_t(L^{3,t}_{x,k})(Z, Z') &= \sum_{i_1, \ldots, i_p} \alpha^{j_1, \ldots, j_q}_{i_1, \ldots, i_p} e^{i_1} \wedge \cdots \wedge e^{i_p} \tilde{F}^{j_1, \ldots, j_q}_{t, i_1, \ldots, i_p} (Z, Z'), \\
\tilde{F}^{j_1, \ldots, j_q}_{t, i_1, \ldots, i_p} &\in (c(N{X^\sigma/X}) \otimes \text{End}(E))_x, \quad 1 \leq i_1 < \cdots < i_p \leq \ell, \quad 1 \leq j_1 < \cdots < j_q \leq \ell, \quad \\
\alpha^{j_1, \ldots, j_q}_{i_1, \ldots, i_p} &\in \Lambda(T^*\mathbb{R}) \otimes i^* \Lambda(T^*B). \nonumber
\end{align}

As explained in Section \[3\], $\ell = \dim X^\sigma$ has the same parity as $n = \dim X$. As in (5.64), put

\begin{align}
(5.91) \quad [\tilde{F}_t(L^{3,t}_{x,k})(Z, Z')]_{\text{max}}^{\text{max}} &= \alpha_{1, \ldots, \ell} \tilde{F}^{1, \ldots, \ell}_{t, 1, \ldots, \ell}(Z, Z'). \nonumber
\end{align}

In other words, $\tilde{F}^{1, \ldots, \ell}_{t, 1, \ldots, \ell}(Z, Z')$ is the coefficient of $e^1 \wedge \cdots \wedge e^\ell$ in (5.90).

The following proposition is an analogue of [13], Proposition 7.25 with the same proof.

**Proposition 5.21.** If $Z \in T_xX$, $|Z| \leq \varepsilon_0/\sqrt{t}$,

\begin{align}
(5.92) \quad t^{(n-\ell)/2} \text{Tr}_s[g\tilde{F}_t(A_{x,k,t})(g^{-1}(\sqrt{t}Z), \sqrt{t}Z)]k'_x(\sqrt{t}Z) \\
= (-i)^{\ell/2} 2^{\ell/2} \text{Tr}_s[S_N \otimes E][g\tilde{F}_t(L^{3,t}_{x,k})(g^{-1}Z, Z)]_{\text{max}}. \nonumber
\end{align}

**Remark 5.22.** As in [12] (1.6) and (1.7)], if $n$ is even,

\begin{align}
(5.93) \quad \text{Tr}_s(c(e_{i_1}) \cdots c(e_{i_p})) = 0, \quad \text{for } p < n, 1 \leq i_1, \ldots, i_p \leq n, \\
\text{Tr}_s(c(e_1) \cdots c(e_n)) = (-2i)^{n/2}; \nonumber
\end{align}
if $n$ is odd,
\begin{equation}
\text{Tr}[1] = 2^{n/2}, \quad \text{Tr}[c(e_1) \cdots c(e_n)] = (-i)^{(n+1)/2} 2^{(n-1)/2},
\end{equation}
and the trace of the other monomials are zero.

If $n = \dim X$ is odd, since (5.94) holds and the total degree of $\tilde{F}_t$ is even, we only take the trace for the odd degree Clifford part. In this case, (5.65) is replaced by
\begin{equation}
\int_{z \in N_n | z \leq \frac{\Delta}{2}} \psi_{\mathbb{R} \times B} \text{Tr}^{\text{odd}}[g \tilde{F}_t(A_{z,K},t)](g^{-1}(x, \sqrt{t}Z), (x, \sqrt{t}Z))
\end{equation}
\begin{equation}
\times k(x, \sqrt{t}Z) dv_2(N)(Z) - \left\{ \tilde{A}_{g,z,K}(TX, \nabla^T X) ch_{g,z,K}(E, \nabla^E) \right\}^{\text{max}} \leq Ct^\delta.
\end{equation}
In particular, since
\begin{equation}
\text{Tr}[c(e_1) \cdots c(e_n)] = (-i)^{(\ell+1)/2} 2^{(\ell-1)/2} t^{\ell/2} \left\{ \text{Tr}^{S_N}[c_t(e_1) \cdots c_t(e_\ell)c(e_{\ell+1}) \cdots c(e_n)] \right\}^{\text{max}},
\end{equation}
the analogue of (5.92) is
\begin{equation}
\int_{\mathbb{R} \times B} \psi_{\mathbb{R} \times B} \text{Tr}^{\text{odd}}[g \tilde{F}_t(A_{z,K},t)](g^{-1}(\sqrt{t}Z), \sqrt{t}Z) k_2'(\sqrt{t}Z)
\end{equation}
\begin{equation}
= (-i)^{(\ell+1)/2} 2^{(\ell-1)/2} t^{\ell/2} \left\{ \text{Tr}^{S_N\otimes E}[g \tilde{F}_t(L_{x,z,K}^3,\ell)](g^{-1}Z, Z) \right\}^{\text{max}}.
\end{equation}

Let $j : W^g \to W$ be the obvious embedding.

**Definition 5.23.** Let $L_{x,z,K}^{3.0}$ be the operator in
\begin{equation}
\Lambda(T^*\mathbb{R} \otimes \pi^*\Lambda(T^*B)) \otimes \Lambda(T^*X^g) \otimes c(N_{x/z}) \otimes \text{End}(E), \otimes \text{Op}_x,
\end{equation}
under the notation (5.68), given by
\begin{equation}
L_{x,z,K}^{3.0} = - \left( \nabla_e + \frac{1}{4} \left( (j^*R_{\nabla^T X} - m_{\nabla^T X}(zK)) Z, e_i \right) \right)^2 + j^* R_x^E - m^E(zK)x.
\end{equation}
In the sequel, we will write that a sequence of differential operators on $T_x X$ converges if its coefficients converge together with their derivatives uniformly on the compact subsets in $T_x X$.

Comparing with [13, Proposition 7.25], by (5.86) and (5.89), we have

**Proposition 5.24.** As $t \to 0$,
\begin{equation}
L_{x,z,K}^{3,\ell} \to L_{x,z,K}^{3,0}.
\end{equation}

### 5.8. A family of norms.

For $x \in W^g$, let $L_{\ell}^q$ be the vector space of smooth sections of $(\Lambda(T^*\mathbb{R} \otimes \pi^*\Lambda(T^*B)) \otimes \Lambda(T^*X^g) \otimes S_N \otimes E)_x$ on $T_x X$, let $I_{(r,q),x}$ be the vector space of smooth sections of
\begin{equation}
\left( \Lambda^1(T^*\mathbb{R}) \otimes \pi^*\Lambda^r(T^*B) \otimes \Lambda^g(T^*X^g) \otimes S_N \otimes E \right)_x
\end{equation}
on $T_x X$. We denote by $I_{x}^0 = \oplus I_{(r,q),x}^0$ the corresponding vector space of square-integrable sections. Put $k = \dim B$.

**Definition 5.25.** If $s \in I_{(r,q),x}$, then $I_{s,\ell,0}$ has compact support, put
\begin{equation}
|s|_s^2, x, 0 \quad = \quad \int_{T_x X} |s|^2 \left( 1 + |Z| \rho \left( \frac{\sqrt{t}Z}{2} \right) \right)^{2(k+\ell-q-r)} dv_{T_x X}(Z).
\end{equation}
Recall that by (5.80), if \( \rho(\sqrt{t}Z) > 0 \), then \( |\sqrt{t}Z| \leq 4\varepsilon_0 \). By the same arguments as in [15] Proposition 11.24], for \( t \in (0, 1] \), the following family of operators acting on \((\operatorname{I}_{x}^0, | \cdot |_{t,x,0})\) are uniformly bounded:

\[
1_{|\sqrt{t}Z|\leq 4\varepsilon_0} \sqrt{t}c_t(e_j), \quad 1_{|\sqrt{t}Z|\leq 4\varepsilon_0} |Z| \sqrt{t}c_t(e_j), \quad 1 \leq j \leq \ell.
\]

**Definition 5.26.** If \( s \in \operatorname{I}_{x} \) has compact support, put

\[
|s|^2_{t,x,1} = |s|^2_{t,x,0} + \sum_{i=1}^{n} |\nabla \epsilon_i s|^2_{t,x,0},
\]

and

\[
|s|_{t,x,-1} = \sup_{0 \neq s^t \in \operatorname{I}_{x}} \frac{|\langle s, s^t \rangle|}{|s|^1_{t,x,1}}.
\]

Let \((\operatorname{I}_{x}^1, | \cdot |_{t,x,1})\) be the Hilbert closure of the above vector space with respect to \( | \cdot |_{t,x,1} \). Let \((\operatorname{I}_{x}^{-1}, | \cdot |_{t,x,-1})\) be the antidual of \((\operatorname{I}_{x}^1, | \cdot |_{t,x,1})\). Then \((\operatorname{I}_{x}^1, | \cdot |_{t,x,1})\) and \((\operatorname{I}_{x}^{-1}, | \cdot |_{t,x,0})\) are densely embedded in \((\operatorname{I}_{x}^0, | \cdot |_{t,x,0})\) and \((\operatorname{I}_{x}^{-1}, | \cdot |_{t,x,1})\) with norms smaller than 1 respectively.

From now on, we fix \( K \in \mathfrak{j}(g) \).

Comparing with [13] Proposition 7.29], by (5.88) and (5.102), we get the following estimates.

**Lemma 5.27.** There exist constants \( C_i > 0 \), \( i = 1, 2, 3, 4 \), such that if \( t \in (0, 1] \), \( z \in \mathbb{C} \), \( |z| \leq 1 \), if \( n \in \mathbb{N} \), \( x \in X^g \), if the support of \( s, s^t \in \operatorname{I}_{x} \) is included in \( \{ Z \in T_x X : |Z| \leq n \} \), then

\[
\operatorname{Re} \langle L_{x,z}^{3,t} s, s \rangle_{t,x,0} \geq C_1 |s|^2_{t,x,1} - C_2 (1 + |nz|^2) |s|^2_{t,x,0},
\]

\[
|\operatorname{Im} \langle L_{x,z}^{3,t} s, s \rangle_{t,x,0}| \leq C_3 \left( (1 + |nz|) |s|^1_{t,x,1} |s|^2_{t,x,0} + |nz|^2 |s|^2_{t,x,0} \right),
\]

\[
|\langle L_{x,z}^{3,t} s, s^t \rangle_{t,x,0}| \leq C_4 (1 + |nz|^2) |s|^1_{t,x,1} |s^t|^1_{t,x,1}.
\]

**Proof.** We only need to observe that the terms containing \( |nz|^2 \) come from terms

\[
\left| \left( \rho(\sqrt{t}Z) (\langle m^{TX}(K)Z, \partial_t \rangle + t^{-1/2} h_t(zK, \sqrt{t}Z) \rangle \right)^2 \langle s, s \rangle_{t,x,0} \right|,
\]

which can be dominated by \( C(1 + |nz|^2) |s|^1_{t,x,0} \).

The proof of Lemma 5.27 is completed. \( \square \)

5.9. The kernel \( \tilde{F}_t(L_{x,K}^{3,t}) \) as an infinite sum. Let \( h \) be a smooth even function from \( \mathbb{R} \) into \([0, 1]\) such that

\[
h(u) = \begin{cases} 1 & \text{if } |u| \leq 1/2; \\ 0 & \text{if } |u| \geq 1. \end{cases}
\]

For \( n \in \mathbb{N} \), put

\[
h_n(u) = h \left( u + \frac{n}{2} \right) + h \left( u - \frac{n}{2} \right).
\]

Then \( h_n \) is a smooth even function whose support is included in \( [-\frac{n}{2}, -\frac{n}{2} + 1] \cup \left[ \frac{n}{2} - 1, \frac{n}{2} + 1 \right] \).

Set

\[
\mathcal{H}(u) = \sum_{n \in \mathbb{N}} h_n(u).
\]
The above sum is locally finite, and \( H(u) \) is a bounded smooth even function which takes positive values and has a positive lower bound on \( \mathbb{R} \).

Put
\[
(5.110) \quad k_n(u) = \frac{h_n(u)}{H(u)}.
\]

Then the \( k_n \) are bounded even smooth functions with bounded derivatives, and moreover
\[
(5.111) \quad \sum_{n \in \mathbb{N}} k_n = 1.
\]

**Definition 5.28.** For \( t \in [0, 1] \), \( n \in \mathbb{N}, a \in \mathbb{C} \), put
\[
(5.112) \quad F_t(n)(a) = \int_{-\infty}^{+\infty} \exp(\sqrt{2} iua) \exp\left( -\frac{u^2}{2} \right) f(\sqrt{tu}) k_n(s) \frac{du}{\sqrt{2\pi}}.
\]

By (5.54), (5.111) and (5.112),
\[
(5.113) \quad F_t(a) = \sum_{n \in \mathbb{N}} F_t(n)(a).
\]

Also, given \( m, m' \in \mathbb{N} \), there exist \( C > 0, C' > 0, C'' > 0 \) such that for any \( n \in \mathbb{N}, c > 0, \lambda \in V_c \),
\[
(5.114) \quad \sup_{a \in \mathbb{C}, |\text{Im}(a)| \leq c} |a|^m \left| F_t^{(m')}(a) \right| \leq C \exp(-C'n^2 + C''c^2).
\]

Let \( \widetilde{F}_t(n)(a) \) be the unique holomorphic function such that
\[
(5.115) \quad F_t(n)(a) = \widetilde{F}_t(n)(a^2).
\]

Recall that \( V_c \) was defined in (5.35). By (5.112), given \( m, m' \in \mathbb{N} \), there exist \( C > 0, C' > 0, C'' > 0 \) such that for any \( n \in \mathbb{N}, c > 0, \lambda \in V_c \),
\[
(5.116) \quad \left| \lambda \right|^m \left| \widetilde{F}_t^{(m')}(\lambda) \right| \leq C \exp(-C'n^2 + C''c^2).
\]

By (5.113),
\[
(5.117) \quad \widetilde{F}_t(a) = \sum_{n \in \mathbb{N}} \widetilde{F}_t(n)(a).
\]

Using (5.117), we get
\[
(5.118) \quad \widetilde{F}_t(L^3_{x,K}) = \sum_{n \in \mathbb{N}} \widetilde{F}_t(n)(L^3_{x,K}).
\]

More precisely, by (5.116) and using standard elliptic estimates, given \( t \in (0, 1] \), we have the identity
\[
(5.119) \quad \widetilde{F}_t(L^3_{x,K})(Z, Z') = \sum_{n \in \mathbb{N}} \widetilde{F}_t(n)(L^3_{x,K})(Z, Z')
\]
and the series in the right-hand side of (5.119) converges uniformly together with its derivatives on the compact sets in \( T_xX \times T_xX \).

**Definition 5.29.** For \( \gamma \) in (5.78), put
\[
(5.120) \quad L^3_{x,K,n} = - \left( 1 - \gamma \left( \frac{|Z|}{2(n+2)} \right) \right) \Delta^{T_X} + \gamma \left( \frac{|Z|}{2(n+2)} \right) L^3_{x,K}.
\]
Observe that if \( k_n(u) \neq 0 \), then \( |u| \leq \frac{\eta}{2} + 1 \). Using finite propagation speed and (5.72), we find that there is \( C > 0 \) such that if \( Z \in T_x X \), the support of \( \tilde{F}_{t,n}(L_{x,zK}^{3,t})(Z, \cdot) \) is included in \( \{ Z' \in T_x X : |Z' - Z| \leq n + 2 \} \). Therefore, given \( p \in \mathbb{N} \), if \( Z \in T_x X \), \( |Z| \leq p \), the support of \( \tilde{F}_{t,n}(L_{x,zK}^{3,t})(Z, \cdot) \) is included in \( \{ Z' \in T_x X : |Z' - Z| \leq n + p + 2 \} \).

If \( |Z| \leq n + p + 2 \), then \( (|Z'|/2(n + p + 2)) = 1 \). Using finite propagation speed again, we see that by (5.120), for \( Z \in T_x X \), \( |Z| \leq p \),

\[
\tilde{F}_{t,n}(L_{x,zK}^{3,t})(Z, Z') = \tilde{F}_{t,n}(L_{x,zK,n+p}^{3,t})(Z, Z').
\]

From Lemma 5.27, we have

\[
\text{Re}(L_{x,zK,n}^{3,t} s, s)_{t,x,0} \geq C_1 |s|_{t,x,1}^2 - C_2 (1 + |nz|^2)|s|_{t,x,0}^2.
\]

(5.122)

\[
|\text{Im}(L_{x,zK,n}^{3,t} s, s)_{t,x,0}| \leq C_3 \left( (1 + |nz|)|s|_{t,x,1} |s|_{t,x,0} + |nz|^2|s|_{t,x,0}^2 \right),
\]

\[
|(L_{x,zK,n}^{3,t} s', s')_{t,x,0}| \leq C_4 (1 + |nz|^2)|s|_{t,x,1} |s'|_{t,x,1}.
\]

Put

\[
L_{x,zK,n}^{3,0} = - \left( 1 - \gamma \left( \frac{|Z|}{2(n + 2)} \right) \right) \Delta^{T X} + \gamma \left( \frac{|Z|}{2(n + 2)} \right) L_{x,zK}^{3,0}.
\]

By Proposition 5.24 as \( t \to 0 \),

\[
L_{x,zK,n}^{3,t} \to L_{x,zK,n}^{3,0}.
\]

By (5.122), the functional analysis arguments in [13, §7.10-7.12] perfectly works here. We have the following uniform estimates, which is formally the same as [13, Theorem 7.38]. In particular, since the estimates in (5.105) and (5.122) are the analogue of [13, (7.131) and (7.148)], the proof of the following theorem is exactly the same as that of [13, Theorem 7.38].

**Theorem 5.30.** There exist \( C' > 0 \), \( C'' > 0 \), \( C''' > 0 \) such that for \( \eta > 0 \) small enough, there is \( c_\eta \in (0, 1) \) such that for any \( m \in \mathbb{N} \), there are \( C > 0 \), \( r \in \mathbb{N} \) such that for \( t \in (0, 1] \), \( |z| \leq c_\eta \), \( n \in \mathbb{N} \), \( x \in X^g \), \( Z, Z' \in T_x X \),

\[
\sup_{|\alpha|, |\alpha'| \leq m} \left| \frac{\partial^{\alpha + |\alpha'|}}{\partial Z^\alpha Z'^{\alpha'}} \tilde{F}_{t,n}(L_{x,zK}^{3,t})(Z, Z') \right| \leq C (1 + |Z| + |Z'|) \eta \exp \left( -C' n^2/4 + 2C'' \eta^2 \sup(|Z|^2, |Z'|^2) - C''' |Z - Z'|^2 \right).
\]

5.10. **A proof of Theorem 5.17.** Remark that as explained in the introduction of [13], \( L_{x,zK}^{3,t} \) does not have a fixed lower bound. So it is not possible to define a priori a honest heat kernel for \( \exp(-L_{x,zK}^{3,t}) \). So we cannot prove Theorem 5.17 following the arguments in [13, §11].

Since \( L_{x,zK,n+p}^{3,0} \) coincides with \( -\Delta^{T X} \) near infinity, the operator \( \tilde{F}_{0,n}(L_{x,zK,n+p}^{3,0}) \) is well defined. Also, by proceeding as in (5.121), if \( |Z| = |Z'| \leq p \), using finite propagation speed, we find that the kernel \( \tilde{F}_{0,n}(L_{x,zK,n+p}^{3,0})(Z, Z') \) does not depend on \( p \). Finally this kernel verifies estimates similar to (5.125) for \( \eta > 0 \) small enough and \( |z| \leq c_\eta \). Therefore we may define the kernel \( \exp(-L_{x,zK}^{3,0})(Z, Z') \) by

\[
\exp(-L_{x,zK}^{3,0})(Z, Z') = \sum_{n \in \mathbb{N}} \tilde{F}_{0,n}(L_{x,zK,n+p}^{3,0})(Z, Z'), \quad \text{for } |Z|, |Z'| \leq p,
\]
and the series in (5.120) converges uniformly on compact subsets of \( T_x X \times T_x X \) together with its derivatives.

Now we have the following uniform estimates, which is formally the same as [13 Theorem 7.43]. From Theorem 5.30 (5.122) and (5.125), the proof of the following theorem is exactly the same as that of [13 Theorem 7.43].

**Theorem 5.31.** There exist \( C'' > 0, C''' > 0 \) such that for \( \eta > 0 \) small enough, there exist \( c_\eta \in (0,1) \) \( r \in \mathbb{N} \), \( C > 0 \), such that for \( t \in (0,1] \), \( z \in \mathbb{C} \), \( |z| \leq c_\eta \), \( x \in X^\eta \), \( Z, Z' \in T_x X \),

\[
(5.127) \quad \left| \left( \tilde{F}_t(L_{x,zK}^{3,t}) - \exp(-L_{x,zK}^{3,0}) \right) (Z, Z') \right| \leq C t^{\frac{1}{4(n+1)}} (1 + |Z| + |Z'|)^r \exp(2C'' \eta^2 \sup |Z|^2 + |Z'|^2 - C'''|Z - Z'|^2 / 2).
\]

Now there is \( C > 0 \) such that if \( Z \in N_{X^\eta/X} \), then

\[
(5.128) \quad |g^{-1} Z - Z| \geq C |Z|.
\]

By (5.127) and (5.128), we find that there exists \( C''' > 0 \) such that if \( Z \in N_{X^\eta/X} \),

\[
(5.129) \quad \left| \left( \tilde{F}_t(L_{x,zK}^{3,t}) - \exp(-L_{x,zK}^{3,0}) \right) (g^{-1} Z, Z) \right| \leq C t^{\frac{1}{4(n+1)}} (1 + |Z|)^r \exp(2C'' \eta^2 |Z|^2 - C'''|Z|^2).
\]

For \( \eta > 0 \) small enough,

\[
(5.130) \quad 2C'' \eta^2 - C''' \leq -C''' / 2.
\]

So by (5.129), if \( Z \in N_{X^\eta/X} \),

\[
(5.131) \quad \left| \left( \tilde{F}_t(L_{x,zK}^{3,t}) - \exp(-L_{x,zK}^{3,0}) \right) (g^{-1} Z, Z) \right| \leq C t^{\frac{1}{4(n+1)}} \exp(-C''' |Z|^2 / 4).
\]

Put

\[
(5.132) \quad H^{TX} = j^* R^{TX} - m^{TX}(zK).
\]

Clearly \( H^{TX} \) splits as

\[
(5.133) \quad H^{TX} = H^{TX^\eta} + H^N.
\]

Using the Mehler’s formula (cf. e.g., [28 (1.34)]), by (5.99), for \( |z| \) small enough,

\[
(5.134) \quad \exp(-L_{x,zK}^{3,0})(g^{-1} Z, Z) = (4\pi)^{-\ell/2} \det^{\frac{1}{2}} \left( \frac{H^{TX}/2}{\sinh(H^{TX}/2)} \right) \cdot \exp \left( -\frac{1}{2} \left( \frac{H^{TX}/2}{\sinh(H^{TX}/2)} (\cosh(H^{TX}/2) - \exp(H^{TX}/2)g^{-1}) Z, Z \right) \right) \cdot \exp(-j^* R^E + m^E(zK)).
\]

Observe that for \( z \in \mathbb{C} \), \( |z| \) small enough, the right hand side of (5.134) is well-defined. Using (5.134), comparing with [28 (1.37)], if \( |z| \) is small enough,

\[
(5.135) \quad \int_N \exp(-L_{x,zK}^{3,0})(g^{-1} Z, Z) dv_N(Z) = (4\pi)^{-\ell/2} \det^{\frac{1}{2}} \left( \frac{H^{TX^\eta}/2}{\sinh(H^{TX^\eta}/2)} \right) \cdot \left( \det^{1/2}(1 - g^{-1}|x|) \det^{1/2}(1 - g \exp(-H^N)) \right)^{-1} \cdot \exp(-j^* R^E + m^E(zK)).
\]
Also

\[(5.136) \quad \text{Tr}_s^{E}[g \exp(-j^* R^E + m^E(zK))] = (-i)^{(n-\ell)/2} \det^{1/2} (1 - g^{-1}|_N) \text{Tr}[\exp(-j^* R^E + m^E(zK))].\]

Using (2.15), (2.16), (4.9), (5.135) and (5.136), we get

\[(5.137) \quad \psi_{R \times B} \int_N (-i)^{\ell/2} 2^{\ell/2} \text{Tr}_s^{E}[g \exp(-L^{3,0}_{x,zK})(g^{-1} Z, Z)] dv_N(Z) = (\hat{A}_{g,zK}(TX, \nabla^T X) \text{ch}_{g,zK}(E, \nabla^E)).\]

From (5.92), (5.131) and (5.137), we obtain Theorem 5.17 for \(n\) even.
If \(n\) is odd, following the explanation in Remark 5.22, the proof is the same.
The proof of Theorem 5.17 is completed.

### 5.11. A proof of Theorem 4.2.

Since \(v \geq t > 0\), we have

\[(5.138) \quad 0 \leq t^{-1} - v^{-1} < t^{-1}.\]

Set

\[(5.139) \quad A'_{K,t} = \left( B_t + \sqrt{t} c(K^X) \left( \frac{1}{t} - \frac{1}{v} \right) + t \cdot dt \wedge \frac{\partial}{\partial t} \right)^2 + \mathcal{L}_K.\]

Let \(A'^{(0)}_{K,t}\) be the piece of \(A'_{K,t}\) which has degree 0 in \(\Lambda(T^*(\mathbb{R} \times B))\). Then from (5.138), \(A'^{(0)}_{K,t}\) satisfies the same estimate in Lemma 5.2 and the estimate (5.47) of \(A_{K,t} - A'^{(0)}_{K,t}\) also holds for \(A'_{K,t} - A'^{(0)}_{K,t}\). Since \(v \geq t\), as \(t \to +\infty\), we have

\[(5.140) \quad \left| \frac{\partial}{\partial t} \left( \frac{\sqrt{t} c(K^X)}{4} \left( \frac{1}{t} - \frac{1}{v} \right) - \frac{c(T^H)}{4 \sqrt{t}} \right) \right| = O(t^{-3/2}).\]

Then the analogue of (5.29) holds. Thus replacing \(A_{zK,t}\) by \(A'_{K,t}\) in the proof of Theorem 5.1, we prove Theorem 4.2.

### 6. A proof of Theorem 4.3

In this section, we prove Theorem 4.3. This section is organized as follows. In Section 6.1, we establish a Lichnerowicz formula for \(B_{K,t,v}\). In Section 6.2, we prove Theorem 4.3 a). In Sections 6.3 - 6.8, we prove Theorem 4.3 b). In Section 6.9, we prove Theorem 4.3 c). In Section 6.10, we prove Theorem 4.3 d). In this section, we use the assumptions and the notations in Section 4.

#### 6.1. A Lichnerowicz formula.

Let \(L\) be a trivial line bundle over \(W\). We equip a connection on \(L\) by

\[(6.1) \quad \nabla^L_v = d - \frac{\partial K}{4v}.\]

Thus

\[(6.2) \quad R^L_v = (\nabla^L_v)^2 = -\frac{d\partial K}{4v}.\]
The corresponding Dirac operator is
\[
\mathcal{D}_v = \sum_{i=1}^{n} c(e_i) \nabla_{v,e_i}^E \otimes L = D - \frac{c(K^X)}{4v}.
\]

Since
\[
\nabla_{v,f^H}^E = \nabla_{v,f^H}^L,
\]
the new Bismut superconnection associated with $E \otimes L$ is
\[
\mathcal{E}_v^L = \mathcal{B}_L - \frac{\sqrt{t c(K^X)}}{4v}.
\]

**Theorem 6.1.** The following identity holds,
\[
\begin{align*}
\mathcal{B}_{K,t,v} &= -t \left( \frac{1}{2} (S(e_i) e_j, f_p^H) c(e_j) f_p^p \right. \\
&\quad + \frac{1}{4t} \langle S(e_i) f_p^H, f_q^H \rangle f_p^p \wedge f_q^q \wedge \left. - \frac{K^X, e_i}{4} \left( \frac{1}{t} + \frac{1}{v} \right)^2 \right) \\
&\quad + \frac{t}{4} H + \frac{t}{2} \left( R^E/S(e_i, e_j) - \frac{1}{v} \langle \nabla^{TX} K^X, e_j \rangle \right) c(e_i) c(e_j) \\
&\quad + \frac{1}{2} \left( R^E/S(f_p^H, f_q^H) - \frac{1}{2v} \langle T(f_p^H, f_q^H), K^X \rangle \right) f_p^p \wedge f_q^q \wedge -m^E/S(K^X) + \frac{1}{4v} |K^X|^2.
\end{align*}
\]

**Proof.** From (5.39), we have
\[
\begin{align*}
\mathcal{B}_{K,t,v} &= \left( \mathcal{B}_L - \frac{\sqrt{t c(K^X)}}{4v} \right)^2 + \mathcal{L}_K = -t \left( \frac{1}{2} (S(e_i) e_j, f_p^H) c(e_j) f_p^p \right. \\
&\quad + \frac{1}{4t} \langle S(e_i) f_p^H, f_q^H \rangle f_p^p \wedge f_q^q \wedge \left. - \frac{K^X, e_i}{4t} \right)^2 \\
&\quad + \frac{t}{4} H + \frac{t}{2} R^E/S(e_i, e_j) c(e_i) c(e_j) \\
&\quad + \frac{1}{2} \left( R^E/S(f_p^H, f_q^H) c(e_i) f_p^p \wedge + \frac{1}{2} R^E/S(f_p^H, f_q^H) f_p^p \wedge f_q^q \wedge -m^E/S(K^X) \right).
\end{align*}
\]

Since $G$ acts trivially on $L$, the corresponding $m^E(K)$ is the sense of (2.4) is given by
\[
m^L(K^X) = -K^X + \nabla_{v,K^X}^L = -\frac{|K^X|^2}{4v}.
\]

Then (6.6) follows from (3.4), (3.5), (6.2), (6.5) and (6.8).

The proof of Theorem 6.1 is completed. \qed

**6.2. A proof of Theorem 4.3 a.** From (2.15), (2.16), (3.1) and (3.2), set
\[
\gamma_{K,v} = -\left\{ \frac{dK}{8v^2} \exp \left( \frac{dK}{8v^2} \right) \right\}^{\max}
\]
Then by (3.6), we have
\[
\tilde{\gamma}_v = \int_{X^g} \gamma_{K,v} d\mu_{X^g}.
\]
We restate Theorem 4.3a) as follows.

**Theorem 6.2.** For $|K|$ small enough, given $v > 0$, when $t \to 0$,

$$\phi \tilde{\tau} \left[ g \frac{\sqrt{\epsilon}(K^X)}{4v} \exp\left(-B_{K,t,v}\right) \right] \to -\tilde{e}_v;$$

Proof. Let $da \wedge$ be odd Grassmann variable. We have

$$\tilde{\tau} \left[ g \frac{\sqrt{\epsilon}(K^X)}{4v} \exp\left(-B_{K,t,v}\right) \right] = \left\{ \tilde{\tau} \left[ g \exp\left(-B_{K,t,v} + da \frac{\sqrt{\epsilon}(K^X)}{4v}\right) \right] \right\}^{da}.$$ 

So from the same argument of Sections 5.2-5.10, letting $L_{x,K}$ in Section 5.7 defined from $B_{K,t,v} - da \frac{\sqrt{\epsilon}(K^X)}{4v}$, by (4.9), if $n$ is even, we have

$$\tilde{\tau} \left[ g \frac{\sqrt{\epsilon}(K^X)}{4v} \exp\left(-L_{x,K}\right) \right] = \tilde{\tau} \left[ g \frac{\sqrt{\epsilon}(K^X)}{4v} \exp\left(-L_{x,K}\right) \right]^{da} = -\tilde{\tau} \left[ g \frac{\sqrt{\epsilon}(K^X)}{4v} \exp\left(-L_{x,K}\right) \right]^{da} = -\tilde{\tau} \left[ g \frac{\sqrt{\epsilon}(K^X)}{4v} \exp\left(-L_{x,K}\right) \right]^{da}.$$ 

which is an analogue of (5.137).

From the analogue of (5.95), (5.97) and (6.13), we get the odd case.

The proof of Theorem 6.2 is completed. \Box

6.3. **Localization of the problem.** The proof of Theorem 1.3b) is devoted to Section 6.3-6.8.

Let $B_{K,t,v}^0$ be the piece of $B_{K,t,v}$ which has degree 0 in $\Lambda(T^*B)$. Then for $t \in (0,1]$, $v \in [t,1]$, by (5.138), $tB_{K,t,v}^0$ satisfies the same estimates as Lemma 5.11 uniformly for $v \in [t,1]$. Thus following the same arguments in the proof of Theorem 5.10, we have

**Theorem 6.3.** There exist $\beta > 0, C > 0, C' > 0$ such that if $K \in \mathfrak{g}, |K| \leq \beta$, $t \in (0,1]$, $v \in [t,1]$,

$$\|\tilde{I}_v(tB_{K,t,v})\|_{(1)} \leq C \exp(-C'/t).$$

So our proof of inequality (4.15) in Theorem 4.3 can be localized near $X^g$. As in Section 5.3, we may and we will assume that $W = B \times X, TX$ is spin and $E = S_X \otimes E$.

6.4. **A rescaling of the normal coordinate to $X^g,K$ in $X^g$.** In the sequel, we fix $g \in G, K_0 \in \mathfrak{g}(g)$ and

$$K = zK_0, \quad z \in \mathbb{R}^*.$$

Recall that $X^g$ and $X^g,K$ are totally geodesic in $X$. Given $\varepsilon > 0$, let $U_{\varepsilon}$ be the $\varepsilon$-neighbourhood of $X^g,K$ in $N_{X^g/K,X^g}$ (cf. the notation in the proof of Lemma 3.1). There exists $\varepsilon''_0 \in (0,1/2]$ such that for $0 < \varepsilon \leq 16\varepsilon''_0$, the map $(y_0,Z_0) \in N_{X^g,K/X^g} \to \exp_{y_0}^{X^g}(Z_0) \in X^g$ is a diffeomorphism from $U''$ into the tubular neighbourhood $V_{\varepsilon}$ of $X^g,K$ in $X^g$. By replacing $\varepsilon_0,\varepsilon''_0$ by inf$(\varepsilon_0,\varepsilon''_0)$, we may and we will assume that $\varepsilon_0'' = \varepsilon_0$.

Since $X^g$ is totally geodesic in $X$, the connection $\nabla^{TX}$ induces the connection $\nabla^{N_{X^g/X}}$ on $N_{X^g/X}$ (cf. (1.31)).
For $(y_0, Z_0) \in \mathcal{U}_\varepsilon'$, we identify $N_{X^g/X}(y_0, Z_0)$ with $N_{X^g/X,y_0}$ by parallel transport along the geodesic $s \in [0, 1] \to sZ_0$ with respect to $\nabla^T_X$. If $y_0 \in X^{g,K}$, $Z_0 \in N_{X^{g,K}/X,y_0}$, $Z \in N_{X^g/X,y_0}$, $|Z_0|$, $|Z| \leq 4\varepsilon_0$, we identify $(y_0, Z_0, Z)$ with $\exp^{X}_{\exp_{y_0}^{X}(Z_0)}(Z) \in X$. Therefore, $(y_0, Z_0, Z)$ defines a coordinate system on $X$ near $X^{g,K}$.

Recall that for $|z|$ small enough, $\bar{e}_v$ is a smooth form on $W^g$, the function $k$ is defined in (5.63) and $t' = \dim (X^{g,K})$.

**Theorem 6.4.** There exist $\beta_1 \in (0, 1]$, $\delta \in (0, 1]$ such that for $p \in \mathbb{N}$, there is $C > 0$ such that if $z \in \mathbb{R}^*$, $|z| \leq \beta_1$, $t \in (0, 1]$, $v \in [t, 1]$, $y_0 \in X^{g,K}$, $Z_0 \in N_{X^{g,K}/X,y_0}$, $|Z_0| \leq \varepsilon_0/\sqrt{v}$, then for $K = zK_0$,

\[
\left|\phi \int_{Z \in N_{X^{g,K}/X,y_0}|Z| \leq \varepsilon_0/\sqrt{v}} \frac{\sqrt{t' e(K)^{X}}}{4v} \widetilde{F}_1 (-B_{K,t,v}) \left( g^{-1}(y_0, \sqrt{v}Z_0, Z), (y_0, \sqrt{v}Z_0, Z) \right) \cdot k(y_0, \sqrt{v}Z_0, Z) d\nu_{N_{X^g/X}}(Z) + \gamma_{K,v}(y_0, \sqrt{v}Z_0) \right| \\
\leq C \left( \frac{1}{(1 + |Z_0|)^p} \left( \frac{t}{v} \right)^{\delta} \right).
\]

**Proof.** Sections 6.5-6.7 will be devoted to the proof of Theorem 6.4 \[\square\]

### 6.5. Getzler rescaling near $X^{g,K}$

Since $g$ preserves geodesics and the parallel transport, in the coordinate system in above subsection,

\[
g(Z_0, Z) = (Z_0, gZ).
\]

By an abuse of notation, we will often write $Z_0 + Z$ instead of $\exp^{X}_{\exp_{y_0}^{X}(Z_0)}(Z)$.

Comparing with (5.73), we set

\[
2\nabla^{E,t} := \nabla^E + \frac{1}{2\sqrt{t}} \left( x^2 \right) e_j f_j h^{p} c(e_j) f^p \wedge \\
+ \frac{1}{4t} \left( x^2 \right) h^{p} f^p \wedge h^q \wedge - \frac{\theta K_n}{4t} \left( 1 + \frac{t}{v} \right).
\]

Now we fix $Z_0 \in N_{X^{g,K}/X,y_0}$, $|Z_0| \leq \varepsilon_0$, and we take $Z \in T_{y_0}X$, $|Z| \leq 4\varepsilon_0$. The curve $s \in [0, 1] \to Z_0 + sZ$ lies in $B_X^{Y}(0, 5\varepsilon_0)$. Moreover we identify $TX_{Z_0} + Z$, $\mathcal{E}_{Z_0} + Z$ with $TX_{Z_0}$, $\mathcal{E}_{Z_0}$ by parallel transport with respect to the connections $\nabla^T_X$, $2\nabla^{E,t}$ along the curve.

When $Z_0 \in N_{X^{g,K}/X,y_0}$ is allowed to vary, we identify $TX_{Z_0}$, $\mathcal{E}_{Z_0}$ with $TX_{y_0}$, $\mathcal{E}_{y_0}$ by parallel transport with respect to the connections $\nabla^T_X$, $2\nabla^{E,t}$ along the curve $s \in [0, 1] \to sZ_0$.

We may and we will assume that $\varepsilon_0$ is small enough so that if $|Z_0| \leq \varepsilon_0$, $|Z| \leq 4\varepsilon_0$, then

\[
\frac{1}{2} g^T_X \leq g^T_{Z_0} \leq \frac{3}{2} g^T_{y_0}.
\]

For $x \in X^g$, we denote by $H_x$ the vector space of smooth sections of $\pi^*(\Lambda(T^*B)) \otimes \mathcal{E}_x$. We still define $\rho_x(Z)$ as in (5.80).

We fix $Z_0 \in N_{X^{g,K}/X,y_0}$, $|Z_0| \leq \varepsilon_0$. The considered trivializations depend explicitly on $Z_0$. We denote by $(B_{K,t,v})_{Z_0}$ the action of $B_{K,t,v}$ centered at $Z_0$, i.e.

\[
(B_{K,t,v})_{Z_0} f(Z) := B_{K,t,v} f(Z + Z_0).
\]
In (6.20), the operator \((B_{K,t,v})_0\) acts on \(H_{Z_0}\). Also \(H_{Z_0}\) is identified with \(H_{y_0}\), so that ultimately, \((B_{K,t,v})_0\) acts on \(H_{y_0}\).

We define \(k'_{(y_0,Z_0)}(Z)\) as in (5.71).

**Definition 6.5.** Let \(L^{1,(t,v)}_{Z_0,K}\) be the differential operator acting on \(H_{y_0}\),

\[
L^{1,(t,v)}_{Z_0,K} = -(1 - \rho^2(Z)(-t\Delta^X) + \rho^2(Z)(B_{K,t,v})_0.
\]

By proceeding as in (5.82), and using (6.17), we find that if \(Z_0 \in N_{X^s,K}/X^{y_0}\), \(Z \in N_{X^s,K}/X^{y_0}\), \(|Z|, |Z_0| \leq \varepsilon_0\),

\[
\tilde{F}_t(B_{K,t,v})(g^{-1}(Z_0, Z),(Z_0, Z))k'_{(y_0,Z_0)}(Z) = \tilde{F}_t(L^{1,(t,v)}_{Z_0,K})(g^{-1}Z, Z).
\]

We still define \(H_t\) as in (5.84). Let \(L^{2,(t,v)}_{Z_0,K}\) be the operator obtained from \(L^{1,(t,v)}_{Z_0,K}\) by

\[
L^{2,(t,v)}_{Z_0,K} = H_t^{-1}L^{1,(t,v)}_{Z_0,K}H_t.
\]

Let \((e_1, \cdots, e_{t'})\), \((e_{t'+1}, \cdots, e_{t})\), \((e_{t+1}, \cdots, e_n)\) be orthonormal oriented basis of \(T_{y_0}X^{y_0,K}, N_{X^s,K}/X^{y_0}, N_{X^s}/X, y_0\) respectively.

**Definition 6.6.** For \(t > 0\), put

\[
c_{v,t}(e_j) = c_t(e_j) := \frac{1}{\sqrt{t}}e_j \land -\sqrt{t}i_{e_j}, \quad 1 \leq j \leq \ell',
\]

\[
c_{v,t}(e_j) = c_{v,t}(e_j) := \frac{\sqrt{v}}{\sqrt{t}}e_j \land -\sqrt{t}i_{e_j}, \quad \ell' + 1 \leq j \leq \ell,
\]

\[
c_{v,t}(e_j) = c(e_j), \quad \ell + 1 \leq j \leq n,
\]

**Definition 6.7.** Let \(L^{3,(t,v)}_{Z_0,K}\) be the differential operator acting on \(H_{y_0}\) obtained from \(L^{2,(t,v)}_{Z_0,K}\) by replacing \(c(e_i)\) by \(c_{v,t}(e_i)\).

Then comparing with (5.88) and (5.89), from (6.6), we have

\[
L^{3,(t,v)}_{Z_0,K} = -(1 - \rho^2(\sqrt{t}Z))\Delta^X
\]

\[
+ \rho^2(\sqrt{t}Z) \cdot \left\{ -g^{ij}(\sqrt{t}Z)(\nabla'_{e_i} \nabla'_{e_j} - \Gamma^k_{ij}(\sqrt{t}Z)\sqrt{t}\nabla_{e_k}) + \frac{t}{4}H_{\sqrt{t}Z}
\]

\[
+ \frac{t}{2} \left( R^{X,S}_{\sqrt{t}Z}(e_i, e_j) - \frac{1}{v}(\nabla^T_{e_i} K^X, e_j) \right) c_{v,t}(e_i)c_{v,t}(e_j)
\]

\[
+ \sqrt{t} \left( R^{X,S}_{\sqrt{t}Z}(e_j, f^H_p) - \frac{1}{2v}(T(e_j, f^H_p, K^X) \right) c_{v,t}(e_i)f^p \land
\]

\[
+ \frac{1}{2} \left( R^{X,S}_{\sqrt{t}Z}(f^H_p, f^H_q) - \frac{1}{4v}(T(f^H_p, f^H_q, K^X) \right) f^p \land f^q \land -m^{X,S}_{\sqrt{t}Z}(K^X) + \frac{1}{4v}|K^X|^2 \right),
\]

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where
\begin{equation}
\nabla'_{e_i} = \nabla_{e_i} + \frac{t}{8} \left( \langle R_{(y_0, Z_0)}^{TX}(e_k, e_l) Z, e_i \rangle + O(\sqrt{t}|Z|^2) \right) c_{e_l}(e_k)c_{e_l}(e_l) \\
+ \frac{\sqrt{t}}{4} \left( \langle R_{(y_0, Z_0)}^{TX}(e_k, f_p^H) Z, e_i \rangle + O(\sqrt{t}|Z|^2) \right) c_{e_l}(e_k)f_p^\land \\
+ \frac{1}{8} \left( \langle R_{(y_0, Z_0)}^{TX}(f_p^H, f_q^H) Z, e_i \rangle + O(\sqrt{t}|Z|^2) \right) f_p^\land f_q^\land \\
+ \left( 1 + \frac{1}{v} \right) \langle m_{(y_0, Z_0)}^{TX}(K) Z, e_i \rangle + \sqrt{th_i}(K, \sqrt{t}Z) \left( \frac{1}{t} + \frac{1}{v} \right).
\end{equation}
Here \( h_i(K, Z) \) is a function depending linearly on \( K \) and \( h_i(K, Z) = O(|Z|^2) \) for \( |K| \) bounded.
Let \( \widetilde{F_i}(L_{Z_0,K}^{3,(l,v)})(Z, Z') \) be the smooth kernel associated with \( \widetilde{F_i}(L_{Z_0,K}^{3,(l,v)}) \) with respect to \( dv_{TX}(Z') \).

The proof of the following proposition is the same as Proposition 5.21.

**Proposition 6.8.** For \( y_0 \in X^g, K \in N_{X^g,X^g,y_0}, \ |Z_0| \leq \varepsilon_0, \ Z \in N_{X^g/X,y_0}, \ |Z| \leq \varepsilon_0/\sqrt{t} \), if \( \ell \) is even,
\begin{equation}
t^{\frac{1}{2} \dim N_{X^g/X}} \text{Tr}_{\text{even}} \left[ \frac{\sqrt{t}c(KX)}{4v} \widetilde{F_i}(B_{K},v)(g^{-1}(Z_0, \sqrt{t}Z), (Z_0, \sqrt{t}Z)) \right] k_{(y_0,Z_0)}(\sqrt{t}Z) \end{equation}
\begin{equation}
= (\omega \ell / 2)^{\frac{\ell}{2}} \frac{1}{v(\ell - 2)/2} \text{Tr}_{\text{even}}^{S_{0} \otimes E} \left[ gM_\ell(K) \widetilde{F_i}(L_{Z_0,K}^{3,(l,v)})(g^{-1}Z, Z) \right]_{\text{max}},
\end{equation}
and if \( \ell \) is odd,
\begin{equation}
t^{\frac{1}{2} \dim N_{X^g/X}} \text{Tr}_{\text{even}} \left[ \frac{\sqrt{t}c(KX)}{4v} \widetilde{F_i}(B_{K},v)(g^{-1}(Z_0, \sqrt{t}Z), (Z_0, \sqrt{t}Z)) \right] k_{(y_0,Z_0)}(\sqrt{t}Z) \end{equation}
\begin{equation}
= (\omega \ell + 1)/2^{(\ell - 1)/2} \frac{1}{v(\ell - 2)/2} \text{Tr}_{\text{even}}^{S_{0} \otimes E} \left[ gM_\ell(K) \widetilde{F_i}(L_{Z_0,K}^{3,(l,v)})(g^{-1}Z, Z) \right]_{\text{max}},
\end{equation}
where
\begin{equation}
M_\ell(K) = \sqrt{t} \left( \sum_{\ell' \leq k \leq \ell} \langle KX, e_k \rangle \left( \frac{\sqrt{v}}{\sqrt{t}} c_{k} - \frac{\sqrt{t}}{\sqrt{v}} c_{k} \right) + \sum_{\ell' \leq k \leq \ell} \langle KX, e_k \rangle c(e_k) \right).
\end{equation}

Recall that \( j : W^g \to W \) is the obvious embedding.

**Definition 6.9.** If \( x \in X^g \), let \( \tilde{L}_{x,K}^{3,(0,v)} \) be the operator in \( (\pi^*A(T^*B) \otimes A(T^*X^g) \otimes \text{End}(E))_x \otimes \text{Op}_x \) given by
\begin{equation}
\nabla e_i = \left( \nabla_{e_i} + \frac{1}{4} (j^*R_{x}^{TX} - m_{TX}(K))Z, e_i \right)^2 \\
+ j^* R_{x}^{E} - m_{x}(K)_x - \frac{1}{4v}(d - i_{Kx})d_{K}.
\end{equation}

Let \( \psi_{e_i} \in \text{End}(\Lambda(T^*X^g)) \) be the morphism of exterior algebras such that
\begin{equation}
\psi_{e_i}(e^j) = e^j, \quad 1 \leq j \leq \ell',
\end{equation}
\begin{equation}
\psi_{e_i}(e^j) = \sqrt{te^j}, \quad \ell' + 1 \leq j \leq \ell,
\end{equation}
Recall that for \( x = (y_0, Z_0) \in X^g, \Lambda(T^*X^g)_{(y_0, Z_0)} \) has been identified with \( \Lambda(T^*X^g)_{y_0} \).
Definition 6.10. Let $L^{3,(0,v)}_{x,K}$ be the operator

$$L^{3,(0,v)}_{x,K} = \psi_a L^{3,(0,v)}_a \psi_a^{-1}. \tag{6.32}$$

From (6.25), we have

Theorem 6.11. Given $Z_0 \in N_{X_{s,K}/X_{s,y_0}}$, $|Z_0| \leq \varepsilon_0$, as $t \to 0$,

$$L^{3,(t,v)}_{x,K} \to L^{3,(0,v)}_{x,K}. \tag{6.33}$$

6.6. A family of norms. For $0 \leq p \leq \ell'$, $0 \leq q \leq \ell - \ell'$, put

$$\Lambda_{(p,q)}(T^*X^q)_y = \Lambda_{(p,q)}(T^*X^q)_{y_0} \oplus \Lambda_{(p,q)}(N_{X_{s,K}/X_{s,y_0}}). \tag{6.34}$$

The various $\Lambda_{(p,q)}(T^*X^q)_y$ are mutually orthogonal in $\Lambda(T^*X^q)_{y_0}$. Let $I_{y_0}$ be the vector space of smooth sections of $(\pi^*\Lambda(T^*B) \otimes \Lambda(T^*X^q) \otimes S_N \otimes E)_y$ on $T_{y_0}X$, let $I_{(p,q),y_0}$ be the vector space of smooth sections of $(\pi^*\Lambda(T^*B) \otimes \Lambda_{(p,q)}(T^*X^q) \otimes S_N \otimes E)_x$ on $T_{y_0}X$. Let $I^0_{y_0}$, $I^0_{(p,q),y_0}$ be the corresponding vector space of square-integrable sections.

Now we imitate constructions in [13, 31].

Definition 6.12. For $t \in [0,1]$, $v \in \mathbb{R}^*$, $y_0 \in X^q,K$, $Z_0 \in N_{X_{s,K}/X_{s,y_0}}$, $|Z_0| \leq \varepsilon_0/\sqrt{v}$, $s \in I_{(p,q),y_0}$, set

$$|s|_{t,v,Z_0,0}^2 = \int_{T_{y_0}X} |s|^2 \left(1 + (|Z_0| + |Z|)\rho \left(\frac{\sqrt{v}Z}{2}\right)\right)^{2(k+\ell'-p-r)} \cdot \left(1 + \sqrt{v}|Z|\rho \left(\frac{\sqrt{v}Z}{2}\right)\right)^{2(\ell'-q)} dv_{TX}(Z). \tag{6.35}$$

Then (6.35) induces a Hermitian product $(\cdot, \cdot)_{t,v,Z_0,0}$ on $I^0_{(p,q),y_0}$. We equip $I^0_{y_0} = \oplus I^0_{(p,q),y_0}$ with the direct sum of these Hermitian metrics.

Recall that by (5.30), if $\rho(\sqrt{v}Z) > 0$, then $|\sqrt{v}Z| \leq 4\varepsilon_0$. The following proposition is proved in [13] Proposition 8.16 (cf. also [15] Proposition 11.24).

Proposition 6.13. For $t \in (0,1]$, $v \in [t,1]$, $y_0 \in X^q,K$, $Z_0 \in N_{X_{s,K}/X_{s,y_0}}$, $|Z_0| \leq \varepsilon_0/\sqrt{v}$, the following family of operators acting on $(I^0_{y_0}, |t,v,Z_0,0|)$ are uniformly bounded:

$$1_{|\sqrt{v}Z| \leq 4\varepsilon_0} \sqrt{t}v_{c,t}(e_j), \quad 1_{|\sqrt{v}Z| \leq 4\varepsilon_0} |Z| \sqrt{t}v_{c,t}(e_j), \quad 1_{|\sqrt{v}Z| \leq 4\varepsilon_0} \sqrt{t/vc,t}(e_j), \quad 1_{|\sqrt{v}Z| \leq 4\varepsilon_0} Z |\sqrt{t/vc,t}(e_j)|, \tag{6.36}$$

for $1 \leq j \leq \ell'$, $1 \leq j \leq \ell' + 1 \leq j \leq \ell$.

Definition 6.14. For $t \in [0,1]$, $v \in \mathbb{R}^*$, $y_0 \in X^q,K$, $Z_0 \in N_{X_{s,K}/X_{s,y_0}}$, $|Z_0| \leq \varepsilon_0/\sqrt{v}$, if $s \in I_{y_0}$ has compact support, set

$$|s|_{t,v,Z_0,1}^2 = |s|_{t,v,Z_0,0}^2 + \frac{1}{t} |\rho(\sqrt{v}Z)|K^X |(\sqrt{v}Z_0 + \sqrt{v}Z)s|_{t,v,Z_0,0}^2 + \sum_{j=1}^n |\nabla v_s|_{t,v,Z_0,0}^2. \tag{6.37}$$

Note that $|s|_{t,v,Z_0,1}$ depends explicitly on $K = zK_0$. In fact, $|s|_{t,v,Z_0,1}$ depends on $z \in \mathbb{R}^*$. If $Z_0 \in N_{X_{s,K}/X_{s,y_0}}$, $|Z_0| \leq \varepsilon_0$, $Z \in T_{y_0}X$, $|Z| \leq 4\varepsilon_0$, if $U \in T_{y_0}X$, let $\tau^{Z_0}U(Z) \in TX_{Z_0+Z}$ be the parallel transport of $U$ along the curve $t \to 2tZ_0$, $0 \leq t \leq 1/2$, $t \to \exp_{Z_0}^Y((2t - 1)Z)$, $1/2 \leq t \leq 1$, with respect to $\nabla^{TX}$. 
Theorem 6.15. There exist constants $C_i > 0$, $i = 1, 2, 3, 4$, such that if $t \in (0, 1]$, $v \in [t, 1]$, $n \in \mathbb{N}$, $y_0 \in X^{y,K}$, $Z_0 \in N_{X^{y,K}/X^{y,30}}$, $|Z_0| \leq \varepsilon_0/\sqrt{v}$, $z \in \mathbb{R}$, $|z| \leq 1$, if the support of $s, s' \in I_{y_0}$ is included in $\{Z \in T_{y_0}X : |Z| \leq n\}$, then

$$
\text{Re}(L^{(t,v)}_{\sqrt{v}Z_0, x} s, s)_{t,v,Z_0,0} \geq C_1 |s|^2_{t,v,Z_0,1} - C_2(1 + |nz|^2)|s|^2_{t,v,Z_0,0},
$$

(6.38)

$$
\text{Im}(L^{(t,v)}_{\sqrt{v}Z_0, x} s, s)_{t,v,Z_0,0} \leq C_3((1 + |nz|)|s|_{t,z,x,1}|s|_{t,v,Z_0,0} + |nz|^2)|s|^2_{t,v,Z_0,0},
$$

(6.45)

$$
|\langle L^{(t,v)}_{\sqrt{v}Z_0, x} s, s' \rangle_{t,v,Z_0,0}| \leq C_4(1 + |nz|^2)|s|_{t,v,Z_0,0}|s'|_{t,v,Z_0,1},
$$

Proof. Comparing with $L^{3,t}_{x,K}$ in (5.38) and (5.105), there are four additional terms in (6.25) which should be estimated:

$$
\frac{1}{2v} |\rho(\sqrt{v}Z)| z K_0^X (\sqrt{v}Z_0 + \sqrt{v}Z)|s|^2_{t,v,Z_0,0},
$$

(6.39)

$$
- \rho^2(\sqrt{v}Z) \frac{t}{2v} \left( \langle \nabla^{TX}_{x_0, z} \nabla \phi \rangle_{t,v,Z_0} z K_0^X (\sqrt{v}Z_0 + \sqrt{v}Z), \varepsilon \rangle_{t,v,Z_0,0} \right),
$$

(6.40)

$$
- \rho^2(\sqrt{v}Z) \frac{t}{2v} \left( \langle T(e_i, f_H^t), z K_0^X (\sqrt{v}Z_0 + \sqrt{v}Z)_{t,v,Z_0} \right),
$$

and

(6.41)

$$
- \rho^2(\sqrt{v}Z) \frac{1}{8v} \left( \langle T(f_H^t, f_H^t), z K_0^X (\sqrt{v}Z_0 + \sqrt{v}Z)_{t,v,Z_0} \right).
$$

(6.42)

The first term is controlled by (6.37) and the second term was estimated in the proof of [13, Theorem 6.15]. We only need to estimate (6.41) and (6.42), which are new terms in the family case.

Observe that the formula (3.20) also works for $Z \in N_{X^{y,K}}$. Thus we have

$$
\rho^2(\sqrt{v}Z) \frac{t}{2v} \left( \langle T(e_i, f_H^t), z K_0^X (\sqrt{v}Z_0 + \sqrt{v}Z)_{t,v,Z_0} \right),
$$

(6.43)

$$
= \rho^2(\sqrt{v}Z) v^{-1} \sqrt{t}|z| \varepsilon \cdot O(||\sqrt{v}Z_0 + \sqrt{v}Z||^2),
$$

and

$$
\rho^2(\sqrt{v}Z) \frac{1}{8v} \left( \langle T(f_H^t, f_H^t), z K_0^X (\sqrt{v}Z_0 + \sqrt{v}Z)_{t,v,Z_0} \right),
$$

(6.44)

$$
= \rho^2(\sqrt{v}Z)v^{-1} |z| \cdot O(||\sqrt{v}Z_0 + \sqrt{v}Z||^2) = \rho^2(\sqrt{v}Z)v^{-1} |z| \cdot O(||Z_0 + \sqrt{v}Z||^2).
$$

Using the fact that $v \leq 1$ and $t/v \leq 1$ and also Proposition 6.13, we find that the operators in (6.43) and (6.44) remain uniformly bounded with respect to $|s|_{t,v,Z_0,0}$.

The proof of Theorem 6.15 is completed. □

Definition 6.16. Put

$$
L^{3,(t,v)}_{Z_0,K,n} = - \left( 1 - \gamma \left( \frac{|Z|}{2(n + 2)} \right) \right) \Delta^{TX} + \gamma \left( \frac{|Z|}{2(n + 2)} \right) f^{(t,v)}_{Z_0,K,n},
$$

(6.45)
Using (6.19) and proceeding as in (5.121), i.e., using finite propagation speed, we see that if $Z \in T_{y_0}X$, $|Z| \leq p$,

\[(6.46) \quad \tilde{F}_{t,n}(L_{Z_0,K}^{3,t,v})(Z, Z') = \tilde{F}_{t,n}(L_{Z_0,K,n+p}^{3,t,v})(Z, Z').\]

Clearly, when replacing $L_{\sqrt{p/c_{Z_0,zK_0}}}^{3,t,v}$ in (6.38) by $L_{\sqrt{p/c_{Z_0,zK_0,n}}}$, the estimates (6.38) still hold when assuming only that $s, s'$ have compact support.

6.7. A Proof of Theorem 6.4. Since $W$ is a compact manifold, there exists a finite family of smooth functions $f_1, \ldots, f_r : W \to [0, 1]$ which have the following properties:

- $W_K = \cap_{j=1}^r \{x \in W : f_j(x) = 0\}$;
- $W_K, df_1, \ldots, df_r$ span $N_{X_\eta,K/y}$.

**Definition 6.17.** Let $Q_{t,v,z_0}$ be the family of operators

\[(6.47) \quad Q_{t,v,z_0} = \{\nabla_{e_i}, 1 \leq i \leq n; \frac{z}{\sqrt{v}}(\sqrt{t}Z)g_j(\sqrt{t}Z_0 + \sqrt{t}Z), 1 \leq j \leq r\}.\]

For $j \in \mathbb{N}$, let $Q_{t,v,z_0}^j$ be the set of operators $Q_1 \cdots Q_j$, with $Q_i \in Q_{t,v,z_0}, 1 \leq i \leq j$.

Following the arguments in [13, §§8.8-8.10], we have the following uniform estimates, which is formally the same as [13, (8.76)]. We only need to take care that in the proof of the analogue of [13, Proposition 8.22 and Theorem 8.24], there are two new terms like (6.41) and (6.42) appear in our family case. However, they are easy to be controlled as in (6.43) and (6.44).

**Theorem 6.18.** There exist $C > 0, C' > 0$ such that given $m > 0$, there exists $\beta_1 > 0$ such that if $t \in (0, 1], v \in [t, 1], z \in \mathbb{R}, |z| \leq \beta_1, y_0 \in X^g,K$, $Z \in N_{X_\eta/K,y_0}, |Z| \leq \varepsilon_0/\sqrt{t}$,

\[(6.48) \quad \left| (\tilde{F}_{t}(L_{\sqrt{p/c_{Z_0,zK_0}}}^{3,t,v}) - \exp(-L_{\sqrt{p/c_{Z_0,zK_0}}}^{3,0,v}))(g^{-1}Z, Z) \right| \leq C \left( \frac{t}{v} \right)^{\frac{1}{4(m+1)}} \cdot \frac{(1 + |Z_0|)^{r+1}}{(1 + |z|Z_0)}m \exp\left(-C'|Z|^2/4\right).\]

The kernel $\exp(-L_{\sqrt{p/c_{Z_0,zK_0}}}^{3,0,v})(g^{-1}Z, Z)$ here is defined in the same way as in (5.126).

Since $K^X$ vanishes on $X^g,K$, we get

\[(6.49) \quad \frac{z}{4v}(K_0^X(\sqrt{t}Z_0 + \sqrt{t}Z) - K_0^X(\sqrt{v}Z_0)) = z\sqrt{t/v}O(\sqrt{t/v}Z).\]

On the other hand, we have

\[(6.50) \quad \left| \frac{z}{4v}K_0^X(\sqrt{t}Z_0) \right| \leq C|z|Z_0|.|Z|\leq \beta_1, then\]

\[(6.51) \quad \left| z^r \phi_{\tilde{T}r} \left[ g_{\tilde{T}r}(X) K_0^X \left( \sqrt{t/v} \right) \exp(-B_{K,t,v}) + \varepsilon_v \right] \right| \leq C \left( \frac{t}{v} \right)^{\delta}.\]

6.8. A Proof of Theorem 4.3(b). Theorem 4.3(b) follows directly from the following theorem.

**Theorem 6.19.** There exist $\beta_1 > 0$, $r \in \mathbb{N}$, $C > 0$, $\delta \in (0, 1]$, such that if $t \in (0, 1], v \in [t, 1], z \in \mathbb{R}^r, |z| \leq \beta_1$, then

\[(6.51) \quad \left| z^r \phi_{\tilde{T}r} \left[ g_{\tilde{T}r}(X) K_0^X \left( \sqrt{t/v} \right) \exp(-B_{K,t,v}) + \varepsilon_v \right] \right| \leq C \left( \frac{t}{v} \right)^{\delta}.\]
Proof. Recall that $U_{\varepsilon}, U'_{\varepsilon}, U''_{\varepsilon}$ are $\varepsilon$-neighborhoods of $X^g, X^g, K, X^g, K$ in $N_{X^g/X}, N_{X^g, K/X}, N_{X^g, K/X}$ respectively. Let $k(y_0, Z_0)$ be the function defined on $X \cap U'_{\varepsilon}$ by the relation

$$dv_{X^g}(y_0, Z_0) = k(y_0, Z_0)dv_{X^g, K}(y_0)dv_{N_{X^g, K/X}}(Z_0).$$

Then

$$k|_{X^g, K} = 1.$$  

Recall that $\overline{F}_1(B_{K,t,v})(g^{-1}x, x)$ vanishes on $X \setminus U_{\varepsilon_0}$. Using (6.71), (6.52), we get

$$\phi \int_{U'_{\varepsilon_0}} \overline{\text{Tr}}' \left[ \frac{g\sqrt{tc(K^X)}}{4v} \exp(-B_{K,t,v})(g^{-1}x, x) \right] dv_X(x) + \int_{X^g \setminus U'_{\varepsilon_0}} \gamma_{K,u} dv_{X^g}$$

$$= \int_{X^g, K} v^{(t-v)/2} \int_{|Z| \leq \varepsilon_0/\sqrt{v}} \left[ \phi \int_{|Z| \leq \varepsilon_0} \overline{\text{Tr}}' \left[ \frac{g\sqrt{tc(K^X)}}{4v} \exp(-B_{K,t,v})(g^{-1}y_0, \sqrt{v}Z) \right] \cdot k(y_0, \sqrt{v}Z)(y_0, \sqrt{v}Z_0) dv_{N_{X^g/X}}(Z) + \gamma_{K,v}(y_0, \sqrt{v}Z_0) \right]$$

$$\overline{\Phi}(y_0, \sqrt{v}Z_0)dv_{N_{X^g, K/X}}(Z_0)dv_{X^g, K}(y_0).$$

Using Theorem 6.4 and (6.54), we find that there exist $C > 0$ and $\beta_1 > 0$ such that for $z \in \mathbb{R}^*$, $|z| \leq \beta_1$,

$$|z|^\gamma \leq \left( \frac{1}{v} \right)^\gamma.$$

Similar estimates can be obtained for

$$\phi \int_{X \setminus U'_{\varepsilon_0}} \overline{\text{Tr}}' \left[ \frac{g\sqrt{tc(K^X)}}{4v} \exp(-B_{K,t,v})(g^{-1}x, x) \right] dv_X(x)$$

$$+ \int_{X^g \setminus U'_{\varepsilon_0}} \gamma_{K,v} dv_{X^g} \leq C \left( \frac{1}{v} \right)^\gamma.$$

In fact, on $X \setminus U'_{\varepsilon_0}$, we observe that $|K^X|^2/2v$ has a positive lower bound. Then we adapt the above techniques to the case where $X^g, K = \emptyset$. The potentially annoying term $\frac{\sqrt{tc(K^X)}}{4v}$ will be controlled by the term $|K^X|^2/2v$.

The proof of Theorem 6.19 is completed. 

6.9. A Proof of Theorem 4.3 c). When $v \in [1, +\infty)$, $1/v$ remains bounded. By using the methods of the last section and of the present section, one sees easily that for $K \in \mathfrak{z}(g)$, $|K|$ small enough, there exists $C > 0$ such that for $t \in (0, 1], v \in [1, +\infty)$,

$$\left| \overline{\text{Tr}}' \left[ g\sqrt{tc(K^X)} \exp(-B_{K,t,v}) \right] \right| \leq C,$$

which is equivalent to Theorem 4.3 c).

The proof of Theorem 4.3 c) is completed.
6.10. A Proof of Theorem 4.3 d). In this subsection, we will prove Theorem 4.3 d) by using the method in [13, §9]. Since the singular term there does not appear here, our proof is in fact much easier.

We fix \( g \in G, K_0 \in \mathfrak{g}(g) \), and take \( K = zK_0 \) with \( z \in \mathbb{R} \).

From Theorem 6.1, we have

\[
(6.58) \quad \mathcal{B}_{K,t,\nu v} = -t \left( \nabla_{e_i}^X + \frac{1}{2\sqrt{t}} \langle S(e_i)e_j, f_p^H \rangle c(e_j) f^p \wedge 
+ \frac{1}{4t} \langle S(e_i)f_p^H, f_q^H \rangle f^p \wedge f^q \wedge -\frac{\langle X^\nu, e_i \rangle}{4t} \left( 1 + \frac{1}{v} \right) \right)^2 
+ \frac{t}{4} H + \frac{t}{2} \left( R^{E/S}(e_i, e_j) - \frac{1}{8v} \langle \nabla_{e_i}^X X^\nu, e_j \rangle \right) c(e_i)c(e_j) 
+ \sqrt{t} \left( R^{E/S}(e_i, f_p^H) - \frac{1}{4} \langle T(e_i, f_p^H), X^\nu \rangle \right) c(e_i) f^p \wedge 
+ \frac{1}{2} \left( R^{E/S}(f_p^H, f_q^H) - \frac{1}{8v} \langle T(f_p^H, f_q^H), X^\nu \rangle \right) f^p \wedge f^q \wedge -m^{E/S}(X^\nu) + \frac{1}{4} |X^\nu|^2. 
\]

As in sections 5.3 and 6.3, the proof of Theorem 4.3 d) can be localized near \( X^\nu \). In the followings, we will concentrate on the estimates near \( X^\nu K \). As in (6.56), the proof of the estimates near \( X^\nu \) and far from \( X^\nu K \) is much easier.

We may assume that for \( \varepsilon_0 \) taken in Section 6.4 if \( \varepsilon \in (0, 8\varepsilon_0] \), the map (\( y_0, Z \) \( N_{X^\nu K}$ \( \times X \) \( \exp_{y_0}^X(Z) \in X \) induces a diffeomorphism from the \( \varepsilon \)-neighborhood \( \mathcal{U}_\varepsilon \) of \( X^\nu K \) in \( N_{X^\nu K} \times X \) on the tubular neighborhood \( \nu \varepsilon \) of \( X^\nu K \) in \( X \).

As in (5.73) and (6.18), we put

\[
(6.59) \quad 3\nabla^{\varepsilon, t} := \nabla^\nu + \frac{1}{2\sqrt{t}} \langle S(\cdot)e_j, f_p^H \rangle c(e_j) f^p \wedge 
+ \frac{1}{4t} \langle S(\cdot)f_p^H, f_q^H \rangle f^p \wedge f^q \wedge -\frac{\partial_k(\cdot)}{4t} \left( 1 + \frac{1}{v} \right). 
\]

Take \( y_0 \in W^{g, K} \) in (2.12). If \( Z \in N_{X^\nu K} \times y_0 \), \( |Z| \leq 4\varepsilon_0 \), we identify \( \pi^\nu \Lambda(T^{*}B) \otimes E_{y_0} \) by parallel transport with respect to the connection \( 3\nabla^{\varepsilon, t} \) along the curve \( u \in [0, 1] \) \( u Z \).

Recall that \( \rho \) is the cut-off function in (5.80). Let

\[
(6.60) \quad L^{1, (t, \nu)}_{y_0, K} = (1 - \rho^2(Z))(-t \Delta^X) + \rho^2(Z)(B_{K,t,\nu v}). 
\]

We still define \( H_t \) as in (5.84) and define \( L^{2, (t, \nu)}_{y_0, K} \) as in (5.85) from \( L^{1, (t, \nu)}_{y_0, K} \). Let \( L^{3, (t, \nu)}_{y_0, K} \) be the operator obtained from \( L^{2, (t, \nu)}_{y_0, K} \) by replacing \( c(e_j) \) by \( c(\nu e_j) \) as in (5.87) for \( 1 \leq j \leq \ell' \), while leaving the \( c(e_j) \)'s unchanged for \( \ell' + 1 \leq j \leq n \).

By (3.3), \( \tilde{T} \) is \( G \)-invariant, thus \( [K^X, \tilde{T}] = 0 \). Since \( m^T(K) \) is skew-adjoint, by (2.5),

\[
(6.61) \quad \frac{\partial}{\partial s}\langle \tilde{T}, K^X \rangle = \langle \nabla^{\nu}_{\tilde{T}} X^\nu, K^X \rangle + \langle \tilde{T}, \nabla^{\nu}_{\tilde{T}} X^\nu \rangle = \langle \nabla^{\nu}_{\tilde{T}} X^\nu, K^X \rangle - \langle \nabla^{\nu}_{K^X} \tilde{T}, Z \rangle. 
\]

Thus

\[
(6.62) \quad \frac{\partial}{\partial s}\langle \tilde{T}, K^X \rangle|_{s=0} = 0. 
\]
From (2.6) and (6.61), we have
\begin{equation}
\begin{aligned}
\frac{\partial^2}{\partial s^2} \langle \tilde{T}, K^X \rangle_{(y_0,sZ)}|_{s=0} &= \frac{\partial}{\partial s} \langle \nabla^T_X \tilde{T}, K^X \rangle_{(y_0,sZ)}|_{s=0} - \frac{\partial}{\partial s} \langle \nabla^T_{K^X} \tilde{T}, Z \rangle_{(y_0,sZ)}|_{s=0} \\
&= \langle \nabla^T_X \tilde{T}, m^T_X(K)Z \rangle_{(y_0,0)} - \langle \nabla^T_X (m^T_X(K)\tilde{T}), Z \rangle_{(y_0,0)} \\
&= - \langle \nabla^T_X m^T_X(K)\tilde{T}, Z \rangle_{(y_0,0)} = \langle R^T_X(K^X, Z)\tilde{T}, Z \rangle_{(y_0,0)} = 0.
\end{aligned}
\end{equation}

From (6.62) and (6.63), we have
\begin{equation}
\langle \tilde{T}, K^X \rangle_{(y_0,Z)} = O(|Z|^3).
\end{equation}

As in (3.19), we have
\begin{equation}
|K^X(y_0,Z)|^2 = |m^T_{y_0}(K)Z|^2 + O(|Z|^3).
\end{equation}

Let $L^j': W^{g,K} \to W$ be the natural embedding. Put
\begin{equation}
L^{3,(0,v)}_{y_0,K} = - \left( \nabla_{e_i} + \frac{1}{4} \left( \left( f^* R^T_X - \left( 1 + \frac{1}{v} \right) m^T_X(K) \right) Z, e_i \right) \right)^2 + f^* R^E_{y_0} - \frac{1}{2v} \sum_{j,k \geq \ell''+1} \langle m^T_X(K)e_j, e_k \rangle_{y_0} c(e_j)c(e_k) + \frac{1}{4v} \sum_{j,k \leq \ell''} \langle R^T_X(m^T_X(K)Z, Z)e_j, e_k \rangle_{y_0} e^j \wedge e^k \wedge + \frac{1}{4v} |m^T_{y_0}(K)Z|^2.
\end{equation}

From (3.18), (6.58) and (6.64)-(6.66), we have
\begin{equation}
L^{3,(t,v)}_{y_0,K} \to L^{3,(0,v)}_{y_0,K}.
\end{equation}

Now we take a new trivialization as in [13, §9.5]. Take $Z_0 \in N_{X^{g,K}/X^{g,y_0}}, |Z_0| \leq \varepsilon_0$. If $Z \in T_{y_0}X, |Z| \leq 4\varepsilon_0$, we identify $\pi^*\Lambda(T^*B)\otimes \mathcal{E}_{Z+Z_0}$ with $\pi^*\Lambda(T^*B)\otimes \mathcal{E}_{Z_0}$ by parallel transport along the curve $u \in [0,1] \to \exp_{y_0}^Z(uZ)$ with respect to the connection $^3\nabla^E_t$. Also we identify $\pi^*\Lambda(T^*B)\otimes \mathcal{E}_{Z_0}$ with $\pi^*\Lambda(T^*B)\otimes \mathcal{E}_{y_0}$ by parallel transport along the curve $u \in [0,1] \to uZ_0$ with respect to the connection $\nabla^E$. Using this trivialization, the analogues of [13, Theorem 9.19 and 9.22] hold here following the same arguments except for replacing the norm in [13, (9.43)] by
\begin{equation}
|s|^2 \left|_{t,Z_0,0} = \int_{T_{y_0}X} |s|^2 \left( 1 + (|Z| + |Z_0|) \rho \left( \frac{\sqrt{|Z|}}{2} \right) \right)^{2(k+t-p-r)} dv_{T^X}(Z),
\end{equation}

here $s$ is a square integrable section of $(\pi^*\Lambda^t(T^*B)\otimes \Lambda^p(T^*X^{g,K})\otimes \mathcal{S}_{X^{g,K}/X} \otimes E)_{y_0}$ over $T_{y_0}X$.

As in [13, (9.52)-(9.57)], if $n$ is even, there exists $\beta > 0$, if $z \in \mathbb{R}^+, |z| \leq \beta$, for $t \to 0$,
\begin{equation}
\int_{X^{g,K}} \int_{(Z_0,Z) \in N_{X^{g,K}/X} \times N_{X^{g,K}/X}, |Z_0|, |Z| \leq \varepsilon_0} \text{Tr}_s \left[ g \left( c^T_X \right) \frac{4}{\sqrt{4v}} \tilde{F}(B_{z,K_{0,t,tv}})(g^{-1}(y_0, Z_0, Z), (y_0, Z_0, Z)) \right] dv_X(y_0, Z_0, Z)
\end{equation}

\begin{equation}
\to \int_{X^{g,K}} \int_{N_{X^{g,K}/X}} (-i)^{\ell'/2} 2^{\ell'/2} \text{Tr}_s \left[ g \left( m^T_{y_0}(K)Z \right) \frac{4}{2v} \exp \left( -L^{3,(0,v)}_{y_0,z K_{0}}(g^{-1}Z, Z) \right) \right] dv_{N_{X^{g,K}/X}}(Z).
\end{equation}
The heat kernel \( \exp \left( -L_{^{3,0}(0,v)}^{3,0}(g^{-1}Z, Z) \right) \) could be calculated in the same way as \( (5.134) \), which is an even function on \( Z \). So the right hand side of \( (6.69) \) is an integral of an odd function on \( Z \) over \( N_{X^{g,K}/X} \), which is zero.

If \( n \) is odd, by Remark \( 5.22 \) from the same argument above, as \( t \to 0 \),

\[
\int_{U''_{g_0}} \text{Tr}^\text{even} \left[ g \frac{c(K^X)}{4\sqrt{tv}} \exp \left( -\mathcal{B}_{K,t,tv} \right) \right] \to 0.
\]

(6.70)

After adapting the above technique to the case where \( X^{g,K} = \emptyset \), for \( z \in \mathbb{R}^* \), \(|z|\) small enough, as \( t \to 0 \), we have

\[
\int_{X \setminus U''_{g_0}} \widetilde{\text{Tr}}^t \left[ g \frac{c(K^X)}{4\sqrt{tv}} \exp \left( -\mathcal{B}_{K,t,tv} \right) \right] \to 0.
\]

(6.71)

The proof of Theorem \( 4.3 \) d) is completed.

References


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