Some congruences related to the \( q \)-Fermat quotients

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Abstract. We give \( q \)-analogues of the following congruences by Z.-W. Sun:

\[
\sum_{k=1}^{p-1} \frac{D_k}{k} \equiv -\frac{2^{p-1} - 1}{p} \pmod{p},
\]

\[
\sum_{k=1}^{p-1} \frac{H_k}{k^{2^k}} \equiv 0 \pmod{p}, \quad p \geq 5,
\]

where \( p \) is an odd prime, \( D_n = \sum_{k=0}^{n} \binom{n+k}{2k} \) are the Delannoy numbers, and \( H_n = \sum_{k=1}^{n} \frac{1}{k} \) are the harmonic numbers. We also prove that, for any positive integer \( m \) and prime \( p > m + 1 \),

\[
\sum_{1 \leq k_1 \leq \cdots \leq k_m \leq p-1} \frac{1}{k_1 \cdots k_m 2^{k_m}} \equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^m} \pmod{p},
\]

which is a multiple generalization of Kohnen’s congruence. Furthermore, a \( q \)-analogue of this congruence is established.

Keywords. Fermat quotients; \( q \)-Fermat quotients; Glaisher’s congruence; Kohnen’s congruence; \( q \)-Delannoy numbers; Dilcher’s identity

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1 Introduction

Fermat’s Little Theorem states that if \( p \) is a prime, then for any integer \( a \) not divisible by \( p \), the number \( a^{p-1} - 1 \) is a multiple of \( p \). Numbers of the form \( (a^{p-1} - 1)/p \) are called Fermat quotients of \( p \) to base \( a \). There are several different congruences for the Fermat quotients \( (2^{p-1} - 1)/p \) in the literature.

Let \( p \geq 3 \) be a prime. Since \( \frac{1}{k} \binom{p}{k} \equiv \frac{(-1)^{k-1}}{k} \pmod{p} \) for \( 1 \leq k \leq p-1 \), the following result is well-known:

\[
\frac{2^{p-1} - 1}{p} \equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^{k-1}}{k} \pmod{p}. \quad (1.1)
\]
A classical Glaisher's congruence (see [8,9]) for the Fermat quotients is

\[ \sum_{k=1}^{p-1} \frac{2^{k-1}}{k} \equiv -\frac{2^{p-1} - 1}{p} \quad (\mod p). \quad (1.2) \]

Kohnen [17] established the following congruence

\[ \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^{k-1}}{k} \quad (\mod p). \quad (1.3) \]

Z.-W. Sun [25] prove that

\[ \sum_{k=1}^{p-1} \frac{D_k}{k} \equiv -\frac{2^{p-1} - 1}{p} \quad (\mod p), \quad (1.4) \]

where \( D_n \) are the (central) Delannoy numbers defined by

\[ D_n = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}. \]

Let \( q \) be an indeterminate. The \( q \)-shifted factorial is defined by \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\), and the \( q \)-integer is defined as \([n] = (1 - q^n)/(1 - q)\). It is well known that

\[ \frac{x^p - 1}{x - 1} = \prod_{k=1}^{p-1} (x - \zeta^k), \quad (1.5) \]

where \( \zeta \) is a \( p \)-th primitive root of unity. Letting \( p \) be an odd prime and \( x = -1 \) in (1.5), we obtain \((-\zeta; \zeta)_{p-1} = 1\), which means that

\[ (-q; q)_{p-1} \equiv 1 \quad (\mod [p]). \quad (1.6) \]

The polynomials

\[ \frac{(-q; q)_{p-1} - 1}{[p]} \]

are called \( q \)-Fermat quotients of \( p \) (see [21] for a more general form).

Pan [21] gave \( q \)-analogues of (1.1) and (1.2) as follows:

\[ \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]} \equiv -\frac{2(-q; q)_{p-1} - 2}{[p]} - \frac{(p - 1)(1 - q)}{2} \quad (\mod [p]), \quad (1.7) \]

\[ \sum_{k=1}^{p-1} \frac{(-q; q)_{kq^k}}{2[k]} \equiv -\frac{(-q; q)_{p-1} - 1}{[p]} - \frac{(p - 1)(1 - q)}{2} \quad (\mod [p]), \quad (1.8) \]
Tauraso [29] obtained the following $q$-analogues of (1.2) and (1.3):

$$
\sum_{k=1}^{p-1} \frac{(-q; q)_{k-1} q^{-\binom{k}{2}}}{[k]} \equiv -\frac{(-q; q)_{p-1} - 1}{[p]} \quad \text{(mod } [p])
$$

$$
\sum_{k=1}^{p-1} \frac{q^k}{[k](-q; q)_k} \equiv \frac{(-q; q)_{p-1} - 1}{[p]} \quad \text{(mod } [p]).
$$

In this paper, we give a $q$-analogue of (1.4) and new $q$-analogues of (1.2) and (1.3).

**Theorem 1.1** For any prime $p \geq 3$, there holds

$$
\sum_{k=1}^{p-1} \frac{D_k(q)}{[k]} \equiv -\frac{(-q; q)_{p-1} - 1}{[p]} + \frac{(p-1)(1-q)}{4} \quad \text{(mod } [p]).
$$

(1.9)

Here the $q$-Delannoy numbers $D_n(q)$ are defined by

$$
D_n(q) = \sum_{k=0}^{n} \frac{1 + q^k \binom{n+k}{2k} \binom{2k}{k} q^{\binom{k}{2} - 2nk}}{2^{n-k}}.
$$

where

$$
\binom{n}{k} = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}
$$

stands for the $q$-binomial coefficient.

**Theorem 1.2** For any prime $p \geq 3$, there hold

$$
\sum_{k=1}^{p-1} \frac{(-q; q)_{k-1} q^k}{[k]} \equiv -\frac{(-q; q)_{p-1} - 1}{[p]} - \frac{(p-1)(1-q)}{2} \quad \text{(mod } [p]),
$$

(1.10)

$$
\sum_{k=1}^{p-1} \frac{q^{\binom{k+1}{2}}}{[k](-q; q)_k} \equiv \frac{(-q; q)_{p-1} - 1}{[p]} + \frac{(p-1)(1-q)}{2} \quad \text{(mod } [p]).
$$

(1.11)

We shall also give a multiple generalization of (1.3) as follows.

**Theorem 1.3** For any positive integer $m$ and prime $p > m + 1$, there holds

$$
\sum_{1 \leq k_1 \leq \cdots \leq k_m \leq p-1} \frac{1}{k_1 \cdots k_m 2^{k_m}} \equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^m} \quad \text{(mod } p).
$$

(1.12)

In particular, if $m$ is even, then

$$
\sum_{1 \leq k_1 \leq \cdots \leq k_m \leq p-1} \frac{1}{k_1 \cdots k_m 2^{k_m}} \equiv 0 \quad \text{(mod } p).
$$

(1.13)
Note that, when \( m = 2 \), the congruence (1.13) can be written as
\[
\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv 0 \pmod{p}, \quad p \geq 5,
\]
where \( H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \) are the harmonic numbers. The congruence (1.14) was first proved by Z.-W. Sun [25] and generalized to the modulus \( p^2 \) case by Sun and Zhao [28]. Some other generalizations and refinements of (1.14) can be found in [18,27].

Let
\[
H_n(q) = \sum_{k=1}^{n} \frac{1}{[k]}
\]
be the \( q \)-harmonic numbers. We shall prove the following neat \( q \)-analogue of (1.14).

**Theorem 1.4** For any prime \( p \geq 5 \), there holds
\[
\sum_{k=1}^{p-1} \frac{H_k(q)q^{(k+1)/2}}{[k](-q;q)_k} \equiv \frac{(p^2 - 1)(1 - q)^2}{24} \pmod{[p]}.
\]

The paper is organized as follows. In the next section, we give a proof of Theorem 1.1 by using some \( q \)-series identities and known \( q \)-congruences. In Section 3, we give proofs of Theorems 1.2–1.4 by first establishing a multiple series generalization of Kohnen’s identity [17]:
\[
\sum_{k=1}^{n} \frac{1}{k} (1 - x)^k = \sum_{k=1}^{n} \frac{(-1)^k}{k} \binom{n}{k} (x^k - 1).
\]
(1.15)

In fact, a \( q \)-analogue of (1.12) will be proved. Some consequences and remarks will be mentioned in the last section.

## 2 Proof of Theorem 1.1

Applying the Lagrange interpolation formula for \( x^r \) at the values \( q^{-k} \) \((0 \leq k \leq n)\) of \( x \), we have the following result (see [12, Theorem 1.1] for a generalization), which will be used in the proof of Theorem 1.1.

**Lemma 2.1** For \( n \geq 1 \) and \( 0 \leq r \leq n \), there holds
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{q^{(k+1)/2} - rk}{1 - xq^k} = \frac{(q;q)_n}{(x; q)_{n+1}} x^r.
\]
(2.1)
Proof of Theorem 1.1. By the $q$-Lucas theorem (see [4, 11, 20]), or by the factorization of $q$-binomial coefficients into cyclotomic polynomials (see [3, 16]), for any prime $p \geq 3$ and $(p - 1)/2 < k < p$, there holds

$$\binom{2k}{k} \equiv 0 \pmod{p}.$$ 

Hence, by the $q$-Chu-Vandermonde identity (see [1, (3.3.10)]), we have

$$\sum_{m=1}^{p-1} D_m(q) - 1 = \sum_{m=1}^{p-1} \sum_{k=1}^{m} \frac{1 + q^k}{2} \binom{2k}{k} \binom{m+k}{2k} q^{(k-2mk}$$

$$= \sum_{k=1}^{p-1} \frac{1 + q^k}{2} \binom{2k}{k} q^{(k)} \sum_{m=k}^{p-1} \sum_{j=k}^{2k} \frac{m}{[m]} \binom{m}{j} \binom{k}{2k-j}$$

$$\equiv \sum_{k=1}^{(p-1)/2} \frac{1 + q^k}{2} \binom{2k}{k} q^{(k)} \sum_{j=k}^{2k} \sum_{m=j}^{p-1} \frac{m}{[m]} \binom{m}{j} \binom{k}{2k-j} \pmod{p}.$$ 

Note that

$$\frac{1}{[m]} \binom{m}{j} = \frac{1}{[j]} \binom{m-1}{j-1},$$

$$\sum_{m=j}^{p-1} \frac{m-1}{[j-1]} q^{-m} = \binom{p-1}{j} q^{-(p-1)j},$$

$$\binom{p-1}{k} = \prod_{j=1}^{k} \frac{1 - q^{p-j}}{1 - q^j} = \prod_{j=1}^{k} \frac{1 - q^{-j}}{1 - q^j} = (-1)^k q^{-(k+1)/2} \pmod{p}. \quad (2.2)$$

For $1 \leq k \leq (p-1)/2$, we have

$$\sum_{j=k}^{2k} \sum_{m=j}^{p-1} q^{-j(2k-j+m)} \frac{m}{[m]} \binom{k}{2k-j} = \sum_{j=k}^{2k} \frac{q^{-j(2k-j)}}{[j]} \binom{k}{j} \sum_{m=j}^{p-1} \frac{m-1}{[j-1]} q^{-m}$$

$$\equiv \sum_{j=k}^{2k} q^{-j(2k-j+p-1)} \frac{k}{[j]} \binom{p-1}{j} \pmod{p}$$

$$\equiv \sum_{j=k}^{2k} (-1)^j q^{(j+1)/2-2j} k \binom{k}{j} \pmod{p}.$$
By Lemma 2.1, we have
\[ \sum_{j=k}^{2k} (-1)^j q^{(j+1) - 2j} \binom{k}{j - k} = (1 - q)q^{-k(3k-1)/2} \sum_{j=k}^{2k} (-1)^j q^{j-k+1} - (j-k)q \binom{k}{j - k} \]
\[ = (1-k)(1-q)q^{-k(3k-1)/2+k^2} \]
\[ = (-1)^k q^{-k} \binom{k}{k+1}. \]

It follows that
\[ \sum_{m=1}^{p-1} \frac{D_m(q) - 1}{m} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^k (1 + q^k)}{2k} \pmod{[p]}. \] (2.3)

Since \( q^{-k} \equiv q^{p-k} \pmod{[p]} \), we have
\[ \frac{(-1)^k q^k}{k} = \frac{(-1)^k q^k (1 - q)}{1 - q^k} = \frac{(-1)^k - (1 - q)}{1 - q^k} = \frac{(-1)^{p-k} (1 - q)}{1 - q^p} = \frac{(-1)^{p-k}}{[p-k]}, \] (2.4)

and so the congruence (2.3) can be written as
\[ \sum_{m=1}^{p-1} \frac{D_m(q) - 1}{m} \equiv \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} + \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{(-1)^{p-k}}{[p-k]} \]
\[ = \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} + \frac{1}{2} \sum_{k=(p+1)/2}^{p-1} \frac{(-1)^k}{k} \]
\[ = \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \pmod{[p]}. \]

The proof then follows from (1.7) and the following congruence due to Andrews [2]:
\[ H_{p-1}(q) \equiv \frac{(p-1)(1-q)}{2} \pmod{[p]}. \] (2.5)

**Remark.** If we define the \( q \)-Delannoy numbers by
\[ \overline{D}_n(q) = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{k}{2k} q^{k-2nk}, \]
then we have the following congruence:
\[ \sum_{m=1}^{p-1} \frac{1 - \overline{D}_m(q)}{m} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} \pmod{[p]}. \] (2.6)

However, it is difficult to determine the right-hand side of (2.6) modulo \([p]\). This is why we need to replace \( \overline{D}_n(q) \) by \( D_n(q) \) in Theorem 1.1.
3 Proofs of Theorems 1.2–1.4

Dilcher [5] established the following identity:

\[
\sum_{1 \leq k_1 \leq \cdots \leq k_m \leq n} \frac{q^{k_1+\cdots+k_m}}{(1-q^{k_1}) \cdots (1-q^{k_m})} = \sum_{k=1}^{n} (-1)^{k-1} \left[ \frac{n}{k} \right] \frac{q^{(k+1)\over 2}}{(1-q^k)^m},
\]

(3.1)

which is a multiple series generalization of Van Hamme’s identity [30]:

\[
\sum_{k=1}^{n} \frac{q^k}{1-q^k} = \sum_{k=1}^{n} (-1)^{k-1} \left[ \frac{n}{k} \right] \frac{q^{(k+1)\over 2}}{1-q^k}.
\]

(3.2)

Further generalizations of Dilcher’s identity (3.1) have been obtained by Fu and Lascoux [6, 7], Zeng [32], Ismail and Stanton [15], Guo and Zhang [13], Gu and Prodinger [10], Guo and Zeng [11], and Xu [31].

In what follows we give a new generalization of Dilcher’s identity (3.1) that also includes Kohnen’s identity (1.15) as a special case.

**Theorem 3.1** For \(m, n \geq 1\), there holds

\[
\sum_{1 \leq k_1 \leq \cdots \leq k_m \leq n} \frac{(x;q)_{k_1+\cdots+k_m}}{(1-q^{k_1}) \cdots (1-q^{k_m})} = \sum_{k=1}^{n} (-1)^{k-1} \left[ \frac{n}{k} \right] \frac{q^{(k+1)\over 2}}{(1-q^k)^m} (x^k - 1).
\]

(3.3)

**Proof.** For \(1 \leq r \leq n\), the coefficient of \(x^r\) in the left-hand side of (3.3) is given by

\[
(-1)^r q^{(r)\over 2} \sum_{k_m=r}^{n} \frac{k_m}{k_{m-1}=r} \cdots \sum_{k_2=r}^{k_1=r} \left[ k_1 \right] \frac{q^{k_1+\cdots+k_m}}{(1-q^{k_1}) \cdots (1-q^{k_m})}.
\]

(3.4)

It is easy to see that

\[
\sum_{k_1=r}^{k_2} \left[ k_1 \right] \frac{q^{k_1}}{1-q^{k_1}} = \frac{1}{1-q^r} \sum_{k_1=r}^{k_2} \left[ k_1 - 1 \right] q^{k_1} = \frac{1}{1-q^r} \left[ k_2 \right] q^r.
\]

(3.5)

By repeatedly using the summation formula (3.5), one sees that (3.4) is equal to

\[
(-1)^r q^{(r)\over 2} \sum_{k_m=r}^{n} \frac{k_m}{k_{m-1}=r} \left[ k_1 \right] \frac{q^{k_1+\cdots+k_m}}{(1-q^{k_1}) \cdots (1-q^{k_m})}.
\]

(3.6)

That is to say, the coefficients of \(x^r\) in both sides of (3.3) are equal for \(1 \leq r \leq n\). Also (3.3) is true for \(x=1\). Therefore it must be true for any \(x\). □

**Remark.** An equivalent form of the fact that (3.4) equals (3.6) has been given by Fu and Lascoux [7, Lemma 2.1]. The proof given here is more straightforward.

When \(m = 1\), we obtain the following result, which is crucial in the proof of Theorem 1.2.
Corollary 3.2 For \( n \geq 1 \), there holds
\[
\sum_{k=1}^{n} \left( \frac{x}{q} \right)_k q^k \frac{1}{1 - q^k} = \sum_{k=1}^{n} (-1)^k q^{\left(\frac{k+1}{2}\right)} \left[ \frac{n}{k} \right] (x^k - 1).
\] (3.7)

Proof of Theorem 1.2. In (3.7) we set \( n = p - 1 \) and \( x = -1 \) and multiply both sides by \( 1 - q \). By (2.2) and (2.5), the equation (3.7) simplifies to
\[
\sum_{k=1}^{p-1} (-1; q)_k q^k \equiv \sum_{k=1}^{p-1} (-1)^k q^{p-1 - 2} - (p - 1)(1 - q) \pmod{[p]},
\] (3.8)
which is equivalent to the congruence (1.10).

Replacing \( k \) by \( p - k \) and applying (2.4) and (1.6), we get
\[
\sum_{k=1}^{p-1} (-1; q)_k q^k \equiv -2 \sum_{k=1}^{p-1} (-1)^k q^{p-1 - 2} \pmod{[p]},
\] (3.9)
Combining (3.8) and (3.9), we complete the proof of (1.11). \( \square \)

By (2.4), we have
\[
\sum_{k=1}^{p} \frac{(-1)^k}{[k]} = \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{[k]} + \sum_{k=1}^{(p-1)/2} \frac{(-1)^{p-k}}{[p-k]} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^k (1 + q^k)}{[k]} \pmod{[p]},
\]
and so, by (1.7), the congruence (1.11) can be written as
\[
\sum_{k=1}^{p-1} q^{\frac{k+1}{2}} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^{k-1} (1 + q^k)}{2[k]} + \frac{(p - 1)(1 - q)}{4} \pmod{[p]},
\]
which clearly reduces to (1.3) when \( q = 1 \).

We now give a special case of Theorem 3.1. Because of its importance, we also label it as a theorem.

Theorem 3.3 For any positive integer \( m \) and prime \( p \geq 3 \), there holds
\[
\sum_{1 \leq k_1 \leq \cdots \leq k_m \leq p-1} \frac{q^{(k_m+1)}}{[k_1] \cdots [k_m] (-q; q)_{k_m}} \equiv (-1)^m \sum_{k=1}^{p-1} \frac{q^{(m-1)k}}{2[k]^m}((-1)^k - 1) \pmod{[p]}. \] (3.10)
Proof. Letting \( n = p - 1 \) and \( x = -1 \), and multiplying both sides by \((1 - q)^m\) in (3.3), we obtain
\[
\sum_{1 \leq k_1 \leq \cdots \leq k_m \leq p-1} \frac{(-1; q)_{k_1}q^{k_1+\cdots+k_m}}{[k_1] \cdots [k_m]} = \sum_{k=1}^{p-1} (-1)^k \binom{p-1}{k} q^{\frac{k(m+1)}{2}} \frac{q^{(k+1)\cdot k_m}}{[k]^{m}}((-1)^k - 1). \tag{3.11}
\]
Replacing \( k_i \) by \( p - k_i \) for \( 1 \leq i \leq m \) in (3.11), noticing that \([p-k_i] \equiv -q^{-k_i}[k_i] \pmod{[p]}\) and
\[
(-1; q)_{p-k_i} = 2(-q; q)_{p-1}^{(p-1)} \equiv \frac{2}{(-q^{-k_1}; q)_{k_1}} = \frac{2q^{(k_1+1)}}{(-q; q)_{k_1}} \pmod{[p]},
\]
and then applying (2.2), we get
\[
(-1)^m \sum_{1 \leq k_m \leq \cdots \leq k_1 \leq p-1} \frac{2q^{(k_1+1)}}{[k_1] \cdots [k_m]}(-q; q)_{k_1} \equiv \sum_{k=1}^{p-1} q^{(m-1)k} \frac{q^{(m-1)k}}{[k]^{m}}((-1)^k - 1) \pmod{[p]}. \tag{3.12}
\]
The proof then follows from reversing the order of \( k_1, \ldots, k_m \) in (3.12).

Proof of Theorem 1.3. Letting \( q = 1 \) in (3.10) and using the classical congruence (see, for example, [14, p. 48, Exercise 11]):
\[
\sum_{k=1}^{p-1} \frac{1}{k^m} \equiv 0 \pmod{p}, \quad \text{for } p > m + 1,
\]
we complete the proof of (1.12).

For \( m \) even, replacing \( k \) by \( p - k \), one sees that
\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k^m} \equiv \sum_{k=1}^{p-1} \frac{(-1)^p}{(p-k)^m} \equiv - \sum_{k=1}^{p-1} \frac{(-1)^k}{k^m} \equiv 0 \pmod{p}.
\]
This proves (1.13).

Proof of Theorem 1.4. When \( m = 2 \), the congruence (3.10) can be written as
\[
\sum_{k=1}^{p-1} \frac{H_k(q)q^{k+1}}{[k](-q; q)_k} \equiv \sum_{k=1}^{p-1} \frac{(-1)^k q^k}{2[k]^2} - \sum_{k=1}^{p-1} \frac{q^k}{2[k]^2} \pmod{[p]}, \quad \text{for } p \geq 5. \tag{3.13}
\]
Since \( q^{-k} \equiv q^{p-k} \pmod{[p]} \), we have
\[
\frac{q^k}{[k]^2} = \frac{q^k(1-q)^2}{(1-q^k)^2} = \frac{q^{-k}(1-q)^2}{(1-q^{-k})^2} \equiv \frac{q^{p-k}(1-q)^2}{(1-q^{p-k})^2} = \frac{q^{p-k}}{[p-k]^2} \pmod{[p]},
\]
and so
\[
\sum_{k=1}^{p-1} \frac{(-1)^k q^k}{[k]^2} \equiv \sum_{k=1}^{p-1} \frac{(-1)^k q^{p-k}}{[p-k]^2} = - \sum_{k=1}^{p-1} \frac{(-1)^k q^k}{[k]^2} \equiv 0 \pmod{[p]}, \tag{3.14}
\]
where in the second step we replaced \( k \) by \( p - k \). On the other hand, Shi and Pan [22, (4)] proved that

\[
\sum_{k=1}^{p-1} q^k \equiv -\frac{(p^2 - 1)(1 - q)^2}{12} \quad \text{(mod \([p]\))}, \quad \text{for } p \geq 5. \tag{3.15}
\]

The proof then follows form combing (3.13)–(3.15).

\[\square\]

4 Some consequences and remarks

**Corollary 4.1** For any prime \( p \geq 3 \), there holds

\[
\sum_{k=1}^{p-1} \frac{kq^k}{1 + q^k} \equiv \frac{p(p - 1)(1 - q)}{2} + p \sum_{k=1}^{p-1} \frac{(-q; q)_{k-1}q^k}{1 - q^k} \quad \text{(mod \([p]\))}. \tag{4.1}
\]

**Proof.** Multiplying both sides of (1.10) by \( 1 - q^p \), we have

\[
\left(-q; q\right)_{p-1} - 1 \equiv -(1 - q^p) \left(\frac{(p - 1)(1 - q)}{2} + \sum_{k=1}^{p-1} \frac{(-q; q)_{k-1}q^k}{[k]}\right) \quad \text{(mod \([p]^2\))}. \tag{4.2}
\]

Differentiating both sides of (4.2) with respect to \( q \), we obtain

\[
\left(-q; q\right)_{p-1} \sum_{k=1}^{p-1} \frac{kq^{k-1}}{1 + q^k} \equiv pq^{p-1} \left(\frac{(p - 1)(1 - q)}{2} + \sum_{k=1}^{p-1} \frac{(-q; q)_{k-1}q^k}{[k]}\right) \quad \text{(mod \([p]\))}. \tag{4.3}
\]

Since \((-q; q)_{p-1} \equiv q^p \equiv 1 \quad \text{(mod \([p]\))}, \) one sees that (4.3) is equivalent to (4.1). \[\square\]

Combining (1.10) and (4.1), we are led to the following result.

**Corollary 4.2** For any prime \( p \geq 3 \), there holds

\[
\frac{(-q; q)_{p-1} - 1}{1 - q^p} \equiv -\frac{1}{p} \sum_{k=1}^{p-1} \frac{kq^k}{1 + q^k} \quad \text{(mod \([p]\))}.
\]

Letting \( q \to 1 \), \( n = p - 1 \), and \( x \in \mathbb{Z} \) in (3.3), we get

**Corollary 4.3** For any integer \( x \), positive integer \( m \), and prime \( p \geq 3 \), there holds

\[
\sum_{1 \leq k_1 \leq \cdots \leq k_m \leq p-1} \frac{(1 - x)^{k_1}}{k_1 \cdots k_m} \equiv \sum_{k=1}^{p-1} \frac{(x - 1)^{k}}{k^m} \quad \text{(mod \(p\))}. \tag{4.4}
\]
Letting \( x = -1 \) or \( x = 2 \) in (4.4), we have

\[
\sum_{1 \leq k_1 \leq \ldots \leq k_m \leq p-1} \frac{2^{k_1}}{k_1 \cdots k_m} \equiv \sum_{k=1}^{p-1} \frac{((-1)^k - 1)}{k} \pmod{p}, \tag{4.5}
\]

\[
\sum_{1 \leq k_1 \leq \ldots \leq k_m \leq p-1} \frac{(-1)^{k_1}}{k_1 \cdots k_m} \equiv \sum_{k=1}^{p-1} \frac{(2^k - 1)}{k} \pmod{p}. \tag{4.6}
\]

It follows from (4.5) and (4.6) that

\[
\sum_{1 \leq k_1 \leq \ldots \leq k_m \leq p-1} \frac{2^{k_1} - (-1)^{k_1}}{k_1 \cdots k_m} \equiv \sum_{k=1}^{p-1} \frac{((-1)^k - 2^k)}{k} \pmod{p}. \tag{4.7}
\]

The \( m = 3 \) case of (4.7) has already appeared in [28]. Z.-W. Sun [24] proved that

\[
\sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \equiv \sum_{k=1}^{\lfloor 3p/4 \rfloor} \frac{(-1)^{k-1}}{k} \pmod{p}, \tag{4.8}
\]

where \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \). On the other hand, Z.-H. Sun [23, Theorem 4.1(iii)] gave a generalization of Kohnen’s congruence (1.3) as follows:

\[
\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv \frac{2^{p-1} - 1}{p} - \frac{(2^{p-1} - 1)^2}{2p} \pmod{p^2}, \tag{4.9}
\]

of which an elementary proof has been given by Meštrović [19].

We end the paper with the following problem.

**Problem 4.4** Are there any \( q \)-analogues of the congruences (4.8) and (4.9)?

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**References**


