Two truncated identities of Gauss

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Abstract. Two new expansions for partial sums of Gauss’ triangular and square numbers series are given. As a consequence, we derive a family of inequalities for the overpartition function $p(n)$ and for the partition function $p_1(n)$ counting the partitions of $n$ with distinct odd parts. Some further inequalities for variations of partition function are proposed as conjectures.

Keywords: Partition function; Overpartition function; Gauss’ identities; Shanks’ identity

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1 Introduction

The partition function $p(n)$ has the generating function

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \cdots. \quad (1.1)$$

Two classical results in the partition theory [1, p. 11] are Euler’s pentagonal number theorem

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} \left( 1 + \sum_{j=1}^{\infty} (-1)^j (q^{(3j-1)/2} + q^{(3j+1)/2}) \right) = 1, \quad (1.2)$$

and Euler’s recursive formula for computing $p(n)$:

$$p(n) + \sum_{j=1}^{\infty} (-1)^j (p(n - j(3j - 1)/2) + p(n - j(3j + 1)/2)) = 0, \quad (1.3)$$

where $p(m) = 0$ for all negative $m$.

Recently, Merca [7] has stumbled upon the following inequality:

$$p(n) - p(n-1) - p(n-2) + p(n-5) \leq 0, \quad (1.4)$$
and then, Andrews and Merca [3] proved more generally that, for \( k \geq 1 \),
\[
(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j (p(n - j(3j + 1)/2) - p(n - (j + 1)(3j + 2)/2)) \geq 0 \quad (1.5)
\]
with strict inequality if \( n \geq k(3k + 1)/2 \).

The proof of (1.5) in [3] is based on the truncated formula of (1.2):
\[
1 + 2 \sum_{j=1}^{\infty} (-1)^j q^{j(3j+1)/2} = 1 + (-1)^k \sum_{n=k}^{\infty} \frac{q^{(k+1)n+\binom{n}{2}}}{(q; q)_n} \left[ \frac{n-1}{k-1} \right]_q, \quad (1.6)
\]
where
\[
(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \quad (a; q)_M = \frac{(a; q)_\infty}{(aq^M; q)_\infty}, \quad \text{and} \quad \left[ \frac{M}{N} \right]_q = \frac{(q; q)_M}{(q; q)_N(q; q)_{M-N}}.
\]

Motivated by Andrews and Merca’s work [3], in this paper we shall prove new truncated forms of two identities of Gauss [1, p. 23]:
\[
1 + 2 \sum_{j=1}^{\infty} (-1)^j q^{j^2} = \frac{(q; q)_\infty}{(-q; q)_\infty}, \quad (1.7)
\]
\[
\sum_{j=0}^{\infty} (-1)^j q^{j(j+1)}(1 - q^{2j+1}) = \frac{(q^2; q^2)_\infty}{(-q^2; q^2)_\infty}, \quad (1.8)
\]
and derive similar overpartition function and special partition function inequalities.

**Theorem 1.** For \(|q| < 1\) and \(k \geq 1\), there holds
\[
\frac{(-q; q)_\infty}{(q; q)_\infty} \left( 1 + 2 \sum_{j=1}^{k} (-1)^j q^{j^2} \right) = 1 + (-1)^k \sum_{n=k+1}^{\infty} \frac{(-q; q)_k(-1; q)_{n-k}q^{(k+1)n}}{(q; q)_n} \left[ \frac{n-1}{k} \right]_q. \quad (1.9)
\]

The overpartition function \( \overline{p}(n) \), for \( n \geq 1 \), denotes the number of ways of writing the integer \( n \) as a sum of positive integers in non-increasing order in which the first occurrence of an integer may be overlined or not, and \( \overline{p}(0) = 1 \) (see Corteel and Lovejoy [5]). It is easy to see that
\[
\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + 40q^6 + \cdots.
\]

**Corollary 2.** For \( n, k \geq 1 \), there holds
\[
(-1)^k \left( \overline{p}(n) + 2 \sum_{j=1}^{k} (-1)^j \overline{p}(n - j^2) \right) \geq 0 \quad (1.10)
\]


with strict inequality if \( n \geq (k + 1)^2 \). For example,

\[
\begin{align*}
\mathfrak{p}(n) - 2\mathfrak{p}(n - 1) & \leq 0, \\
\mathfrak{p}(n) - 2\mathfrak{p}(n - 1) + 2\mathfrak{p}(n - 4) & \geq 0, \\
\mathfrak{p}(n) - 2\mathfrak{p}(n - 1) + 2\mathfrak{p}(n - 4) - 2\mathfrak{p}(n - 9) & \leq 0.
\end{align*}
\]

Theorem 3. For \(|q| < 1\) and \( k \geq 1 \), there holds

\[
\frac{(-q; q^2)^\infty}{(q^2; q^2)^\infty} \sum_{j=0}^{k-1} (-1)^j q^{(2j+1)}(1 - q^{2j+1})
= 1 + (-1)^{k-1} \sum_{n=k}^{\infty} \frac{(-q; q^2)_k(-q; q^2)_{n-k}q^{2(k+1)n-k} [n-1]}{(q^2; q^2)_n} [k-1]_{q^2}.
\]

Corollary 4. For \( n, k \geq 1 \), there holds

\[
(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j \left( p_1(n - j(2j + 1)) - p_1(n - (j + 1)(2j + 1)) \right) \geq 0
\]

with strict inequality if \( n \geq (2k + 1)k \). For example,

\[
\begin{align*}
p_1(n) - p_1(n - 1) - p_1(n - 3) + p_1(n - 6) & \leq 0, \\
p_1(n) - p_1(n - 1) - p_1(n - 3) + p_1(n - 6) + p_1(n - 10) - p_1(n - 15) & \geq 0.
\end{align*}
\]

A nice combinatorial proof of (1.3) was given by Bressoud and Zeilberger [4]. It would be interesting to find a combinatorial proof of (1.5), (1.10) and (1.13). A combinatorial proof of (1.11) will be given in Section 4.

2 Proof of Theorem 1

Generalizing Shanks’ work [8, 9], Andrews, Goulden, and Jackson [2, Theorem 1] established the following identity

\[
\sum_{j=0}^{n} \frac{(b; q)_j(1 - bq^{2j})(b/a; q)_j a^j q^{2j}}{(1 - b)(q; q)_j (aq; q)_j} = \frac{(bq; q)_n}{(aq; q)_n} \sum_{j=0}^{n} \frac{(b/a; q)_j a^j q^{(n+1)j}}{(q; q)_j}.
\]

When \( b = 1 \) and \( a = -1 \), the identity (2.1) reduces to
1 + 2 \sum_{j=1}^{n} (-1)^j q^{j^2} = \sum_{j=0}^{n} (-1)^j \frac{(-1; q)_j (q; q)_n q^{(n+1)j}}{(q; q)_j (-q; q)_n}. \tag{2.2}

By (2.2) and the \( q \)-binomial theorem (see [1, Theorem 2.1]), we have

\[
\frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left( 1 + 2 \sum_{j=1}^{k} (-1)^j q^{j^2} \right) = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{j=0}^{k} (-1)^j \frac{(-1; q)_j (q; q)_k q^{(k+1)j}}{(-q; q)_j (q; q)_{\infty}} \\
= \sum_{j=0}^{k} (-1)^j \frac{(-1; q)_j (-q^{k+1}; q)_{\infty} q^{(k+1)j}}{(q; q)_j (q^{k+1}; q)_{\infty}} \\
= \sum_{j=0}^{k} (-1)^j \sum_{i=0}^{\infty} \frac{(-1; q)_j (-1; q)_{k+1}(i+j)}{(q; q)_j(q; q)_i}. \tag{2.3}
\]

By induction on \( k \), it is easy to see that, for \( n \geq 1 \),

\[
\sum_{j=0}^{k} (-1)^j \frac{(-1; q)_j (-1; q)_{n-j}}{(q; q)_j(q; q)_{n-j}} = (-1)^k \frac{(-q; q)_k (-1; q)_{n-k}}{(1-q^n)(q; q)_{n-k-1}(q; q)_k}.
\]

Hence, letting \( i + j = n \), the right-hand side of (2.3) can be written as

\[
\sum_{n=0}^{\infty} \sum_{j=0}^{k} (-1)^j \frac{(-1; q)_j (-1; q)_{n-j} q^{(k+1)n}}{(q; q)_j(q; q)_{n-j}} \\
= 1 + (-1)^k \sum_{n=1}^{\infty} \frac{(-q; q)_k (-1; q)_{n-k} q^{(k+1)n}}{(1-q^n)(q; q)_{n-k-1}(q; q)_k} \\
= 1 + (-1)^k \sum_{n=k+1}^{\infty} \frac{(-q; q)_k (-1; q)_{n-k} q^{(k+1)n}}{(q; q)_n} \left[ \frac{n-1}{q} \right]_k,
\]

as desired.

### 3 Proof of Corollary 2

By Theorem 1, we see that the generating function for the sequence

\[
\left\{ (-1)^k \left( p(n) + 2 \sum_{j=1}^{k} (-1)^j p(n-j^2) \right) \right\}_{n=0}^{\infty}
\]

is given by

\[
(-1)^k + \sum_{n=k+1}^{\infty} \frac{(-q; q)_k (-1; q)_{n-k} q^{(k+1)n}}{(q; q)_n} \left[ \frac{n-1}{q} \right]_k,
\]

which clearly has nonnegative coefficients of \( q^n \) for \( m \geq 1 \) and has positive coefficients of \( q^m \) for \( m \geq (k+1)^2 \). This completes the proof.
4 A combinatorial proof of (1.11)

Let \( \mathcal{P}_n \) denote the set of all overpartitions of \( n \). We now construct a mapping \( \phi : \mathcal{P}_n \to \mathcal{P}_{n-1} \) as follows: For any \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_n \), let

\[
\phi(\lambda) = \begin{cases} 
(\lambda_1, \ldots, \lambda_{k-1}), & \text{if } \lambda_k = 1, \\
(\lambda_1, \ldots, \lambda_{k-1}, \lambda_k - 1), & \text{if } \lambda_k \neq 1, \hat{1}, \\
(\lambda_1, \ldots, \lambda_{k-1}, 1, \ldots, 1), & \text{if } \lambda_k = \hat{1},
\end{cases}
\]

where \( \hat{1} = \hat{T} \) if \( \lambda_{k-1} \) is overlined and \( \hat{1} = 1 \) otherwise.

It is easy to see that \( |\phi^{-1}(\mu)| \leq 2 \) for any \( \mu \in \mathcal{P}_{n-1} \). For example, for \( n = 4 \), the mapping \( \phi \) gives

\[
\begin{align*}
4 & \mapsto 3, \quad 3 \mapsto 3, \quad (3, 1) \mapsto 3, \quad (3, \hat{T}) \mapsto (1, 1, 1), \quad (\hat{T}, 1) \mapsto (1, 1), \quad (3, 1) \mapsto 3, \\
(2, 2) & \mapsto (2, 1), \quad (\hat{T}, 2) \mapsto (\hat{T}, 1), \quad (2, 1, 1) \mapsto (2, 1), \quad (2, \hat{T}, 1) \mapsto (2, \hat{T}), \quad (\hat{T}, \hat{T}, 1) \mapsto (\hat{T}, \hat{T}), \\
(\hat{T}, 1, 1) & \mapsto (\hat{T}, 1), \quad (1, 1, 1, 1) \mapsto (1, 1, 1), \quad (\hat{T}, 1, 1, 1) \mapsto (\hat{T}, 1, 1).
\end{align*}
\]

This proves that \( p(n) \leq 2p(n-1) \).

5 Proof of Theorem 3 and Corollary 4

Shanks [8,9] proved that

\[
1 + \sum_{j=1}^{n} (q^j(q^{2j} + q^{4j}(2j+1))) = \sum_{j=0}^{n} \frac{(q; q^2)_j(q^2; q^2)_n q^{j(2n+1)}}{(q^2; q^2)_j(q^2; q^2)_n},
\]

or equivalently,

\[
\sum_{j=0}^{n-1} q^j(q^{2j+1})(1 + q^{2j+1}) = \sum_{j=0}^{n-1} \frac{(q; q^2)_j(q^2; q^2)_n q^{j(2n+1)}}{(q^2; q^2)_j(q^2; q^2)_n}. \quad (5.1)
\]

By (5.1) (with \( q \) replaced by \( -q \)) and the q-binomial theorem (see [1, Theorem 2.1]), we have

\[
\frac{(-q; q^2)^\infty}{(q^2; q^2)^\infty} \sum_{j=0}^{k-1} (-1)^j q^j(2j+1)(1 - q^{2j+1})
\]

\[
= \frac{(-q; q^2)^\infty}{(q^2; q^2)^\infty} \sum_{j=0}^{k-1} (-1)^j \frac{(-q; q^2)_j(q^2; q^2)_k q^{(2k+1)j}}{(q^2; q^2)_j(-q; q^2)_k}
\]

\[
= \sum_{j=0}^{k-1} (-1)^j \frac{(-q; q^2)_j(-q^{2k+1}; q)_\infty q^{(2k+1)j}}{(q^2; q^2)_j(q^{2k+2}; q)_\infty}
\]

\[
= \sum_{j=0}^{k-1} (-1)^j \sum_{i=0}^{\infty} \frac{(-q; q^2)_j(-q^{-1}; q^2)_i q^{(2k+1)(i+j)+i}}{(q^2; q^2)_j(q^2; q^2)_i}. \quad (5.2)
\]
By induction on \( k \), it is easy to see that, for \( n \geq 1 \),
\[
\sum_{j=0}^{k-1} (-1)^j \frac{(-q; q^2)_j(-q^{-1}; q^2)_{n-j}q^{n-j}}{(q^2; q^2)_j(q^2; q^2)_{n-j}} = (-1)^{k-1} \frac{(-q; q^2)_k(-q; q^2)_{n-k}q^{n-k}}{(1 - q^{2n})(q^2; q^2)_{n-k}(q^2; q^2)_{k-1}}.
\]
Hence, letting \( i + j = n \), the right-hand side of (5.2) can be written as
\[
\sum_{n=0}^{\infty} \sum_{j=0}^{k-1} (-1)^j \frac{(-q; q^2)_j(-q^{-1}; q^2)_{n-j}q^{2(k+1)n+n-j}}{(q^2; q^2)_j(q^2; q^2)_{n-j}}
\]
\[
= 1 + (-1)^{k-1} \sum_{n=1}^{\infty} \frac{(-q; q^2)_k(-q; q^2)_{n-k}q^{2(k+1)n-k}}{(1 - q^{2n})(q^2; q^2)_{n-k}(q^2; q^2)_{k-1}}.
\]
\[
= 1 + (-1)^{k-1} \sum_{n=k}^{\infty} \frac{(-q; q^2)_k(-q; q^2)_{n-k}q^{2(k+1)n-k}}{(q^2; q^2)_n} \left\lfloor \frac{n - 1}{k - 1} \right\rfloor q^2.
\]
This proves Theorem 3. The proof of Corollary 4 is similar to that of Corollary 2 and is omitted here.

6 Open problems

In this section, we propose a common generalization of (1.5), (1.10) and (1.13). Let \( m, r \) be positive integers with \( 1 \leq r \leq m/2 \). Consider the generalized partition function \( J_{m,r}(n) \) defined by
\[
\sum_{n=0}^{\infty} J_{m,r}(n)q^n = \frac{1}{(q^r; q^m)^{\infty}(q^{m-r}; q^m)^{\infty}(q^m; q^m)^{\infty}}. \tag{6.1}
\]
It is easy to see that
\[
J_{2,1}(n) = p(n), \quad J_{3,1}(n) = p(n), \quad J_{4,1}(n) = p_1(n).
\]
Moreover, if \( r < m/2 \), then \( J_{m,r}(n) \) can be understood as the number of partitions of \( n \) into parts congruent to 0, ±\( r \) modulo \( m \). Now, Jacobi’s triple product identity implies (see [6, p. 375]) that
\[
(q^r; q^m)^{\infty}(q^{m-r}; q^m)^{\infty}(q^m; q^m)^{\infty} = 1 + \sum_{j=1}^{\infty} (-1)^j (q^{j(mj+m-2r)/2} + q^{j(mj+m+2r)/2}). \tag{6.2}
\]
It follows from (6.1) and (6.2) that \( J_{m,r}(n) \) satisfies the recurrence formula:
\[
J_{m,r}(n) + \sum_{j=1}^{\infty} (-1)^j (J_{m,r}(n - j(mj - m + 2r)/2) + J_{m,r}(n - j(mj + m - 2r)/2)) = 0, \tag{6.3}
\]
where \( J_{m,r}(s) = 0 \) for all negative \( s \).
Conjecture 5. For $m, n, k, r \geq 1$ with $r \leq m/2$, there holds
\[
(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j \left( J_{m,r}(n - j(mj + m - 2r)/2) - J_{m,r}(n - (j + 1)(mj + 2r)/2) \right) \geq 0
\] (6.4)
with strict inequality if $n \geq k(mk + m - 2r)/2$.

In fact, when $m = 2$ and $r = 1$, the conjectural inequality (6.4) is equivalent to
\[
(-1)^{k-1} \left( \bar{p}(n) + 2 \sum_{j=1}^{k-1} (-1)^j \bar{p}(n - j^2) \right) \geq \bar{p}(n - k^2)
\] (6.5)
with strict inequality if $n \geq k^2$. It is clear that (6.5) is stronger than the proved inequality (1.10) (with $k$ replaced by $k - 1$).

Finally, along the same line of thinking, we consider the sequence $\{t(n)\}_{n \geq 0}$ (see A000716 in Sloane’s database of integer sequences [10]) defined by
\[
\sum_{n=0}^{\infty} t(n)q^n = \frac{1}{(q; q)^3}\]
\[
= 1 + 3q + 9q^2 + 22q^3 + 51q^4 + 108q^5 + 221q^6 + 429q^7 + 810q^8 + 1479q^9 + \cdots .
\]
Clearly, the number $t(n)$ counts partitions of $n$ into 3 kinds of parts. Now, invoking the identity of Jacobi [6, p. 377]:
\[
(q; q)^3 = \sum_{j=0}^{\infty} (-1)^j (2j + 1)q^{j(j+1)/2},
\]
we derive the recurrence formula:
\[
\sum_{j=0}^{\infty} (-1)^j (2j + 1)t(n - j(j + 1)/2) = 0,
\] (6.7)
where $t(m) = 0$ for all negative $m$.

We end the paper with the following conjecture:

Conjecture 6. For $n, k \geq 1$, there holds
\[
(-1)^k \sum_{j=0}^{k} (-1)^j(2j + 1)t(n - j(j + 1)/2) \geq 0
\]
with strict inequality if $n \geq (k + 1)(k + 2)/2$. For example,
\[
t(n) - 3t(n - 1) \leq 0,
\]
\[
t(n) - 3t(n - 1) + 5t(n - 3) \geq 0,
\]
\[
t(n) - 3t(n - 1) + 5t(n - 3) - 7t(n - 6) \leq 0.
\]
References


