A RESULT ON THE BRUHAT ORDER OF A COXETER GROUPS

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Let $W = (W, S)$ be a Coxeter group with $S$ the set of its Coxeter generators. Let $\leq$ be the Bruhat order of $W$. That is, we denote $y \leq w$ for $y, w \in W$ if there exist reduced expressions $w = s_1s_2 \cdots s_r$ and $y = s_{i_1}s_{i_2} \cdots s_{i_t}$ with $s_{i_j} \in S$, $1 \leq j \leq r$, and $i_1, i_2, \cdots, i_t$ a subsequence of $1, 2, \ldots, r$. Let $\ell(x)$ be the length function on $W$. To each $w \in W$, we associate two subsets of $S$:

$$\mathcal{L}(w) = \{s \in S \mid sw < w\} \quad \text{and} \quad \mathcal{R}(w) = \{s \in S \mid ws < w\}.$$ 

Now we can state our main result as follows.

**Theorem 1.** Suppose that $x, y \in W$ and $s \in S$ satisfy the condition $s \not\in \mathcal{L}(y) \cup \mathcal{R}(x)$. Then $xy < xsy$.

By a Coxeter transformation on an expression $s_1s_2 \cdots s_t$ with $s_i \in S$, we mean that it is one of the following transformations.

(A) If there exist some $a, b \in S$ and $i, j \in \mathbb{Z}$ with $a \neq b$ and $1 \leq i < j \leq t$ such that

$$s_is_{i+1} \cdots s_j \equiv aba \cdots \quad \text{and} \quad o(ab) = j - i + 1.$$ 

then we define a transformation

$$s_1s_2 \cdots s_t \mapsto s_1s_2 \cdots s_{i-1} \underbrace{bab \cdots}_{o(ab) \text{ factors}} s_{j+1} \cdots s_t,$$

where the notation $o(x)$ stands for the order of the element $x$.

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(B) If there exists some \( i \in \mathbb{Z} \), \( 1 \leq i < t \), such that \( s_i = s_{i+1} \), then we define a transformation
\[
s_1 s_2 \cdots s_t \mapsto s_1 s_2 \cdots s_{i-1} s_{i+2} \cdots s_t.
\]
(C) For any \( i \), \( 0 \leq i \leq t \), and \( a \in S \), we define a transformation
\[
s_1 s_2 \cdots s_t \mapsto s_1 s_2 \cdots s_i(a) s_{i+1} \cdots s_t.
\]

**Remark 2.** Given any two expressions \( s_1 s_2 \cdots s_k \) and \( t_1 t_2 \cdots t_m \) of \( W \) with \( s_i, t_j \in S \) and \( s_1 s_2 \cdots s_k \not\equiv t_1 t_2 \cdots t_m \), it is well known that the expression \( s_1 s_2 \cdots s_k \) can be passed to \( t_1 t_2 \cdots t_m \) by a succession of Coxeter transformations. In particular, in the case when \( t_1 t_2 \cdots t_m \) is a reduced expression, \( s_1 s_2 \cdots s_k \) can be passed to \( t_1 t_2 \cdots t_m \) by only performing the Coxeter transformations of kinds (A) and (B).

Define a set of triples
\[
T = \{ (x, s, y) \mid x, y \in W, s \in S, s \not\in \mathcal{L}(y) \cup \mathcal{R}(x) \}.
\]
For \( i \in \mathbb{Z} \), define
\[
T_i = \{ (x, s, y) \in T \mid \ell(x) + \ell(y) + 1 - \ell(xsy) = i \}.
\]
Clearly, if \( T_i \neq \emptyset \) then \( i \geq 0 \) and \( i \in 2\mathbb{Z} \). Thus we have a decomposition:
\[
T = \bigcup_{j \geq 0} T_{2j}.
\]
If \( (x, s, y) \in T_i \) then we define \( p(x, s, y) = i \). Then Theorem 1 can be reformulated as follows.

**Theorem 3.** If \( (x, s, y) \in T \) then \( xy < xsy \).

It is obvious that

**Lemma 4.** If \( (x, s, y) \in T_0 \) then \( xy < xsy \).

Let \( g \equiv s_1 s_2 \cdots s_k s_1 t_1 t_2 \cdots t_m \) be an expression of \( W \) with \( s_i, s_j \in S \) satisfying the following conditions:
(a) Both \( x \equiv s_1 s_2 \cdots s_k \) and \( y \equiv t_1 t_2 \cdots t_m \) are reduced expressions.
(b) \( (x, s, y) \in T_{2n} \) for some \( n > 0 \).

Let \( g' \) be an expression obtained from the expression \( g \) by a Coxeter transformation \( f \) of kind \( \neq (C) \). Then \( f \) must have kind (A). Suppose that \( f \) does not involve the factor \( s \). Let \( s' = s, x' \equiv s'_1 s'_2 \cdots s'_k \), and \( y' \equiv t'_1 t'_2 \cdots t'_m \)
be such that \( g' \equiv x' s' y' \). Then \( x' \) and \( y' \) are also reduced expressions with \( x' = x \) and \( y' = y \).

Now suppose that \( f \) involves the factor \( s \). Then we have
\[
g \equiv g_1 (aba \cdots) g_2 \quad \text{and} \quad g' \equiv g_1 (bab \cdots) g_2
\]
for some subexpressions \( g_1, g_2 \) of \( g \), where \( a, b \in S \) satisfy \( a \neq b \) and \( r = o(ab) \), and \( s \) is the \( i \)-th factor in the parentheses of the expression \( g \) with \( 1 \leq i \leq r \).

Then by choosing \( s' \) to be the \((r + 1 - i)\)-th factor in the parentheses of the expression \( g' \), we have the expression
\[
g' \equiv g_1 (bab \cdots) g_2 \equiv s'_1 s'_2 \cdots s'_k t'_1 t'_2 \cdots t'_m
\]
with \( s'_i, t'_j \in S \) satisfying the following conditions:
(i) \( k' = k + r + 1 - 2i, m' = m + 2i - r - 1 \).
(ii) Let \( x \equiv s'_1 s'_2 \cdots s'_k \) and \( y' \equiv t'_1 t'_2 \cdots t'_m \). Then \( xy' = x \) and \( (x', s', y') \in T_{2h} \) for some \( h \leq n \). The equality \( h = n \) holds if and only if both the expressions \( x' \) and \( y' \) are reduced.

Sum up, we have the following result.
Lemma 5. Let \( g \equiv s_1s_2 \cdots s_kst_1t_2 \cdots t_m, x, y, n \) be as above. Let \( g' \) be an expression obtained from the expression \( g \) by a Coxeter transformation \( f \) of kind \( \neq (C) \). Then we can choose a factor \( s' \) in the expression \( g' \) such that

(a) \( g' = s_1's_2' \cdots s_k't_1't_2' \cdots t_m' \) with \( s_i', t_j' \in S \).

(b) Let \( x' \equiv s_1's_2' \cdots s_k' \) and \( y' \equiv t_1't_2' \cdots t_m' \). Then \( x'y' = xy \) and \( (x', s', y') \in T_{2h} \) for some \( h \leq n \). The equality \( h = n \) holds if and only if both the expressions \( x' \) and \( y' \) are reduced.

Again assume that \( g \equiv s_1s_2 \cdots s_kst_1t_2 \cdots t_m, x, y, n \) are as above. If \( n > 0 \), then by Remark 2, there exists a sequence of expressions \( g_0 \equiv g, g_1, \ldots, g_h \) of \( xsy \) for some \( h > 0 \) such that for every \( i, 1 \leq i \leq h \), \( g_i \) is obtained from \( g_{i-1} \) by a Coxeter transformation of kind \( \neq (C) \) and \( g_h \) is a reduced expression. By Lemma 5, we see that there must exist some integer \( u \) with \( 1 \leq u < h \) such that

(i) \( g_i \equiv (s(1)i) \cdots s(i, k_i) s(i)t(i, 1) \cdots t(i, m_i) \) with \( s(i), j \), \( t(i, j) \in S \) for all \( i \), \( 0 \leq i < u \).

(ii) The expressions \( s(i, 1) \cdots s(i, k_i) \) and \( t(i, 1) \cdots t(i, m_i) \) are reduced for all \( i \), \( 0 \leq i < u \).

(iii) Either \( s(u, 1) \cdots s(u, k_u) \) or \( t(u, 1) \cdots t(u, m_u) \) is not a reduced expression.

(iv) Let \( x_i = s(i, 1) \cdots s(i, k_i) \) and \( y_i = t(i, 1) \cdots t(i, m_i) \) for \( 0 \leq i \leq u \). Then \( x_iy_i = xy \) and \( (x_i, s(i), y_i) \in T_{2n} \) with \( n_i = n \) for \( 0 \leq i < u \) and \( n_u < n \).

Given \( (x, s, y) \in T \), we call a sequence of expressions, say \( g_0, g_1, \ldots, g_u \) for some \( u \geq 0 \), of \( xsy \) a declining sequence with respect to \( (x, s, y) \), if the following conditions are satisfied.

(a) For every \( i, 1 \leq i \leq u \), \( g_i \) is obtained from \( g_{i-1} \) by a Coxeter transformation of kind \( \neq (C) \).

(b) The above conditions (i)-(iv) are satisfied, where \( n = p(x, s, y) \) and \( (x, s, y) = (x_0, s(0), y_0) \).

Thus the above discussion shows that

Lemma 6. For any \( (x, s, y) \in T \) with \( p(x, s, y) > 0 \), there exists some declining sequence with respect to \( (x, s, y) \).

Proof of Theorem 3: Apply induction on \( p(x, s, y) \geq 0 \). If \( p(x, s, y) = 0 \) then it is just the assertion of Lemma 4. Now assume that \( p(x, s, y) > 0 \) and that the result has been shown for any \( (x', s', y') \in T \) with \( p(x', s', y') < p(x, s, y) \). By Lemma 6, there exists a declining sequence \( g_0, g_1, \ldots, g_u \) with respect to \( (x, s, y) \) as described above. Thus \( p(x_u, s(u), y_u) < p(x, s, y) \). By inductive hypothesis, we have \( xsy = x_ys(u)y_u > x_uy_u = xy \). So our result is shown. \( \square \)

Thus Theorem 1 follows as it is equivalent to Theorem 3.

By noting \( \ell(xsy) \equiv \ell(xy)(\mod 2) \), we see that in Theorem 1 we have

\[
\ell(xsy) > \ell(xy) + 1.
\]

For any subset \( J \) of \( S \), let \( W_J \) be the subgroup of \( W \) generated by \( J \).

Theorem 7. Given \( x, y \in W, J \subseteq S - R(x), \) and \( w \in W_J \), if \( \ell(wy) = \ell(w) + \ell(y) \), then \( xwy \leq xsy \) and \( \ell(xwy) \geq \ell(x) + \ell(y) \). In particular, in the case where \( \ell(xy) = \ell(x) + \ell(y) \), we have \( \ell(xwy) = \ell(x) + \ell(y) + \ell(w) \).

§2. SOME APPLICATIONS OF THEOREM 1.

We shall make some applications of Theorem 1 in the present section. The first one is concerned with some multiplication properties of a Hecke algebra. Let \( H \) be the Hecke algebra over \( A = \mathbb{Z}[u, u^{-1}] \) associated to a Coxeter group \( (W, S) \) as below. \( H \) is a free \( A \)-module with a basis \( \{T_w \mid w \in W\} \) and its multiplication satisfies the rule.
The rule (1) are equivalent to the rules

\begin{equation}
T_s T_w = \begin{cases} 
(u^{-1} - u)T_w + T_{sw}, & \text{if } s \in \mathcal{L}(w), \\
T_{sw}, & \text{if } s \notin \mathcal{L}(w).
\end{cases}
\end{equation}

For any \(x, y, z \in W\), define an element \(f_{x,y,z} \in A\) by

\begin{equation}
T_x T_y = \sum_z f_{x,y,z} T_z.
\end{equation}

It is easily seen from (2) that \(f_{x,y,z}\) is a polynomial in \(v = u^{-1} - u\) of positive coefficients.

Define a subset \(\Lambda(x, y) = \{z \in W \mid f_{x,y,z} \neq 0\}\) of \(W\) for any \(x, y \in W\). It is well known that there exists a unique maximal element, written \(\lambda(x, y)\) in \(\Lambda(x, y)\) with respect to the Bruhat order \(\leq\) (see [2]). That is,

\begin{equation}
\lambda(x, y) \geq z, \quad \forall z \in \Lambda(x, y).
\end{equation}

Here we shall apply Theorem 1 to show that there also exists a unique minimal element in \(\Lambda(x, y)\) with respect to the same partial order.

**Theorem 8.** For any \(x, y \in W\), we have \(xy \in \Lambda(x, y)\) and

\begin{equation}
xy \leq z, \quad \forall z \in \Lambda(x, y).
\end{equation}

**Proof.** Apply induction on \(\ell(x) \geq 0\). It is trivial in case \(\ell(x) = 0\). Now assume \(\ell(x) > 0\). Let \(s \in \mathcal{R}(x)\) and \(x' = xs\). Then

\[T_x T_y = T_{x'} T_s T_y.\]

If \(s \notin \mathcal{L}(y)\), then \(T_x T_y = T_{x'} T_{sy}\) and so \(\Lambda(x, y) = \Lambda(x', sy)\). Since \(\ell(x') < \ell(x)\), this implies by the inductive hypothesis that

\[x' sy \leq z, \quad \forall z \in \Lambda(x', sy).\]

So we get (5) in this case. If \(s \in \mathcal{R}(y)\) then

\[T_x T_y = T_{x'} (v \cdot T_y + T_{sy}) = v \cdot T_{x'} T_y + T_{x'} T_{sy}.\]

By the positivity of the coefficients of the \(f_{x,y,z}\)'s, we have

\[\Lambda(x, y) = \Lambda(x', y) \cup \Lambda(x', sy).\]

By the inductive hypothesis and the fact that \(\ell(x') < \ell(x)\), we get

\[x'y \leq z, \quad \forall z \in \Lambda(x', y)\]

and

\[x'sy \leq z', \quad \forall z' \in \Lambda(x', sy).\]

But \(s \notin \mathcal{R}(x') \cup \mathcal{L}(sy)\). This implies from Theorem 1 that

\[xy = x's(y) < x's(y) = x'y.\]

Therefore we again get (5). \(\square\)

Let \(\deg f_{x,y,z}\) be the degree of the polynomial \(f_{x,y,z}\) in \(v = u^{-1} - u\). Here we state a property of \(f_{x,y,z}\).
Corollary 9. For any $x, y \in W$ and $z \in \Lambda(x,y)$, we have $\deg f_{x,y,z} \geq 0$. The equality holds if and only if $z = xy$. The constant of $f_{x,y,z}$ in $v$ is equal to zero if $z \neq xy$ and is equal to 1 if $z = xy$.

Proof. This follows directly from the multiplication rule of the Hecke algebra $H$ and Theorem 8. □

The second application of Theorem 1 is to verify a conjecture of L. K. Jones. Let $g$ be an expression of $W$. For any $s \in S$, let $n_s(g)$ be the number of the factor $s$ occurring in $g$. For any $w \in W$, let $I(w)$ be the set of all reduced expressions of $w$. Define a number

$$N_s(w) = \min\{n_s(g) \mid g \in I(w)\}.$$ 

Then Jones made the following conjecture which plays a crucial role in his paper.

Conjecture 10. Let $(W, S)$ be a symmetric group. Suppose that $x, y \in W$ and $s \in S$ satisfy the conditions $s \notin \mathcal{L}(y) \cup \mathcal{R}(x)$. Then for any $t \in S - \{s\}$, we have $N_s(xy) \leq N_s(xsy)$.

To show the above conjecture, we shall prove the following result which includes this conjecture as a special case.

Proposition 11. Let $(W, S)$ be a Coxeter group. Let $x, y \in W$ and $s \in S$ be such that $s \notin \mathcal{L}(y) \cup \mathcal{R}(x)$. Then for any $t \in S$, we have $N_s(xy) \leq N_s(xsy)$.

Proof. It is well known that for any reduced expression $s_1s_2 \cdots s_r$ of $w$ with $s_i \in S$, there exists some subsequence $i_1, i_2, \cdots, i_t$ of $1, 2, \cdots, r$ such that $s_{i_1}s_{i_2} \cdots s_{i_t}$ is a reduced expression of $y$. This implies that

$$N_s(y) \leq N_s(w), \quad \text{for any } y \leq w \text{ in } W \text{ and } s \in S.$$ 

So by Theorem 1, our result follows. □

References