ON TWO PRESENTATIONS OF THE AFFINE WEYL GROUPS OF CLASSICAL TYPES

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ABSTRACT. The main result of the paper is to get the transition formulae between the alcove form and the permutation form of $w \in W_a$, where $W_a$ is an affine Weyl group of classical type. On the other hand, we get a new characterization for the alcove form of an affine Weyl group element which has a much simpler form compared with that in [10]. As applications, we give an affirmative answer to a conjecture of H. Eriksson and K. Eriksson in [4] concerning the characterization of the inverse table of $w \in W_a$; we also describe the number $\pi_s(w)$ in terms of permutation form of $w \in W_a$.

Introduction.

Affine Weyl groups, as a family of infinite crystallographic Coxeter groups, play a more and more important role in various fields of mathematics, such as Kac-Moody algebras, algebraic groups and their representation theory, combinatorial and geometric group theory, etc. [2; 3; 5; 7; 9; 14].

Besides the presentations as Coxeter groups (i.e., the ones by generators and relations over the pairs of generators), there are many other presentations for the affine Weyl groups $W_a$, in particular for those of the classical types, i.e., types $\widetilde{A}_l$ ($l \geq 1$), $\widetilde{B}_m$ ($m \geq 3$), $\widetilde{C}_n$ ($n \geq 2$) and $\widetilde{D}_k$ ($k \geq 4$). Two of their presentations are particularly useful, one is to regard $W_a$ as a certain permutation group over the integer set $\mathbb{Z}$ (only applied for the classical types); the other is to identify $W_a$ with the set of alcoves in a euclidean space $E$ after removing a certain set of hyperplanes (applied for all types) [1; 8; 9; 10; 11; 12; 13].

Key words and phrases. affine Weyl group of classical types, presentations, alcove forms, permutation forms.
The relations between these two presentations has been explicitly described for the case of type $\tilde{A}_l$ (see [9]). but not yet for the other classical types so far. The aim of the present paper is to study these relations for the cases of types $\tilde{B}_m$, $\tilde{C}_n$ and $\tilde{D}_k$. Our main result is to obtain the transition formulae between the permutation form and the alcove form of $w \in W_a$ when the type of $W_a$ belongs to these three families (see Theorems 4.1, 5.2, 5.4 and 5.6).

The alcove form $(k(w; \alpha))_\alpha$ of $w \in W_a$ adopted here is essentially the same as that in [10], but with two changes as below. The one is that the relation $k(w; -\alpha) = -k(w; \alpha)$ in [10] is replaced by $k(w; -\alpha) = -k(w; \alpha) - 1$ for all the positive roots (and hence for all the roots) $\alpha$. The other is that in dealing with the case of the classical types, we replace the root system $\Phi$ by the corresponding coroot system $\Phi^\vee$ as the index set of the alcove form of $w$. The reason for these changes is to make our transition formulae simpler. The another reason for the second change is that we have got a new characterization for the alcove form of $w \in W_a$, which is equivalent to the original one in [10, Theorem 5.2] but has a much simpler form when stated in terms of coroot system (see Theorem 1.3).

In general, the permutation form of $w \in W_a$ is not unique. This depends on the way of embedding the group $W_a$ into the permutation group on $\mathbb{Z}$ (see [1; 13]). The permutation form of $w \in W_a$ we take in the present paper has the advantage that when restricting to the corresponding Weyl group, the permutation form of an element is just the usual one.

We give two applications of our results. One is to show that when an affine Weyl group $W_a$ has classical type, the entries of the alcove form of $w \in W_a$ exactly comprise the inverse table of $w$ defined by H. Eriksson and K. Eriksson (see [4, §8.] and Theorem 4.1). Thus our new characterization for the alcove form of $w \in W_a$ gives a characterization for the inverse table of $w$ (see Theorem 4.5), the latter was conjectured by H. Eriksson and K. Eriksson in the case of type $\tilde{C}_l$ in [4, 8.3.].

Let $(W, S)$ be a Coxeter system. For any $w \in W$ and $s \in S$, let $\pi_s(w)$ be the minimal possible multiplicity of the factor $s$ occurring in a reduced expression of $w$. There is no general formula for the number $\pi_s(w)$. Then our second application is to deduce very simple formulae for $\pi_s(w)$ in terms of permutation form of $w$ when $(W, S)$ is an affine Weyl group of classical type (see 5.8-5.10).

Throughout this paper, an affine Weyl group is always assumed irreducible, i.e., its Coxeter graph is connected.

The contents of the paper are organized as follows. In Section 1, we recall some
pay our attention only to the affine Weyl groups $W_a$ of the classical types. We describe
the alcove form and the permutation form of $w \in W_a$ in Sections 2 and 3 respectively.
Then we show our main results of the paper in Sections 4 and 5, where we obtain the
transition formulae between the alcove form and the permutation form of $w \in W_a$ in
the cases of the classical types.

\section{The alcove forms.}

In this section, we collect some results concerning the alcove form of an element $w$
in an affine Weyl group $W_a$. As mentioned in the Introduction, we make some formal
changes in the definition of the alcove form of $w$. Thus we have to reformulate some
results in \cite{10}. One result is new (i.e., Theorem 1.3), which gives a new characterization
for an alcove in a euclidean space.

1.1. Let $\Phi$ be an irreducible root system of rank $l > 1$. Let $E$ be the euclidean space
spanned by $\Phi$ with inner product $\langle \ , \rangle$ such that $|\alpha|^2 = \langle \alpha, \alpha \rangle = 1$ for any short root $\alpha$ of $\Phi$. Choose a simple root system $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ of $\Phi$ and let $\Phi^+, \Phi^-$ be the
corresponding positive and negative root systems. Let $-\alpha_0$ be the highest short root
of $\Phi$. Denote by $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$ the coroot of $\alpha \in \Phi$ and by $\Phi^\vee$
the coroot system \{\(\alpha^\vee \mid \alpha \in \Phi\}\}. Note that a coroot system $\Phi^\vee$ itself is also a root system, but with its
type dual to that of $\Phi$. This fact will be important later when we make a change for
the index set of the alcove form of an affine Weyl group element.

For any $\alpha \in \Phi$ and $k \in \mathbb{Z}$, define a hyperplane
\begin{equation}
H_{\alpha; k} = \{v \in E \mid \langle v, \alpha^\vee \rangle = k\}
\end{equation}
and a strip
\begin{equation}
H^1_{\alpha; k} = \{v \in E \mid k < \langle v, \alpha^\vee \rangle < k + 1\}.
\end{equation}
Then we have $H_{-\alpha, k} = H_{\alpha; -k}$ and $H^1_{-\alpha; k} = H^1_{\alpha; -k-1}$ for any $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Note
that the definition of a strip $H^1_{\alpha; k}$ for $\alpha \in \Phi^-$ given here slightly differs from that in
\cite{10}. According to \cite{10}, we would have $H^1_{-\alpha, k} = H^1_{\alpha; -k}$. Thus by this new definition, we
have to re-examine some results of \cite{10}. Note that it makes no change for the results
only involving positive roots. We call any non-empty connected simplex of
\begin{equation}
E - \bigcup_{\alpha \in \Phi^+, k \in \mathbb{Z}} H_{\alpha; k}
\end{equation}
an alcove of $E$. Each alcove of $E$ has the form $\bigcap_{\alpha \in \Phi^+} H^1_{\alpha; k_\alpha}$ for some $\Phi^+$-tuple $(k_\alpha)_{\alpha \in \Phi^+}$.
One should note that not every $\Phi^+$-tuple $(k_\alpha)_{\alpha \in \Phi^+}$ over $\mathbb{Z}$ gives rise to an alcove of $E$ as above. In fact, it is well known that $\bigcap_{\alpha \in \Phi^+} H^1_{\alpha;k_\alpha}$ is an alcove of $E$ if and only if for any $\alpha, \beta \in \Phi^+$ with $\alpha + \beta \in \Phi^+$, the inequality

$$|\alpha|^2k_\alpha + |\beta|^2k_\beta + 1 \leq |\alpha + \beta|^2(k_{\alpha+\beta} + 1) \leq |\alpha|^2(k_{\alpha+1}) + |\beta|^2(k_{\beta+1}) + |\alpha + \beta|^2 - 1$$

holds (see [10, Theorem 5.2]). Now we give a new and simpler form for this result.

**Theorem 1.3.** $\bigcap_{\alpha \in \Phi^+} H^1_{\alpha;k_\alpha}$ is an alcove of $E$ if and only if for any $\alpha, \beta \in \Phi^+$ with $\gamma = (\alpha^\vee + \beta^\vee)^\vee \in \Phi^+$, the inequality

$$k_\alpha + k_\beta \leq k_\gamma \leq k_\alpha + k_\beta + 1.$$

holds.

**Proof.** By [10, Theorem 5.2], it is enough to show that the inequality systems (1.2.1) and (1.3.1) are equivalent for any irreducible positive root system $\Phi^+$ of rank 2. Note that when $\alpha, \beta, \alpha + \beta \in \Phi^+$ satisfy $|\alpha| = |\beta| = |\alpha + \beta|$, the inequality (1.2.1) becomes

$$k_\alpha + k_\beta \leq k_{\alpha + \beta} \leq k_\alpha + k_\beta + 1,$$

and $\gamma = (\alpha^\vee + \beta^\vee)^\vee = \alpha + \beta$. Thus the inequalities (1.2.1) and (1.3.1) are the same in this case. So we need only consider the case that $\Phi^+$ has type $B_2$ or $G_2$ and that the lengths of the three roots involved in the inequalities are not all the same. We shall only deal with the case of type $B_2$ here. The case of type $G_2$ can be done similarly and hence is left to the reader. Now let $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$. Then the inequality system (1.2.1) is

$$\begin{cases}
    k_\alpha + 2k_\beta \leq k_{\alpha + \beta} \leq k_\alpha + 2k_\beta + 2, \\
    k_\alpha + k_{\alpha + \beta} \leq 2k_{2\alpha + \beta} + 1 \leq k_\alpha + k_{\alpha + \beta} + 2.
\end{cases}$$

and (1.3.1) is

$$\begin{cases}
    k_\alpha + k_\beta \leq k_{2\alpha + \beta} \leq k_\alpha + k_\beta + 1, \\
    k_\beta + k_{2\alpha + \beta} \leq k_{\alpha + \beta} \leq k_\beta + k_{2\alpha + \beta} + 1.
\end{cases}$$

We have to show the equivalence of the inequality systems (1.3.3) and (1.3.4). First assume (1.3.3). Adding two inequalities of (1.3.3) together on the corresponding sides and then subtracting from all sides of the resulting inequality:

$$\begin{cases}
    k_\alpha + 2k_\beta \leq k_{\alpha + \beta} \leq k_\alpha + 2k_\beta + 2, \\
    k_\alpha + k_{\alpha + \beta} \leq 2k_{2\alpha + \beta} + 1 \leq k_\alpha + k_{\alpha + \beta} + 2.
\end{cases}$$

and (1.3.4) is

$$\begin{cases}
    k_\alpha + k_\beta \leq k_{2\alpha + \beta} \leq k_\alpha + k_\beta + 1, \\
    k_\beta + k_{2\alpha + \beta} \leq k_{\alpha + \beta} \leq k_\beta + k_{2\alpha + \beta} + 1.
\end{cases}$$

We have to show the equivalence of the inequality systems (1.3.3) and (1.3.4). First assume (1.3.3). Adding two inequalities of (1.3.3) together on the corresponding sides and then subtracting from all sides of the resulting inequality:

$$\begin{cases}
    k_\alpha + 2k_\beta \leq k_{\alpha + \beta} \leq k_\alpha + 2k_\beta + 2, \\
    k_\alpha + k_{\alpha + \beta} \leq 2k_{2\alpha + \beta} + 1 \leq k_\alpha + k_{\alpha + \beta} + 2.
\end{cases}$$
Then the first inequality of (1.3.4) follows from (1.3.5). Rewrite the second inequality of (1.3.3) into the form
\[ 2k_{2\alpha+\beta} \leq k_\alpha + k_{\alpha+\beta} + 1 \leq 2k_{2\alpha+\beta} + 2. \]

Then adding (1.3.6) to the first inequality of (1.3.3) on the corresponding sides and subtracting \( k_\alpha \) from all sides of the resulting inequality, we get
\[ 2(k_{2\alpha+\beta} + k_\beta) \leq 2k_{\alpha+\beta} + 1 \leq 2(k_{2\alpha+\beta} + k_\beta) + 4. \]

The second inequality of (1.3.4) follows from (1.3.7). Next assume (1.3.4). Then the first inequality of (1.3.3) is obtained by adding two inequalities of (1.3.4) together on the corresponding sides, followed by subtracting \( k_{\alpha+\beta} \) from all sides of the resulting inequality. Rewrite the second inequality of (1.3.4) into the form
\[ k_{\alpha+\beta} \leq k_\beta + k_{2\alpha+\beta} + 1 \leq k_{\alpha+\beta} + 1. \]

Then the second inequality of (1.3.3) is obtained by adding (1.3.8) to the first inequality of (1.3.4) on the corresponding sides, followed by subtracting \( k_\beta \) from all sides of the resulting inequality. □

Note that the result still holds if we replace all the notations \( \Phi^+ \) by \( \Phi \) in Theorem 1.3.

1.4. Let \( W \) be the Weyl group of \( \Phi \) generated by all the reflections \( s_\alpha, \alpha \in \Phi \), on \( E \), where \( s_\alpha \) sends \( x \) to \( x - \langle x, \alpha^\vee \rangle \alpha \). Let \( Q \) denote the root lattice \( \mathbb{Z}\Phi \), and \( N \) the group consisting of all the translations \( T_\lambda \) on \( E \) for \( \lambda \in Q \), where \( T_\lambda \) sends \( x \) to \( x + \lambda \). We denote by \( W_a \) the group of affine transformations of \( E \) generated by \( N \) and \( W \). It is well known that \( W_a \) is the semidirect extension of \( W \) by the normal subgroup \( N \) on which the action of \( W \) is known.

For linear and affine transformations, we shall write operations on the right and compose them accordingly. With this convention, we define \( s_0 = s_{\alpha_0}T_{-\alpha_0} \) and \( s_i = s_{\alpha_i}, \quad 1 \leq i \leq l \). It is known that \( W_a \) (resp. \( W \)) is a Coxeter group on the generators \( s_0, s_1, \ldots, s_l \) (resp. \( s_1, \ldots, s_l \)). We write \( \tilde{S} = \{ s_0, s_1, \ldots, s_l \} \) and \( S = \{ s_1, \ldots, s_l \} \). The group \( W_a \) is called an affine Weyl group of type \( \tilde{X} \), where \( X \) is the type of the coroot system \( \Phi^\vee \).

1.5. It is well known that the right action of \( W_a \) on \( E \) gives rise to permutations of the set \( \{ H_{\alpha;k} \mid \alpha \in \Phi, \; k \in \mathbb{Z} \} \). So it induces permutations of the set \( \mathcal{U} \) of alcoves of \( E \). It is also well known that \( \mathcal{U} \) is simply transitive under \( W_a \) (see [2; 14]).

Let \( w = \bar{w} \cdot T_\lambda \in W_a \) with \( \bar{w} \in W \) and \( \lambda \in Q \). We consider the alcove
For \( v \in \bigcap_{\alpha \in \Phi} H^1_{\alpha;k_a} \), we have \( (v)w = (v)\bar{w} + \lambda \). This implies that

\[
\langle (v)w, \alpha^\vee \rangle = \langle v, ((\alpha)\bar{w}^{-1})^\vee \rangle + \langle \lambda, \alpha^\vee \rangle.
\]

So we get

\[
(1.5.1) \quad h_\alpha = k(\alpha)\bar{w}^{-1} + \langle \lambda, \alpha^\vee \rangle
\]

for \( \alpha \in \Phi \).

We see that \( A_1 = \bigcap_{\alpha \in \Phi} H^1_{\alpha;\epsilon_a} \) is an alcove of \( E \), where \( \epsilon_\alpha = 0 \) if \( \alpha \in \Phi^+ \), and \( \epsilon_\alpha = -1 \) if \( \alpha \in \Phi^- \) (comparing with [10, Lemma 1.1]). Denote \( A_w = (A_1)w \) for any \( w \in W_a \). Thus any alcove of \( \mathfrak{U} \) has the form \( A_w = \bigcap_{\alpha \in \Phi^+} H^1_{\alpha;k(w;\alpha)} \) or \( A_w = \bigcap_{\alpha \in \Phi} H^1_{\alpha;k(w;\alpha)} \) for some \( w \in W_a \), with the convention that

\[
(1.5.2) \quad k(w; -\alpha) = -k(w; \alpha) - 1 \quad \text{for any} \quad \alpha \in \Phi^+
\]

(comparing with [10, §1.1]). Note that (1.5.2) actually holds for any \( \alpha \in \Phi \). We shall identify \( W_a \) with \( \mathfrak{U} \) as sets under the correspondence \( w \mapsto A_w \), and call \( (k(w; \alpha))_{\alpha \in \Phi^+} \) or \( (k(w; \alpha))_{\alpha \in \Phi} \) the alcove form of \( w \). From (1.5.1), we get

\[
(1.5.3) \quad k(w; \alpha) = \epsilon(\alpha)\bar{w}^{-1} + \langle \lambda, \alpha^\vee \rangle
\]

for \( \alpha \in \Phi \), where the decomposition \( w = \bar{w} \cdot T_\lambda \) is as above (comparing with [10, Theorem 3.3]). This is also equivalent to a formula given by Iwahori and Matsumoto [6].

1.6. The action of an element \( s_j \in \tilde{S} \) on \( \mathfrak{U} \) can be described as follows. For \( w \in W_a \), we have by (1.5.1) that

\[
(1.6.1) \quad k(ws_j, \alpha) = \begin{cases} k(w; (\alpha)s_j), & \text{if } j \neq 0, \\ k(w; (\alpha)s_0) + \langle -\alpha_0, \alpha^\vee \rangle, & \text{if } j = 0. \end{cases}
\]

for any \( \alpha \in \Phi \) (comparing with [10, Proposition 4.2]).

1.7. We can define the left action of \( W_a \) on \( \mathfrak{U} \) as below. For \( y, w \in W_a \), we set \( y(A_w) = A_yw \). Then for \( w \in W_a \) and \( s_j \in \tilde{S} \), we have

\[
(1.7.1) \quad k(s_jw, \alpha) = \begin{cases} k(w; \alpha), & \text{if } \alpha \neq \pm(\alpha_j)\bar{w}, \\ k(w; \alpha) \mp 1, & \text{if } \alpha = \pm(\alpha_j)\bar{w}, \end{cases}
\]

where \( \bar{w} \) is the image of \( w \) under the natural map \( W_a \rightarrow W_a/N \cong W \) (comparing with [10, Proposition 4.1]).

1.8. To any \( w \in W_a \), we associate two subsets of \( \tilde{S} \): \( \mathcal{L}(w) = \{ s \in \tilde{S} \mid sw \leq w \} \) and \( \mathcal{P}(w) = \{ s \in \tilde{S} \mid ws \leq w \} \), where the notation \( \leq \) stands for the Chevalley-Bruhat order.
Proposition 1.9 (comparing with [10, Propositions 3.4 and 4.3]). Let \( (k(w; \alpha))_{\alpha \in \Phi} \) be the alcove form of \( w \in W_a \). Then

1. \( \ell(w) = \sum_{\alpha \in \Phi^+} |k(w; \alpha)| \).
2. \( \mathcal{R}(w) = \{ s_j \in \tilde{S} \mid \text{either } j \neq 0, k(w; \alpha_j) < 0, \text{ or } j = 0, k(w; -\alpha_0) > 0 \} \).
3. \( \mathcal{L}(w) = \{ s_j \in \tilde{S} \mid k(w; (\alpha_j)\tilde{w}) > 0 \} \).
4. \( k(w^{-1}, \alpha) = \begin{cases} k(w; -\alpha)\tilde{w}, & \text{if either } \alpha, -(\alpha)\tilde{w} \in \Phi^+ \text{ or } \alpha, -(\alpha)\tilde{w} \in \Phi^-, \\ k(w; -\alpha)\tilde{w} + 1, & \text{if } \alpha, (\alpha)\tilde{w} \in \Phi^+, \\ k(w; -\alpha)\tilde{w} - 1, & \text{if } \alpha, (\alpha)\tilde{w} \in \Phi^- \end{cases} \).

§2. The cases of the classical types.

In this section, we shall first give an explicit description for all the irreducible root systems of the classical types. Then we reformulate some results on the alcove forms of \( w \in W_a \) in these cases. A significant change is made for the index set of the alcove form of \( w \in W_a \), where the root system \( \Phi \) is replaced by the corresponding coroot system \( \Phi^\vee \).

2.1. We set the notations \( \Omega_l = \{ A_{l-1}, B_l, C_l, D_l \} \) and \( \Omega'_l = \{ B_l, C_l, D_l \} \) for \( l \geq 2 \), which will be used quite often later on.

2.2. We know that there is a natural bijective map \( \alpha \mapsto \alpha^\vee \) from a root system \( \Phi \) to the corresponding coroot system \( \Phi^\vee \). By Theorem 1.3, it is more convenient to express the alcove form \( (k(w; \alpha))_{\alpha \in \Phi} \) of \( w \in W_a \) by \( (k(w; \alpha))_{\alpha \in \Phi^\vee} \), where we set \( k(w; \alpha) = k(w; \alpha^\vee) \) for any \( \alpha \in \Phi^\vee \) (note \( (\alpha^\vee)^\vee = \alpha \)). We make such a change from now on. Then for an affine Weyl group \( W_a \) of type \( \widetilde{X} \), \( X \in \Omega_l \), the index set of the alcove form of \( w \in W_a \) will be the root system of type \( X \). Note that \( -\alpha_0^\vee \) is the highest coroot in \( \Phi^\vee \).

2.3. Let \( e_1, e_2, \ldots, e_l \) be an orthonormal basis of a euclidean space \( E \). Set \( e_{-i} = -e_i, 1 \leq i \leq l \), and \( e_0 = 0 \). Define \( (i, j) = e_j - e_i \) for \( i, j \in [-l, l] \), where the notation \([a, b]\) stands for the interval \([a, a + 1, \ldots, b]\) for any \( a \leq b \) in \( \mathbb{Z} \). Then the relations

\[(2.3.1) \quad (i, j) = (i, t) + (t, j) \quad \text{and} \quad (i, j) = (-j, -i) \]

hold for any \( i, t, j \in [-l, l] \).

2.4. The root system of type \( A_{l-1}, l \geq 2 \), can be described by

\[ \Phi(A_{l-1}) = \{(i, j) \mid 1 \leq i, j \leq l; i \neq j \} \].

The roots \((t - 1, t), 1 < t \leq l\), form a simple root system, and \( \Phi^+(A_{l-1}) = \{(i, j) \mid 1 \leq i < j \leq l\} \) is the corresponding positive root system. The highest root is \((1, l)\).
The roots \((t-1,t), 1 \leq t \leq l\), form a simple root system, and \(\Phi^+(B_l) = \{(i,j) \in \Phi(B_l) \mid i < j\}\) is the corresponding positive root system. The highest root is \((-l + 1, l)\).

2.6. The root system of type \(C_l\), \(l \geq 2\), can be described by

\[
\Phi(C_l) = \{(i,j) \mid i, j \in [-l,l] \setminus \{0\}; i \neq j\}.
\]

The roots \((t-1,t), 1 < t \leq l\), and \((-1,1)\) form a simple root system, and \(\Phi^+(C_l) = \{(i,j) \in \Phi(C_l) \mid i < j\}\) is the corresponding positive root system. The highest root is \((-l, l)\).

2.7. The root system of type \(D_l\), \(l \geq 4\), can be described by

\[
\Phi(D_l) = \{(i,j) \mid i, j \in [-l,l] \setminus \{0\}; i \neq \pm j\}.
\]

The roots \((t-1,t), 1 < t \leq l\), and \((-1,2)\) form a simple root system, and \(\Phi^+(D_l) = \{(i,j) \in \Phi(D_l) \mid i < j\}\) is the corresponding positive root system. The highest root is \((-l + 1, l)\).

2.8. Fix \(w \in W_a(\tilde{X}), X \in \Omega_l\). Let \(\Phi\) be the root system of type \(X\). In the alcove form \((k(w; \alpha))_{\alpha \in \Phi}\) of \(w\), we shall denote the entry \(k(w; (i,j))\) simply by \(k(w; i,j)\) for \((i,j) \in \Phi\). Then the entry \(k(w; (i,j)\bar{x})\) becomes \(k(w; (i)\bar{x}, (j)\bar{x})\) for any element \(\bar{x}\) of the corresponding Weyl group, the latter is regarded as a certain permutation group on the set \([-l,l]\). By (1.5.2), (2.3.1), Theorem 1.3 and the note immediately after the Theorem, we get

Lemma 2.9. Let \(X \in \Omega_l\) for an appropriate integer \(l\) and let \(\Phi\) be the root system of type \(X\). Then for any \(w \in W_a(\tilde{X})\) and \((i,j), (i,t), (t,j) \in \Phi\), we have

1. \(k(w; j,i) = -k(w; i,j) - 1 = k(w; -i,-j)\),
2. \(k(w; i,t) + k(w; t,j) \leq k(w; i,j) \leq k(w; i,t) + k(w; t,j) + 1\).

§3. The permutation forms.

3.1. Let \(A_l\) be the affine Weyl group of type \(\tilde{A}_{l-1}\), \(l \geq 2\). It is well known that \(A_l\) could be regarded as a group of certain permutations on \(\mathbb{Z}\):

\[
A_l = \left\{ w : \mathbb{Z} \longrightarrow \mathbb{Z} \mid (i+l)w = (i)w + l \text{ for } i \in \mathbb{Z}; \sum_{i=1}^{l} (i)w = \sum_{i=1}^{l} i \right\}.
\]

Its simple reflection set \(\tilde{S} = \{r_0, r_1, \ldots, r_{l-1}\}\) is given by

\[
\tilde{S} = \{\bar{r}_0, \bar{r}_1, \ldots, \bar{r}_{l-1}\} = \{0, 1, 2, \ldots, l-1\}.
\]
for \(i \in \mathbb{Z}\) and \(0 \leq t < l\). Each element \(w \in A_l\) is determined entirely by the images \((a_1)w, (a_2)w, \ldots, (a_l)w\) of any given \(l\) integers \(a_1, \cdots, a_l\) under \(w\), which are pairwisely incongruent modulo \(l\).

**3.2.** As a Coxeter group, the affine Weyl group \(W_a(\widetilde{X})\) of type \(X \in \Omega'_l\) has the following presentation:

\[
W_a(\widetilde{X}) = \langle s_i \mid 0 \leq i \leq l, (s_is_j)^{m_{ij}} = 1, \text{ for } 0 \leq i \leq j \leq l \rangle,
\]

where \(m_{ij} = 1\) for \(i = j\); \(m_{ij} = 3\) in one of the following cases:

(i) \(2 < i + 1 = j \leq l\),

(ii) \((i, j) = (0, l - 1)\) and \(X = B_l\),

(iii) \((i, j) = (1, 3), (0, l - 1)\) and \(X = D_l\);

\(m_{ij} = 4\) in one of the following cases:

(i) \((i, j) = (1, 2)\) and \(X = B_l\),

(ii) \((i, j) = (1, 2), (0, l)\) and \(X = C_l\);

\(m_{ij} = 2\) in all the other cases. Note that the labelings of the nodes in the corresponding extended Dynkin diagrams slightly differ from the usual ones (see [2; 5] for examples).

**3.3.** The affine Weyl group \(W_a(\widetilde{C}_l)\) \((l \geq 2)\) can be embedded into \(A_{2l+2}\) by an injective homomorphism which sends \(s_t\), \(1 < t \leq l\), to \(r_{t-1}r_{2l+2-t}\), \(s_1\) to \(r_0r_{2l+1}r_0\), and \(s_0\) to \(r_1r_{l+1}r_l\). Thus by identifying with its image, we can regard \(W_a(\widetilde{C}_l)\) as a subgroup of \(A_{2l+2}\). The group \(W_a(\widetilde{C}_l)\) can also be regarded as the set of all the fixed points of \(A_{2l+2}\) under the involutive automorphism \(\phi\) which sends \(r_i\) to \(r_{2l+1-i}\) for all \(i, 0 \leq i \leq 2l + 1:\)

\[
(3.3.1) \quad W_a(\widetilde{C}_l) = \{w \in A_{2l+2} \mid \phi(w) = w\} = \{w \in A_{2l+2} \mid (-i)w = -(i)w, \forall i \in \mathbb{Z}\}.
\]

The simple reflections \(s_t, 0 \leq t \leq l\), of \(W_a(\widetilde{C}_l)\) are the permutations on \(\mathbb{Z}\) satisfying that for \(i \in \mathbb{Z}\) and \(1 < t \leq l\), we have

\[
(i)s_t = \begin{cases} 
  i, & \text{if } i \not\equiv t, \pm(t - 1) \pmod{2l + 2}, \\
  i + 1, & \text{if } i \equiv -t, t - 1 \pmod{2l + 2}, \\
  i - 1, & \text{if } i \equiv t, -t + 1 \pmod{2l + 2}.
\end{cases}
\]

\[
(i)s_1 = \begin{cases} 
  i, & \text{if } i \not\equiv \pm 1 \pmod{2l + 2}, \\
  i + 2, & \text{if } i \equiv -1 \pmod{2l + 2}, \\
  i - 2, & \text{if } i \equiv 1 \pmod{2l + 2}.
\end{cases}
\]
3.4. The group $W_a(\tilde{B}_l)$ ($l \geq 3$) can be embedded into $A_{2l+2}$ by an injective homomorphism which sends $s_t$, $1 < t \leq l$, to $r_{t-1}r_{2l+2-t}$, $s_1$ to $r_0r_{2l+1}r_0$ and $s_0$ to $r_1r_{l+1}r_l-1r_l+2r_lr_{l+1}r_1$. Thus we can regard $W_a(\tilde{B}_l)$ as a subgroup of $A_{2l+2}$ by identifying with its image. We can also regard $W_a(\tilde{B}_l)$ as a certain fixed point set of $A_{2l+2}$ under the automorphism $\phi$ (see 3.3):

\[(3.4.1)\]

$$W_a(\tilde{B}_l) = \{w \in A_{2l+2} \mid \phi(w) = w; N_l(w) \equiv 0 \pmod{2}\},$$

$$= \{w \in A_{2l+2} \mid (-i)w = -(i)w, \forall i \in \mathbb{Z}; N_l(w) \equiv 0 \pmod{2}\},$$

where $N_l(w)$ is the number of the integers $j$ with $j < l+1$ and $(j)w > l+1$. The simple reflections $s_t$, $0 \leq t \leq l$, of $W_a(\tilde{B}_l)$ are the permutations on $\mathbb{Z}$ as below. For $i \in \mathbb{Z}$ and $1 < t \leq l$, we have

$$(i)s_t = \begin{cases} 
  i, & \text{if } i \not\equiv \pm t, \pm (t-1) \pmod{2l+2}, \\
  i+1, & \text{if } i \equiv -t, t-1 \pmod{2l+2}, \\
  i-1, & \text{if } i \equiv t, -t+1 \pmod{2l+2}.
\end{cases}$$

$$(i)s_1 = \begin{cases} 
  i, & \text{if } i \not\equiv \pm 1 \pmod{2l+2}, \\
  i+2, & \text{if } i \equiv -1 \pmod{2l+2}, \\
  i-2, & \text{if } i \equiv 1 \pmod{2l+2}.
\end{cases}$$

$$(i)s_0 = \begin{cases} 
  i, & \text{if } i \not\equiv \pm (l-1), \pm l \pmod{2l+2}, \\
  i+3, & \text{if } i \equiv l-1, l \pmod{2l+2}, \\
  i-3, & \text{if } i \equiv -l, -l+1 \pmod{2l+2}.
\end{cases}$$

3.5. The affine Weyl group $W_a(\tilde{D}_l)$ ($l \geq 4$) can be regarded as a subgroup of $A_{2l+2}$ by an injective homomorphism which sends $s_t$, $1 < t \leq l$, to $r_{t-1}r_{2l+2-t}$, $s_1$ to $r_0r_{2l+1}r_0r_1r_{2l+1}r_0$ and $s_0$ to $r_1r_{l+1}r_l-1r_l+2r_lr_{l+1}r_1$, or equivalently, regarded as a certain fixed point set of $A_{2l+2}$ under the automorphism $\phi$ (see 3.3):

\[(3.5.1)\]

$$W_a(\tilde{D}_l) = \{w \in A_{2l+2} \mid \phi(w) = w; N_0(w) \equiv N_l(w) \equiv 0 \pmod{2}\},$$

$$= \{w \in A_{2l+2} \mid (-i)w = -(i)w, \forall i \in \mathbb{Z}; N_0(w) \equiv N_l(w) \equiv 0 \pmod{2}\},$$

where $N_0(w)$ is the number of the integers $j$ with $j < 0$ and $(j)w > 0$. The simple reflections $s_t$, $0 \leq t \leq l$, of $W_a(\tilde{D}_l)$ are the permutations on $\mathbb{Z}$ as below. For $1 < t \leq l$ and $i \in \mathbb{Z}$, we have

\[(\text{case a, } i \not\equiv \pm (l-1), \pm l \pmod{2l+2})\]
On two presentations of the affine Weyl groups

\[ (i) s_1 = \begin{cases} 
  i, & \text{if } i \not\equiv \pm 1, \pm 2 \pmod{2l+2}, \\
  i + 3, & \text{if } i \equiv -1, -2 \pmod{2l+2}, \\
  i - 3, & \text{if } i \equiv 1, 2 \pmod{2l+2}.
\end{cases} \]

\[ (i) s_0 = \begin{cases} 
  i, & \text{if } i \not\equiv \pm l, \pm (l - 1) \pmod{2l+2}, \\
  i + 3, & \text{if } i \equiv l - 1, l \pmod{2l+2}, \\
  i - 3, & \text{if } i \equiv -l, -l + 1 \pmod{2l+2}.
\end{cases} \]

From the above discussion, we get the following result easily.

**Lemma 3.6.** Let \( X \in \Omega_l \) and \( w \in W_a(\tilde{X}) \) for an appropriate integer \( l \). Define an integer \( m_X \) to be \( l \) if \( X = A_{l-1} \), and \( 2l + 2 \) if \( X \in \Omega'_l \). Then regarding \( w \) as an element of \( A_{m_X} \), we have

1. \((i)w \not\equiv (j)w \pmod{m_X}\) for any \( i, j \in \mathbb{Z} \) with \( i \not\equiv j \pmod{m_X} \).
2. In the case of \( X \in \Omega'_l \), we have
   - \((-i)w = -(i)w\) for any \( i \in \mathbb{Z} \).
   - \((h(l+1))w = h(l+1)\) for any \( h \in \mathbb{Z} \).
   - \((i)w \not\equiv 0, l + 1 \pmod{2l+2}\) for any \( i \in \mathbb{Z} \) with \( i \not\equiv 0, l + 1 \pmod{2l+2} \).

**Remark 3.7.** (i) Our definition of \( W_a(\tilde{C}_l) \) as a group of permutations on \( \mathbb{Z} \) is slightly different from that given by R. Bédard (see [1]). According to R. Bédard, the group \( W_a(\tilde{C}_l) \) was embedded into the group \( A_{2l+1} \) instead of \( A_{2l+2} \). The advantage of our definition is that the symmetry between the generators \( s_0 \) and \( s_1 \) in \( W_a(\tilde{C}_l) \) could be seen more explicitly in form.

(ii) The descriptions (3.3.1), (3.4.1), (3.5.1) of the groups \( W_a(\tilde{X}) \), \( X \in \Omega'_l \), can be shown easily by applying induction on \( \ell(w) \geq 0 \) and by the fact that the number of inversions of \( w \) is increasing when \( \ell(w) \) is getting larger.

(iii) Let \( X \in \Omega'_l \). Then regarded as a permutation on \( \mathbb{Z} \), an element \( w \in W_a(\tilde{X}) \) is determined uniquely by the images \((a_1)w, \ldots, (a_l)w\) of any given \( l \) integers \( a_1, \ldots, a_l \) under \( w \), provided that they are pairwisely incongruent modulo \( 2l + 2 \), none of which is divisible by \( l + 1 \), and no pair of which have the sum divisible by \( 2l + 2 \).

**§4. Transition from the permutation forms to the alcove forms.**

Let \( X \in \Omega_l \) for an appropriate integer \( l \) (see 2.1.). In this section, we shall show the transition formulae from the permutation form to the alcove form of \( w \in W_a(\tilde{X}) \). Let \( \Phi = \Phi(X) \) be the root system of type \( X \). Let \( m_X \) be the integer defined as in Lemma 3.6.

**Theorem 4.1.** For any \((i, j) \in \Phi \) and \( w \in W_a(\tilde{X}) \), we have
Thus in this case, we get from (1.6.1) that (1.8). By inductive hypothesis, the result is true for roots $(i, j)$. Proof. For any $i, j \in \mathbb{Z}$ with $i \not\equiv j \pmod{m_X}$, we have the relation

\begin{equation}
(4.1.2) \quad \left[ \frac{(i)w^{-1} - (j)w^{-1}}{m_X} \right] = -\left[ \frac{(j)w^{-1} - (i)w^{-1}}{m_X} \right] - 1
\end{equation}

by Lemma 3.6, (1). Also, we have $(i, j) \in \Phi^+ \iff (j, i) \in \Phi^-$. Thus by Lemma 2.9, (1), we need only show our result in the case when $(i, j)$ is a positive root. The result is known for the case of $X = A_{l-1}$ (see [9, Proposition 6.2.1]). Now assume $X \neq A_{l-1}$ and hence $m_X = 2l + 2$. We see that the set

$$\Phi^+ = \{(i, j) \in \Phi \mid i < j; |i| \leq |j|\}.$$ 

forms a positive root system of $\Phi$. Apply induction on $\ell(w) \geq 0$. When $\ell(w) = 0$, i.e., $w = 1$, we have $((1)1^{-1}, \ldots, (l)1^{-1}) = (1, 2, \ldots, l)$ and $k(w; i, j) = 0$ for all the positive roots $(i, j)$. The result is obviously true. Now assume $\ell(w) > 0$. Take $s_t \in R(w)$ (see 1.8). By inductive hypothesis, the result is true for $ws_t$. First assume $1 < t \leq l$. When $(i, j) \neq (t-1, t), (t-1, t)$ with $(i, j) \in \Phi^+$, we have $(i)s_t < (j)s_t$ and $|(i)s_t| \leq |(j)s_t|$. Thus in this case, we get from (1.6.1) that

$$k(w; i, j) = k(ws_t; (i)s_t, (j)s_t) = \left[ \frac{(j)s_t(ws_t)^{-1} - (i)s_t(ws_t)^{-1}}{2l + 2} \right]$$

On the other hand, by (1.6.1), (2.3.1), (4.1.2), Lemmas 2.9 and 3.6, we have

$$k(w; t-1, t) = -k(ws_t; t-1, t) - 1 = -\left[ \frac{(t)ws_t^{-1} - (t-1)ws_t^{-1}}{2l + 2} \right] - 1$$

and

$$k(w; -t + 1, t) = k(ws_t; -t, t-1) = k(ws_t; -t + 1, t)$$

Next consider the cases $t = 0, 1$. Here we only deal with the case of $X = B$. The proofs:
Now assume \( t = 1 \). The result can be checked easily when \( i, j \neq \pm 1 \). On the other hand, for \( 1 < j \leq l \), we have

\[
k(w; \pm 1, j) = k(ws_1; \mp 1, j) = \left[ \frac{\pm 1}{2l + 2} \right] = \left[ \frac{(j)w_1^{-1} - (\pm 1)w_1^{-1}}{2l + 2} \right].
\]

We also have

\[
k(w; 0, 1) = k(ws_1; 0, -1) = -k(ws_1; 0, 1) - 1
\]

\[
= - \left[ \frac{(1)(w_0)^{-1} - (0)(w_0)^{-1}}{2l + 2} \right] - 1 = - \left[ \frac{(-1)w_1^{-1}}{2l + 2} \right] - 1
\]

\[
= \left[ \frac{(1)w_1^{-1} - (0)w_1^{-1}}{2l + 2} \right].
\]

Finally assume \( t = 0 \). We need only check the cases when \( \{i, j\} \cap \{l - 1, l\} \neq \emptyset \) for otherwise the result is obvious. For \( i \in [-l + 2, l - 2] \), we have

\[
k(w; i, l - 1) = k(ws_0; i, -l) + 1 = -k(ws_0; -i, l)
\]

\[
= - \left[ \frac{(l)(w_0)^{-1} - (-i)(w_0)^{-1}}{2l + 2} \right] = - \left[ \frac{(l + 3)w_1^{-1} - (-i)w_1^{-1}}{2l + 2} \right]
\]

\[
= - \left[ \frac{1}{2l + 2} \right] = \left[ \frac{(l - 1)w_1^{-1} - (i)w_1^{-1}}{2l + 2} \right].
\]

The equation \( k(w; i, l) = \left[ \frac{(l)w_1^{-1} - (i)w_1^{-1}}{2l + 2} \right] \) can be shown similarly. Also, we have

\[
k(w; l - 1, l) = k(ws_0; -l, -l + 1) = k(ws_0; l - 1, l)
\]

\[
= \left[ \frac{(l)(w_0)^{-1} - (l - 1)(w_0)^{-1}}{2l + 2} \right] = \left[ \frac{(l + 3)w_1^{-1} - (l + 2)w_1^{-1}}{2l + 2} \right]
\]

\[
= \left[ \frac{(-l + 1)w_1^{-1} - (l)w_1^{-1}}{2l + 2} \right] = \left[ \frac{(l)w_1^{-1} - (l - 1)w_1^{-1}}{2l + 2} \right]
\]

and

\[
k(w; -l + 1, l) = k(ws_0; l, -l + 1) + 2 = -k(ws_0; -l + 1, l) + 1
\]

\[
= - \left[ \frac{(l)(w_0)^{-1} - (-l + 1)(w_0)^{-1}}{2l + 2} \right] + 1
\]

\[
= - \left[ \frac{(l + 3)w_1^{-1} - (-l - 2)w_1^{-1}}{2l + 2} \right] + 1
\]

\[
= \left[ \frac{(-l + 1)w_1^{-1} - (l)w_1^{-1}}{2l + 2} \right] + \left[ \frac{(l)w_1^{-1} - (l + 1)w_1^{-1}}{2l + 2} \right].
\]
4.2. By Theorem 4.1, it is natural to extend the alcove form of \( w \in W_a(\tilde{X}) \) to \((k(w; i, j))_{i,j \in \mathbb{Z}}\) by setting

\[
(4.2.1) \quad k(w; i, j) = \left[ \frac{(j)w^{-1} - (i)w^{-1}}{m_X} \right]
\]

for any \( X \in \Omega_l \) and \( i, j \in \mathbb{Z} \). Call \((k(w; i, j))_{i,j \in \mathbb{Z}}\) the extended alcove form of \( w \). The relations in Lemma 2.9 remain valid for any \( i, t, j \in \mathbb{Z} \) with \( i \neq j \pmod{m_X} \). Clearly, we have

\[
(4.2.2) \quad k(w; i - m_X, j) = k(w; i, j + m_X) = k(w; i, j) + 1
\]

for any \( i, j \in \mathbb{Z} \). We call \( k(w; i, j) \) a basic entry of the alcove form of \( w \) if \((i, j)\in \Phi(X)\).

It is straightforward to deduce the following

**Corollary 4.3.** Let \( X \in \Omega_l \) for an appropriate integer \( l \). Then we have

\[
(4.3.1) \quad k(wy; i, j) = k(w; (i)y^{-1}, (j)y^{-1})
\]

for any \( i, j \in \mathbb{Z} \) and \( w, y \in W_a(\tilde{X}) \).

Note that in the above corollary, the element \( y \) is regarded as a permutation on \( \mathbb{Z} \).

4.4. The right-hand side of (4.1.1) was named the inverse table of \( w \) by H. Eriksson and K. Eriksson in [4, §8.] as \((i, j)\) ranges over a certain subset of \( \mathbb{Z} \times \mathbb{Z} \). They proposed a conjecture in the case of \( X = C_l \) in order to characterize the inverse table of \( w \in W_a(\tilde{C}_l) \) as a family of integers. Then they claimed that they would be able to formulate a similar conjecture in the cases of \( X = B_l, D_l \). Now Theorems 1.3, 4.1 and Lemma 2.9 together give the following characterization for the inverse table of \( w \in W_a(\tilde{X}), X \in \Omega_l \).

**Theorem 4.5.** Let \( X \in \Omega_l \) for an appropriate integer \( l \) and let \( \Phi \) be the root system of type \( X \). Then a family of integers \((k_{ij})_{(i,j)\in \Phi}\) is the inverse table of some element in \( W_a(\tilde{X}) \) if and only if the following conditions are satisfied:

1. \( k_{ji} = -k_{ij} - 1 \) for any \((i, j)\in \Phi\);
2. \( k_{it} + k_{tj} \leq k_{ij} \leq k_{it} + k_{tj} + 1 \) for any \( i, t, j \) with \((i, j), (i, t), (t, j)\) \in \( \Phi \).

This theorem gives an affirmative answer to the conjecture in [4, 8.3].

§5. Transition from the alcove forms to the permutation forms.

Keep the notations in 4.2. Denote the integer \( m = m_X \) (see Lemma 3.6) and the root system \( \Phi = \Phi(X) \) (see 2.4-2.7) for \( X \in \Omega_l \). Fix \( w \in W_a(\tilde{X}) \). The aim of the present section is to express the numbers \((t)w^{-1}, 1 \leq t \leq l\) in terms of the basic entries of
5.1. By (4.1.1), we can write

\[(j)w^{-1} - (i)w^{-1} = m \cdot k(w; i, j) + r(w; i, j)\]

for any \((i, j) \in \mathbb{Z} \times \mathbb{Z}\), where the integer \(r(w; i, j)\) is taken from the interval \([0, m - 1]\).

Fix \(t, 1 \leq t \leq l\). By taking the sums over \(j = 1, 2, \ldots, m\) on both sides of (5.1.1), we get

\[(5.1.2) \quad m \sum_{j=1}^{m} ((j)w^{-1} - (t)w^{-1}) = m \cdot \sum_{j=1}^{m} k(w; t, j) + \sum_{j=1}^{m} r(w; t, j).\]

By the fact

\[(5.1.3) \quad m + \sum_{j=1}^{m} r(w; t, j) = \sum_{j=1}^{m} j = \sum_{j=1}^{m} (j)w^{-1},\]

we get from (5.1.2) that

\[(5.1.4) \quad (t)w^{-1} = 1 - \sum_{j=1}^{m} k(w; t, j)\]

\[= 1 - \sum_{j=t+1}^{l} k(w; t, j) + \left( t - 1 + \sum_{j=1}^{t-1} k(w; j, t) \right) - \sum_{h=0}^{m-l-1} k(w; t, m-h)\]

\[= t + \sum_{j=1}^{t-1} k(w; j, t) - \sum_{j=t+1}^{l} k(w; t, j) + \sum_{h=0}^{m-l-1} k(w; -h, t).\]

The last equality follows by the fact \(k(w; t, m-h) = k(w; t, -h) + 1 = -k(w; -h, t)\) for any \(t \not\equiv h \pmod{m}\) in \(\mathbb{Z}\). When \(X = A_{l-1}\), we have \(m = l\) and then the last sum of the right-hand side of (5.1.4) vanishes. We get the formula in [9, Corollary 6.2.2, (i)]. Now assume \(X \in \Omega'_l\). Then \(m = 2l + 2\) and (5.1.4) becomes

\[(5.1.5) \quad (t)w^{-1} = t + \sum_{j=1}^{t-1} k(w; j, t) - \sum_{j=t+1}^{l} k(w; t, j) + \sum_{1 \leq h \leq l} k(w; -h, t) + 2k(w; -t, t),\]

where we use the relation

\[(5.1.6) \quad k(w; 0, t) + k(w; -l - 1, t) = k(w; -t, t).\]
by (5.1.1) and Lemma 3.6, (3). Note that \( k(w; 0, t) \) is a basic entry of the alcove form of \( w \) only when \( X = B_l \).

We see that all the \( k(w; a, b) \)'s occurring on the right-hand side of (5.1.5) are the basic entries of the alcove form of \( w \) except for \( k(w; -t, t) \). On the other hand, \( k(w; -t, t) \) is a basic entry of the alcove form of \( w \) only when \( X = C_l \). So we get

**Theorem 5.2.** Let \( w \in W_a(\tilde{X}), l \geq 2 \) and \( t \in [1, l] \). Then we have the following transition formulae on \( w \).

1. When \( X = A_{l-1} \), we have
   \[
   (t)w^{-1} = t + \sum_{j=1}^{t-1} k(w; j, t) - \sum_{j=t+1}^{l} k(w; t, j),
   \]
   (5.2.1)

2. When \( X = C_l \), we have
   \[
   (t)w^{-1} = t + \sum_{j=1}^{t-1} k(w; j, t) - \sum_{j=t+1}^{l} k(w; t, j) + \sum_{h=1}^{l} k(w; -h, t) + k(w; -t, t),
   \]
   (5.2.2)

Since \( k(w; -t, t) \) is not a basic entry of the alcove form of \( w \) when \( X = B_l, D_l \). So in these cases, we have to calculate \( k(w; -t, t) \) via the basic entries of the alcove form of \( w \) in order to get the required transition formulae.

**5.3.** Assume \( X = B_l, l \geq 3 \). By (5.1.7), we need only determine \( r(w; 0, t) \) in order to calculate \( (t)w^{-1} \). By (4.2.1), we have

\[
(5.3.1) \quad k(w; -t, t) = 2k(w; 0, t) + \epsilon(w, t),
\]

where \( \epsilon(w, t) = 0 \) or 1 according to \( r(w; 0, t) < l + 1 \) or \( > l + 1 \) respectively (note that \( r(w; 0, t) \neq l + 1 \) by Lemma 3.6, (3), (4)). Let

\[
(5.3.2) \quad \Delta(w, t) = t + \sum_{j=1}^{t-1} k(w; j, t) - \sum_{j=t+1}^{l} k(w; t, j) + \sum_{1 \leq h \leq l \atop h \neq t} k(w; -h, t) + 4k(w; 0, t).
\]

Then by (5.1.5) and (5.3.1), we have

\[
(5.3.3) \quad (t)w^{-1} = \Delta(w, t) + 2\epsilon(w, t),
\]

where \( \Delta(w, t) \) is determined entirely by the basic entries of the alcove form of \( w \). Let
by the definition of $\epsilon(w,t)$ and by the fact $1 \leq r(w;0,t) \leq 2l + 1$ (see Lemma 3.6, (3), (4)). So $\epsilon(w,t)$ can also be determined uniquely by $\delta(w,t)$ except for the case of $\delta(w,t) = l$. When $\delta(w,t) = l$, $r(w;0,t)$ could be either $l$ or $l + 2$ and hence $\epsilon(w,t)$ could be either $0$ or $1$. But we see from Lemma 3.6, (1), (2) that there is exactly one by the definition of $\epsilon_t$ (5.4.1) ($\ast$).

In the above setup, we have Theorem 5.4.

(5.3.4), the above results can be summarized as follows.

Next assume 5.5. Let

\begin{equation}
\Delta'(w,t) = t + \sum_{j=1}^{t-1} k(w;j,t) - \sum_{j=t+1}^{l} k(w;t,j) + \sum_{1 \leq i \leq l, i \neq t} k(w;-i,t).
\end{equation}

Then $\Delta'(w,t)$ is determined entirely by the basic entries of the alcove form of $w$. By (5.1.5) and (4.2.1), we have

\begin{equation}
(t)w^{-1} - 2 \left[ \frac{2 \cdot (t)w^{-1}}{2l + 2} \right] = \Delta'(w,t).
\end{equation}

This implies by (5.1.7) that

\begin{equation}
k(w;0,t)(2l - 2) + r(w;0,t) - 2\epsilon'(w,t) = \Delta'(w,t),
\end{equation}

where $\epsilon'(w,t) = 0$ or $1$ according to $r(w;0,t) < l + 1$ or $> l + 1$ respectively. Note that $k(w;0,t)$ is not a basic entry of the alcove form of $w$. We see from Lemma 3.6, (1) that $r(w;0,t) - 2\epsilon'(w,t)$ is in $[1, 2l - 1]$. Let $\delta'(w,t)$ be the remainder of $\Delta'(w,t)$ divided by $2l - 2$. Then we see from (5.5.2) that when the remainder $\delta'(w,t) \neq 1$, the numbers

Note that we have $1 \leq \delta(w,t) \leq 2l - 1$ for any $t \in [1, l]$ by (5.3.4). So by (5.1.7) and (5.3.4), the above results can be summarized as follows.

**Theorem 5.4.** In the above setup, we have

\begin{equation}
(t)w^{-1} = \begin{cases} 
    k(w;0,t)(2l + 2) + \delta(w,t), & \text{if } 1 \leq \delta(w,t) < l, \\
    k(w;0,t)(2l + 2) + \delta(w,t) + 2, & \text{if } l < \delta(w,t) \leq 2l - 1, \\
    k(w;0,t)(2l + 2) + l + 1 \pm 1, & \text{if } \delta(w,t) = l.
\end{cases}
\end{equation}

for any $w \in W_a(\widetilde{B}_l)$, $l \geq 3$. In the last case, the value $(t)w^{-1}$ is determined eventually by the evenness of the number $N_l(w^{-1})$.
If \( \delta'(w, t) = 1 \), then \( r(w; 0, t) = 1 \) or \( 2l + 1 \); also, if \( \delta'(w, t) = l \), then \( r(w; 0, t) = l \) or \( l + 2 \). By Lemma 3.6, (1), (2), we see that there are exactly two integers \( t', t'' \in [1, l] \) satisfying \( r(w; 0, t') \in \{1, 2l + 1\} \) and \( r(w; 0, t'') \in \{l, l + 2\} \) respectively. Thus all the \( (t)w^{-1} \) (1 \( \leq \) \( t \leq \) l) but two are determined entirely by \( \Delta'(w, t) \) and hence by the basic entries of the alcove form of \( w \). Then by (5.5.3) and (5.1.7), we see that the evenness of the values \( N_0(w^{-1}) \) and \( N_1(w^{-1}) \) (see 3.5 and 3.4) can determine the values of these two exceptional \( (t')w^{-1} \) and \( (t'')w^{-1} \).

Note that we have \( \delta'(w, t) \in [0, 2l - 3] \) for any \( t \in [1, l] \). Let \( \lambda(w, t) = \left\lfloor \frac{\Delta'(w, t)}{2l - 2} \right\rfloor \). Then by (5.1.7) and (5.5.3), we can summarize our results as follows.

**Theorem 5.6.** In the above setup, we have

\[
(t)w^{-1} = \begin{cases} 
\lambda(w, t)(2l + 2) + \delta'(w, t), & \text{if } 1 < \delta'(w, t) < l, \\
\lambda(w, t)(2l + 2) + \delta'(w, t) + 2, & \text{if } l < \delta'(w, t) \leq 2l - 3, \\
\lambda(w, t)(2l + 2) - 2, & \text{if } \delta'(w, t) = 0, \\
\lambda(w, t)(2l + 2) + 1, & \text{if } \delta'(w, t) = 1, \\
\lambda(w, t)(2l + 2) + l + 1 \pm 1, & \text{if } \delta'(w, t) = l.
\end{cases}
\]

for any \( w \in W_0(\bar{\Delta}_l), l \geq 4 \). In the last two cases, the values \( (t)w^{-1} \) are determined by the evenness of the numbers \( N_0(w^{-1}) \) and \( N_1(w^{-1}) \) respectively.

**5.7.** Now we consider the functions \( N_0(w) \) and \( N_1(w) \) of \( W_a(\bar{X}) (X \in \Omega_l) \). Write, for \( 1 \leq t \leq l \),

\[
(t)w = k_t(2l + 2) + r_t = k'_t(2l + 2) - (l + 1) + r'_t
\]

with \( k_t, k'_t, r_t, r'_t \in \mathbb{Z} \) and \( 0 \leq r_t, r'_t < 2l + 2 \) (By Lemma 3.6 (4), we have \( 1 \leq r_t, r'_t < 2l + 2 \)). Then

\[
N_0(w) = \sum_{i=1}^{l} |k_t| \quad \text{and} \quad N_1(w) = \sum_{i=1}^{l} |k'_t|.
\]

By (5.7.1), we get the inequality

\[
N_0(w) - l \leq N_1(w) \leq N_0(w) + l.
\]

By the relations \((-i)w = -(i)w \) and \((i)w + (2l + 2 - i)w = 2l + 2 \) \( (i \in \mathbb{Z}) \), we also get \( N_0(w) = N_0(w^{-1}) \) and \( N_1(w) = N_1(w^{-1}) \) for any \( w \in W_a(\bar{X}) \).

**5.8.** We conclude the paper by an application of the permutation form of \( W_a(\bar{X}) \), \( X \in \Omega_l \). Let \( \pi_i(w) \) (0 \( \leq \) \( i \leq \) \( l \)) be the minimal possible multiplicity of the factor \( s_i \) occurring in a reduced expression of \( w \in W_a(\bar{X}) \). We want to describe the number \( \pi_i(w) \). We have
Now assume $X = D_l$. Let $b_1, b_2, \ldots, b_l$ (resp. $b'_1, b'_2, \ldots, b'_l$) be the rearrangement of $|k_1|, |k_2|, \ldots, |k_l|$ (resp. $|k'_1|, |k'_2|, \ldots, |k'_l|$) in (weakly) decreasing order. Denote $c_0(w) = b_1$, $c'_0(w) = b'_1$, $c_1(w) = \sum_{i=2}^l b_i$ and $c'_1(w) = \sum_{i=2}^l b'_i$. Let $c(w) = \max\{0, c_0(w) - c_1(w)\}$ and $c'(w) = \max\{0, c'_0(w) - c'_1(w)\}$. Then $c(w), c'(w) \in 2\mathbb{Z}$. We have the following formula

$$\pi_1(w) = \frac{N_0(w) + c(w)}{2}$$

(5.9.1)

This can be shown by applying induction on $\ell(w) \geq 0$ and by the following facts:

1. $N_0(s_kw) = N_0(w), c(s_kw) = c(w)$ and $\pi_1(s_kw) = \pi_1(w)$ for $k \neq 1$;
2. If $\mathcal{L}(w) = \{s_1\}$, then $\pi_1(s_1w) = \pi_1(w) - 1$, and that either $N_0(s_1w) = N_0(w) - 2, c(s_1w) = c(w)$, or $N_0(s_1w) = N_0(w), c(s_1w) = c(w) - 2$ hold.

This, together with (5.7.2), implies

$$\pi_1(w) = \begin{cases} \frac{1}{2}N_0(w), & \text{if } c(w) = 0, \\ c_0(w), & \text{if } c(w) > 0. \end{cases}$$

or equivalently,

$$\pi_1(w) = \max\{\frac{1}{2}N_0(w), c_0(w)\}.$$  

(5.9.2)

Similarly, we have

$$\pi_0(w) = \max\{\frac{1}{2}N_1(w), c'_0(w)\}.$$  

(5.9.3)

This, together with (5.7.1) and (5.7.3), implies

$$\pi_1(w) - \frac{l}{2} \leq \pi_0(w) \leq \pi_1(w) + \frac{l}{2}.$$  

(5.9.4)

The formula (5.9.3) holds also in the case of $X = B_l$.

5.10. We have described the numbers $\pi_i(w)$ in the case where $X \in \Omega'$ and $i = 0, 1$. This number can be described in a much simpler way in all the remaining cases:

$$\pi_i(w) = \begin{cases} N'_i(w), & \text{if } X = A_{l-1}, 0 \leq i < l, \\ N'_{i-1}(w), & \text{if } X \in \Omega'_i, 1 < i \leq l, \end{cases}$$  

(5.10.1)

where $N'_i(w) = \#\{k \in \mathbb{Z} \mid k \leq i, (k)w > i\}$ for $i \in \mathbb{Z}$. This can be shown by applying induction on $\ell(w) \geq 0$.

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