A NEW ALGORITHM FOR FINDING AN L.C.R. SET IN CERTAIN TWO-SIDED CELLS

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Abstract. Let \((W, S)\) be an irreducible Weyl or affine Weyl group. In [18], we constructed an algorithm for finding a representative set of left cells (or an l.c.r. set for short) of \(W\) in a two-sided cell \(\Omega\). In the present paper, we introduce a new simpler algorithm for finding an l.c.r. set of \(W\) in \(\Omega\) when the subset \(F(\Omega)\) of \(\Omega\) is known (see Algorithm 3.11). Some technical tricks are introduced by some examples for applying the algorithm and for finding the set \(F(\Omega)\). The resulting set \(E(\Omega)\) of Algorithm 3.4 is useful in verifying a conjecture of Lusztig that any left cell in an affine Weyl group is left-connected.

Let \(W\) be an irreducible Weyl or affine Weyl group with \(S\) its Coxeter generator set. For a two-sided cell \(\Omega\) of \(W\) (in the sense of Kazhdan–Lusztig, see [7]), we introduced an algorithm for finding an l.c.r. set of \(W\) in \(\Omega\) in [18]. The algorithm has been efficiently applied in many cases (see [3], [4], [13], [18], [19], [22], [23], [26], [27], [28], [29], etc) since then. The algorithm consists of three processes \((A)\), \((B)\), \((C)\) on a distinguished set \(F\) (see 3.1), where process \((C)\) is the most difficult part among the three in which one need to find, for any given \(x \in F\), all elements \(y\) satisfying \(y - x, y < x, R(y) \not\subseteq R(x)\) and \(a(y) = a(x)\) (see 1.1. and 1.3. for the notation). It becomes extremely difficult as the length of \(x\) is getting larger.

For any two-sided cell \(\Omega\) of \(W\), let \(F(\Omega)\) be the set of all \(w \in \Omega\) such that \(a(sw), a(wt) < a(w)\) for any \(s \in L(w)\) and \(t \in R(w)\). We shall introduce a new

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algorithm for finding an l.c.r. set of $W$ in a two-sided cell $\Omega$, provided that the subset $F(\Omega)$ of $\Omega$ is known (see 3.2). The processes in our new algorithm amount to the mixture of processes (A) and (B) in the original algorithm, hence avoiding process (C).

Theorem 3.5 guarantees that our new algorithm will terminate after a finite number of steps, while Theorem 3.12 shows that the resulting set $E_0(\Omega)$ of Algorithm 3.11 forms an l.c.r. set of $W$ in $\Omega$ each element of $E_0(\Omega)$ is shortest in the left cell of $W$ containing it.

Our new algorithm has been applied successfully for the description of the left cells of $a$-values 4, 5, 6 in the affine Weyl groups $E_i$, $i = 6, 7, 8$ (see [6], [8], [24]).

It is desirable to find the subset $F(\Omega)$ explicitly for more two-sided cells $\Omega$ in an irreducible Weyl and affine Weyl group in order to apply our new algorithm.

It is relatively easier to describe the set $F(\Omega)$ when $F(\Omega)$ consists of elements of the form $w_J$ for some $J \subseteq S$, where the subgroup $W_J$ of $W$ generated by $J$ is finite and $w_J$ is the longest element in $W_J$ (see 4.6).

We can also find the sets $F(\Omega)$ for some two-sided cells $\Omega$ when $\Omega$ contains some elements not of the form $w_J$, $J \subseteq S$ (see 4.1 and 4.7–4.10).

Some technical tricks are needed in applying Algorithm 3.11. We explain by some examples the way to find the set $E_0(\Omega)$ from $F_0(\Omega)$, the set $F_0(\Omega)$ from $F(\Omega)$, the set $F(\Omega)$ from $F(W(i))$ (see 1.3 and 4.7) with $a(\Omega) = i$, and the set $F(W(i))$ from some known conditions (see Section 4).

Lusztig conjectured in [1] that any left cell of an affine Weyl group $W$ is left-connected. For a two-sided cell $\Omega$ of $W$, the resulting set $E(\Omega)$ of Algorithm 3.4 is useful in the verification of left-connectedness for a left cell of $W$ (see 3.6-3.7).

The contents of the paper are organized as follows. We collect some results on cells of affine Weyl groups $W$ in Section 1 and on the alcove form of elements of $W$ in Section 2. The most results stated in those sections are known already except for Proposition 2.3. Then we introduce a new algorithm for finding an l.c.r. set of
A new algorithm for finding an l.c.r. set

W in a two-sided cell Ω of W in Section 3. Finally, in Section 4, we explain some technical tricks in applying the algorithm.

§1. Some results on cells of affine Weyl groups.

In this section, we collect some known results on cells of affine Weyl groups.

1.1. Let $W = (W, S)$ be a Coxeter group with $S$ its Coxeter generator set. Let $\leq$ be the Bruhat-Chevalley order on $W$. For $w \in W$, we denote by $\ell(w)$ the length of $w$. Let $A = \mathbb{Z}[u, u^{-1}]$ be the ring of Laurent polynomials in an indeterminate $u$ with integer coefficients. Let $\mathcal{H}(W)$ be the associated Hecke algebra of $W$, i.e., an associative $A$-algebra which is free as an $A$-module with a basis $\{T_w \mid w \in W\}$, subject to the multiplication rule:

$$T_x T_y = T_{xy}, \quad \text{if } \ell(x) + \ell(y) = \ell(xy);$$

$$(T_s - u^{-1})(T_s + u) = 0 \quad \text{for any } s \in S.$$

$\mathcal{H}(W)$ has another $A$-basis $\{C_w \mid w \in W\}$ given by

$$C_w = \sum_{y \leq w} u^{\ell(w) - \ell(y)} P_{y,w}(u^{-2})T_y,$$

where the $P_{y,w} \in \mathbb{Z}[u]$ $(y, w \in W)$ are the celebrated Kazhdan-Lusztig polynomials satisfying: $P_{w,w} = 1$, $P_{y,w} = 0$ if $y \not< w$, and $\deg P_{y,w} \leq (1/2)(\ell(w) - \ell(y) - 1)$ if $y < w$ (see [7]). For $y < w$ in $W$, let $\mu(w, y) = \mu(y, w)$ be the coefficient of $u^{(1/2)(\ell(w) - \ell(y) - 1)}$ in $P_{y,w}$. We denote $y \rightleftharpoons w$ if $\mu(y, w) \neq 0$.

Checking the relation $y \rightleftharpoons w$ for $y, w \in W$ usually involves very complicated computation of Kazhdan-Lusztig polynomials. But it is easier in some special case:

$$\text{If } x, y \in W \text{ satisfy } y < x \text{ and } \ell(y) = \ell(x) - 1, \text{ then } y \rightleftharpoons x.$$

1.2. The preorders $\leq_L$, $\leq_R$, $\leq_{LR}$ and the associated equivalence relations $\sim_L$, $\sim_R$, $\sim_{LR}$ on $W$ are defined as in [7]. The equivalence classes of $W$ with respect to $\sim_L$
(respectively, $\sim^L$, $\sim^R$, $\sim^{LR}$) are called left cells (respectively, right cells, two-sided cells). The preorder $\leq_L$ (respectively, $\leq_R$, $\leq^{LR}$) induces a partial order on the set of left cells (respectively, right cells, two-sided cells) of $W$.

From now on, we always assume $W$ an irreducible Weyl or affine Weyl group unless otherwise specified.

1.3. In [9], Lusztig defined a function $a: W \rightarrow \mathbb{N}$ which satisfies the following properties:

(1) If $x \leq^L y$ then $a(x) \geq a(y)$. In particular, if $x \sim^L y$ then $a(x) = a(y)$. So we may define the $a$-value $a(\Gamma)$ to be $a(x)$ for any $x \in \Gamma$, where $\Gamma$ is a left, right or two-sided cell of $W$ (see [9]).

(2) If $a(x) = a(y)$ and $x \leq_R y$ (respectively, $x \leq^R y$) then $x \sim_R y$ (respectively, $x \sim^R y$) (see [10]).

(3) For any $I \subseteq S$ with $|W_I| < \infty$, we have $a(w_I) = \ell(w_I)$.

For any $w \in W$, set

$$L(w) = \{ s \in S \mid sw < w \} \quad \text{and} \quad R(w) = \{ s \in S \mid ws < w \}.$$  

(4) If $x \leq^L y$ (respectively, $x \leq^R y$), then $R(x) \supseteq R(y)$ (respectively, $L(x) \supseteq L(y)$). In particular, if $x \sim^L y$ (respectively, $x \sim^R y$), then $R(x) = R(y)$ (respectively, $L(x) = L(y)$) (see [7, Proposition 2.4]).

By the notation $x = y \cdot z$ ($x, y, z \in W$), we mean $x = yz$ and $\ell(x) = \ell(y) + \ell(z)$. In this case, we call $x$ a left (respectively, right) extension of $z$ (respectively, $y$), and call $z$ (respectively, $y$) a left (respectively, right) retraction of $x$. When $w = x \cdot y \cdot z$, we call $w$ an extension of $y$ and call $y$ a retraction of $w$.

(5) If $x = y \cdot z$ then $x \leq^L z$ and $x \leq^R y$. Hence $a(x) \geq a(y), a(z)$ by (2). In particular, if $I \in \{ R(x), L(x) \}$, then $a(x) \geq \ell(w_I)$ (see [9]).

Let $W_{(i)} = \{ w \in W \mid a(w) = i \}$ for any $i \in \mathbb{N}$. Then by (2), $W_{(i)}$ is a union of some two-sided cells of $W$.

(6) For any $x \in W$, let $\Sigma(x)$ be the set of all left cells $\Gamma$ of $W$ satisfying that there
exists some $y \in \Gamma$ with $y - x$, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(y) = a(x)$. Then $x \sim y$ in $W$ if and only if $\mathcal{R}(x) = \mathcal{R}(y)$ and $\Sigma(x) = \Sigma(y)$ (see [18, Theorem 2.1], [21] and [22, Section 5]).

(7) If $x - y$ in $W$ and $s \in S$ satisfy $s \in \mathcal{L}(y) \setminus \mathcal{L}(x)$ (respectively, $s \in \mathcal{R}(y) \setminus \mathcal{R}(x)$) then either $y = s \cdot x$ (respectively, $y = x \cdot s$) or $y < x$ (see [7, Subsections 2.3e and 2.3f]). In particular, we have $\ell(y) \leq \ell(x) + 1$.

(8) The number of left cells in $W$ is finite (see [10, Theorem 2.2]).

(9) If a left cell $L$ and a right cell $R$ are in the same two-sided cell of $W$, then $L \cap R \neq \emptyset$ (see [11, Subsection 3.1 (k), (l)]).

1.4. To each $x \in W$, we denote by $M(x)$ the set of all $y \in W$ such that there is a sequence $x_0 = x, x_1, \ldots, x_r = y$ in $W$ with some $r \geq 0$, where for every $1 \leq i \leq r$, the conditions $x_{i-1}^{-1}x_i \in S$ and $\mathcal{R}(x_{i-1}) \not\subseteq \mathcal{R}(x_i)$ are satisfied.

A graph $\mathcal{M}(x)$ associated to an element $x \in W$ is defined as follows. Its vertex set is $M(x)$, each $y \in M(x)$ is labelled by the set $\mathcal{R}(y)$; its edge set consists of all two-elements subsets $\{y, z\} \subseteq M(x)$ with $y^{-1}z \in S$ and $\mathcal{R}(y) \not\subseteq \mathcal{R}(z)$.

By a path in the graph $\mathcal{M}(x)$, we mean a sequence $z_0, z_1, \ldots, z_r$ in $M(x)$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathcal{M}(x)$ for any $1 \leq i \leq r$. Two elements $x, x' \in W$ have the same right generalized $\tau$-invariants, if for any path $z_0 = x, z_1, \ldots, z_r$ in $\mathcal{M}(x)$, there is a path $z'_0 = x', z'_1, \ldots, z'_r$ in $\mathcal{M}(x')$ with $\mathcal{R}(z'_i) = \mathcal{R}(z_i)$ for any $0 \leq i \leq r$, and if the same condition holds when the roles of $x$ and $x'$ are interchanged.

Then the following result is known.

**Proposition 1.5.** (see [17, Section 3]) Any $x, y \in W$ with $x \sim_L y$ have the same right generalized $\tau$-invariants.

1.6. For $s, t \in S$ with $m = o(st) > 2$, each of the sequences

$$zt, zts, ztst, \ldots \quad \text{and} \quad zs, zst, zsts, \ldots$$

(each contains $m - 1$ terms)

is called a right $\{s, t\}$-strings, where $z \in W$ satisfies $\mathcal{R}(z) \cap \{s, t\} = \emptyset$. 
Two elements \(x, y \in W\) form a right primitive pair, if there exist two sequences \(x_0 = x, x_1, \ldots, x_r\) and \(y_0 = y, y_1, \ldots, y_r\) in \(W\) satisfying that

(a) For each \(1 \leq i \leq r\), there exist some \(s_i, t_i \in S\) with \(o(s_i t_i) > 2\) such that any of the pairs \(x_{i-1}, x_i\) and \(y_{i-1}, y_i\) are neighboring terms in a right \(\{s_i, t_i\}\)-string.

(b) \(x_i - y_i\) for all \(0 \leq i \leq r\).

(c) Either \(R(x) \not\subseteq R(y)\) and \(R(y_r) \not\subseteq R(x_r)\), or \(R(y) \not\subseteq R(x)\) and \(R(x_r) \not\subseteq R(y_r)\).

In particular, if \(\{w, y\}\) is an edge in a graph \(\mathcal{M}(x)\) for some \(x \in W\) then \(w, y\) form a right primitive pair by taking \(r = 0\) in the above definition.

Similarly, we can define a left \(\{s, t\}\) string and a left primitive pair.

**Proposition 1.7.** (see [17, Section 3]) Any right (respectively, left) primitive pair \(x, y \in W\) satisfy \(x \sim_R y\) (respectively, \(x \sim_L y\)). In particular, the set \(M(x)\) is contained in a right cell of \(W\).

**1.8.** An affine Weyl group \(W\) is a Coxeter group which can be realized geometrically as follows. Let \(G\) be a connected, adjoint reductive algebraic group over \(\mathbb{C}\). Fix a maximal torus \(T\) of \(G\). Let \(X\) be the group of characters \(T \rightarrow \mathbb{C}\) and let \(\Phi \subset X\) be the root system with \(\Delta = \{\alpha_1, \ldots, \alpha_\ell\}\) a choice of simple system. Then \(E = X \otimes_{\mathbb{Z}} \mathbb{R}\) is a euclidean space with an inner product \(\langle , \rangle\) such that the Weyl group \((W_0, S_0)\) of \(G\) with respect to \(T\) acts naturally on \(E\) and preserves its inner product, where \(S_0\) is the set of simple reflections \(s_i = s_{\alpha_i}, 1 \leq i \leq \ell\). We denote by \(N\) the group of all translations \(T_\lambda\) \((\lambda \in X)\) on \(E\): \(T_\lambda\) sends \(x\) to \(x + \lambda\). Then the semidirect product \(W = W_0 \ltimes N\) is called an affine Weyl group. Let \(\widehat{K}\) be the dual of the type of \(G\). Then we define the type of \(W\) by \(\widehat{K}\). Sometimes we denote \(W\) just by \(\widehat{K}\). There is a canonical homomorphism from \(W\) to \(W_0\): \(w \mapsto \bar{w}\).

Let \(-\alpha_0\) be the highest short root in \(\Phi\). Set \(s_0 = s_{\alpha_0} T_{-\alpha_0}\) with \(s_{\alpha_0}\) the reflection corresponding to \(\alpha_0\). Then \(S = S_0 \cup \{s_0\}\) forms a Coxeter generator set of \(W\).

**Theorem 1.9.** ([12, Theorem 4.8]) In the setup of 1.8, there exists a bijection
A new algorithm for finding an l.c.r. set

$\mathbf{c}: \mathbf{u} \mapsto \mathbf{c(u)}$ from the set of unipotent conjugacy classes in $G$ to the set $\text{Cell}(W)$ of two-sided cells in $W$ such that $\alpha(\mathbf{c(u)}) = \dim \mathcal{B}_u$, where $u$ is any element in $\mathbf{u}$, and $\dim \mathcal{B}_u$ is the dimension of the variety of Borel subgroups of $G$ containing $u$.

§2. Alcove forms for elements in affine Weyl groups.

Keep the setup of 1.8 for an affine Weyl group $W$.

2.1. The alcove form of an element $w \in W$ is, by definition, a $\Phi$-tuple $(k(w; \alpha))_{\alpha \in \Phi}$ over $\mathbb{Z}$, subject to the following conditions:

(a) $k(e; \alpha) = 0$ for any $\alpha \in \Phi$, where $e$ is the identity of $W$;

(b) For $0 \leq i \leq l$, we have $k(s_i; \alpha) = 0$ if $\alpha \neq \pm \alpha_i$; $k(s_i; \alpha) = \mp 1$ if $\alpha = \pm \alpha_i$.

(c) Let $w' = ws_i$ for some $0 \leq i \leq l$. Then $k(w'; \alpha) = k(w; (\alpha\bar{s}_i)) + k(s_i; \alpha)$, where $\bar{s}_i = s_i$ if $1 \leq i \leq l$, and $\bar{s}_0 = s_{\alpha_0}$ (see [15, Proposition 4.2]).

It is easily checked that $k(w; -\alpha) = -k(w; \alpha)$ for any $\alpha \in \Phi$. Let $\Phi^+$ be the positive system of $\Phi$ containing $\Pi$. Then the $\Phi$-tuple $(k(w; \alpha))_{\alpha \in \Phi}$ is entirely determined by the $\Phi^+$-tuple $(k(w; \alpha))_{\alpha \in \Phi^+}$. We can identify $(k(w; \alpha))_{\alpha \in \Phi}$ with $(k(w; \alpha))_{\alpha \in \Phi^+}$ and call the latter also the alcove form of $w$.

Recall the definition for a left extension of an element $x \in W$ in 1.3. The following results on the alcove form $(k(w; \alpha))_{\alpha \in \Phi}$ of $w \in W$ are known.

**Proposition 2.2.** (see [15, Proposition 4.7]) Let $w = (k(w; \alpha))_{\alpha \in \Phi} \in W$. Write $w = \bar{w}T_\lambda$ with $\bar{w} \in W_0$ and $\lambda \in Q$.

(a) For $\alpha \in \Phi^+$, we have $k(\bar{w}; \alpha) = 0$ if $(\alpha)\bar{w}^{-1} \in \Phi^+$ and $k(\bar{w}; \alpha) = -1$ if $(\alpha)\bar{w}^{-1} \in \Phi^-.$

(b) $\mathcal{R}(w) = \{s_j \in S \mid k(w; \alpha_j) < 0\}$.

(c) Let $w' = ws_j$ with $w \in W$ and $0 \leq j \leq l$. Then for any $\alpha \in \Phi$, we have

$$k(w'; \alpha) = k(w; (\alpha)s_j) + k(s_j; \alpha).$$

(d) Let $w' = (k(w'; \alpha))_{\alpha \in \Phi} \in W$. Then $w'$ is a left extension of $w$ if and only if the inequalities $k(w'; \alpha)k(w; \alpha) \geq 0$ and $|k(w'; \alpha)| \geq |k(w; \alpha)|$ hold for any $\alpha \in \Phi.$
The following result is crucial in the proof of Theorem 3.5.

**Proposition 2.3.** Let \( x_0, x_1, \ldots, x_r, \ldots \) be an infinite sequence of elements in \( W \) such that \( x_i \) is a right extension of \( x_{i-1} \) for every \( i \geq 1 \). Then there are some \( q > p \geq 0 \) such that \( x_q \) is a left extension of \( x_p \).

**Proof.** By Proposition 2.2 (a)–(c), we see that there are permutations \( \tau_{ij}, i, j > 0 \), on the set \( \Phi \) satisfying:

(i) \( (-\alpha)\tau_{ij} = -(\alpha)\tau_{ij} \) for any \( \alpha \in \Phi \);
(ii) \( |k(x_j; \alpha)| \geq |k(x_i; (\alpha)\tau_{ij})| \) for any \( \alpha \in \Phi \) and \( j > i \);
(iii) \( \tau_{hi}\tau_{ij} = \tau_{hj} \) for any \( h, i, j > 0 \).

Since \( |\Phi| < \infty \), the permutation group on \( \Phi \) is finite. So there exists an infinite subsequence \( h_1, h_2, \ldots, h_t, \ldots \) of \( 1, 2, 3, \ldots \) with \( \tau_{0,h_a} = \tau_{0,h_b} \) for any \( a, b \geq 0 \). Hence \( |k(x_{h_a}; \alpha)| \geq |k(x_{h_b}; \alpha)| \) for any \( a > b \geq 0 \) and \( \alpha \in \Phi \). Then by the finiteness of the set \( \Phi \), there exist some \( q > p \geq 0 \) in \( \{h_1, h_2, \ldots\} \) such that \( |k(x_q; \alpha)| \geq |k(x_p; \alpha)| \) and \( k(x_q; \alpha) \cdot k(x_p; \alpha) \geq 0 \) for any \( \alpha \in \Phi \). This implies that \( x_q \) is a left extension of \( x_p \) by Proposition 2.2 (d). \( \square \)

§3. A new algorithm for finding an l.c.r. set in a two-sided cell.

3.1. Call a non-empty set \( F \subseteq W \) distinguished if \( |\Gamma \cap F| \leq 1 \) for any left cell \( \Gamma \) of \( W \). Call \( F \) a representative set of left cells (or an l.c.r. set for short) of \( W \) in a two-sided cell \( \Omega \) if \( F \subseteq \Omega \) and \( |\Gamma \cap F| = 1 \) for any left cell \( \Gamma \) of \( W \) in \( \Omega \).

3.2. For any two-sided cell \( \Omega \) of \( W \), set

\[
F(\Omega) = \{ z \in \Omega \mid a(sz), a(zt) < a(z) \text{ for any } s \in L(z) \text{ and } t \in R(z) \},
\]

\[
E(\Omega) = \{ z \in \Omega \mid a(sz) < a(z) \text{ for any } s \in L(z) \}.
\]

Clearly, \( F(\Omega) \subseteq E(\Omega) \). Also, \( w \in F(\Omega) \) if and only if \( w^{-1} \in F(\Omega) \). We have the following result.

**Lemma 3.3.** (1) Any \( w \in \Omega \) has an expression \( w = x \cdot z \cdot y \) for some \( x, y \in W \) and \( z \in F(\Omega) \).
(2) An element \( w \in \Omega \) is in \( E(\Omega) \) if and only if \( x = 1 \) in any expression of the form \( w = x \cdot z \cdot y \) with \( z \in F(\Omega) \).

**Proof.** If \( w \in F(\Omega) \), then take \( x = y = 1 \) and \( z = w \). If \( w \notin F(\Omega) \), then by 1.3 (2), either \( w = s \cdot w' \) for some \( s \in L(w) \), or \( w = w' \cdot t \) for some \( t \in R(w) \), where \( w' \in \Omega \). By applying induction on \( \ell(w) \), we may write \( w' = x \cdot z \cdot y \) for some \( x, y \in W \) and some \( z \in F(\Omega) \). Hence \( w \) is equal to either \( sx \cdot z \cdot y \) or \( x \cdot z \cdot yt \). This implies (1).

Then (2) follows by (1) and 1.3 (2), (5). \( \Box \)

The following algorithm is for finding the set \( E(\Omega) \) from \( F(\Omega) \).

**Algorithm 3.4.**

(3.4.1) Set \( Y_0 = F(\Omega) \);

Let \( k \geq 0 \). Suppose that the set \( Y_k \) has been found.

(3.4.2) If \( Y_k = \emptyset \), then the algorithm terminates;

(3.4.3) If \( Y_k \neq \emptyset \), then find the set \( Y_{k+1} = \{ xs \mid x \in Y_k; s \in S \setminus R(x); xs \in E(\Omega) \} \).

By Lemma 3.3 (2), we have \( E(\Omega) = \bigcup_{i \geq 0} Y_i \).

Then the following result shows that Algorithm 3.4 must terminate after a finite number of steps, that is, \( E(\Omega) = \bigcup_{k=0}^t Y_k \) for some \( t \in \mathbb{N} \).

**Theorem 3.5.** Let \( Y_j, j \geq 0 \), be obtained from the set \( F(\Omega) \) by Algorithm 3.4.

1. There exists some \( t \in \mathbb{N} \) such that \( Y_j = \emptyset \) and \( Y_h = \emptyset \) for \( 0 \leq j \leq t < h \);

2. \( E(\Omega) = \bigcup_{k=0}^t Y_k \).

**Proof.** It is easily seen that if \( Y_i = \emptyset \) for some \( i \geq 1 \) then \( Y_j = \emptyset \) for any \( j \geq i \), or equivalently, if \( Y_i = \emptyset \) for some \( i \geq 0 \) then \( Y_j = \emptyset \) for any \( 0 \leq j \leq i \). Since \( Y_0 \neq \emptyset \), to prove (1), it is enough to prove the existence of an integer \( i > 0 \) with \( Y_i = \emptyset \).

We argue by contrary. Suppose \( Y_i = \emptyset \) for any \( i \geq 0 \). By the finiteness for the number of left cells of \( W \) in \( \Omega \) (see 1.3 (8)), there are infinite sequences \( w_1, w_2, \ldots \) in \( E(\Omega) \) and \( 0 \leq i_1 < i_2 < \cdots < i_n < \cdots \) in \( \mathbb{N} \) satisfying that

(i) \( w_{j+1} \) is a right extension of \( w_j \) for any \( j \geq 1 \) (see 1.3) ;

(ii) \( w_1 \overset{L}{\sim} w_2 \overset{L}{\sim} w_3 \overset{L}{\sim} \cdots \);
(iii) \( w_j \in Y_{i_j} \) for \( j \geq 1 \).

By (i) and Proposition 2.2 (b)–(c), we see that there are permutations \( \tau_{ij}, \ i, j \geq 1, \) on \( \Phi \) such that
\[
|k(w_j; \alpha)| \geq |k(w_i; (\alpha)\tau_{ij})|
\]
for any \( \alpha \in \Phi, \ j > i \geq 1, \) and that \( \tau_{hi} \tau_{ij} = \tau_{hj} \) for any \( h, i, j \geq 1 \) (see the proof of Proposition 2.3).

Then by Propositions 2.3, there are some \( q > p \geq 1 \) such that \( w_q \) is a proper left extension of \( w_p \). Since \( w_p \sim w_q \) are in \( \Omega \), this implies that \( w_q \) is not in \( E(\Omega) \), a contradiction. This proves (1). Then (2) follows by (1) and the definition of the set \( E(\Omega) \).

\[\square\]

**Remark 3.6.** We say that a subset \( K \) of \( W \) is left-connected, if for any \( x, y \in K \), there exists a sequence of elements \( x_0 = x, x_1, ..., x_r = y \) in \( K \) with some \( r \geq 0 \) such that \( x_{i-1} x_i^{-1} \in S \) for every \( 1 \leq i \leq r \). Lusztig conjectured in [1] that any left cell of an affine Weyl group is left-connected. Although it has been verified in many special cases (see [14], [16], [25], [26]), the conjecture is still open in general. Now the resulting set \( E(\Omega) \) of Algorithm 3.4 is useful in dealing with the conjecture. In fact, to verify the left-connectedness for a left cell \( \Gamma \) of \( W \) in a two-sided cell \( \Omega \), we need only to construct a graph \( M(\Gamma) \) with \( \Gamma \setminus E(\Omega) \) as its vertex set. We join two vertices \( x \neq y \) in \( \Gamma \cap E(\Omega) \) with an edge once we find a sequence of elements \( x_0 = x, x_1, ..., x_r = y \) in \( \Gamma \) with some \( r > 0 \) such that \( x_{i-1} x_i^{-1} \in S \) for any \( 1 \leq i \leq r \). Then we complete the proof for \( \Gamma \) being left-connected once the graph \( M(\Gamma) \) we are constructing becomes connected.

**Example 3.7.** The following example is provided by Q. Huang, one of my Ph. D. students. Let \( W = \tilde{E}_8 \) be with \( S = \{s_i \mid 0 \leq i \leq 8\} \) its distinguished generator set such that \( o(s_1 s_3) = o(s_3 s_4) = o(s_2 s_4) = o(s_4 s_5) = o(s_5 s_6) = o(s_6 s_7) = o(s_7 s_8) = o(s_8 s_0) = 3 \). Let \( \Omega \) be the two-sided cell of \( W \) containing the element \( s_2 s_3 s_4 s_2 s_3 s_4 \). Then we can get the set \( E(\Omega) \) by Algorithm 3.4. We observe that the elements \( w_1 = s_2 s_3 s_4 s_2 s_3 s_4 s_1 s_5 s_4 s_6 \) and \( w_2 = s_1 s_4 s_3 s_1 s_4 s_3 \cdot s_2 s_4 s_5 s_4 s_6 \) and \( w_3 = s_3 s_5 s_4 s_3 s_5 s_4 \cdot s_1 s_2 s_3 s_4 s_5 s_6 \) in \( E(\Omega) \) have the same right generalized \( \tau \)-invariants among themselves and have different right generalized \( \tau \)-invariants from
any other element in $E(\Omega)$. On the other hand, the element $y := s_2 \cdot w_2 = s_3 s_1 \cdot w_1$ is a common left extension of both $w_1$ and $w_2$. Since $L(y) = \{s_1, s_2, s_3\}$ and $L(w_2) = \{s_1, s_3, s_4\}$ and $L(s_1 w_1) = \{s_1, s_2, s_4\}$ and $L(w_1) = \{s_2, s_3, s_4\}$, the sequence $w_1, s_1 w_1, y, w_2$ is contained in some left cell (say $\Gamma$) of $W$ by Proposition 1.7. Also, the element $x := s_4 s_2 s_1 \cdot w_3 = s_2 s_4 s_5 \cdot y$ is a common left extension of both $w_2$ and $w_3$. Since $L(w_3) = \{s_3, s_4, s_5\}$ and $L(s_1 w_3) = \{s_1, s_4, s_5\}$ and $L(s_2 s_1 w_3) = \{s_1, s_2, s_5\}$ and $L(x) = \{s_1, s_2, s_4\}$ and $L(s_5 y) = \{s_1, s_2, s_3, s_5\}$ and $L(s_4 s_5 y) = \{s_1, s_4\}$, we see that $y, s_5 y$ form a left primitive pair and so the sequence $w_2, y, s_5 y, s_4 s_5 y, x, s_2 s_1 w_3, s_1 w_3, w_3$ is contained in $\Gamma$ by Proposition 1.7. So $\Gamma \cap E(\Omega) = \{w_1, w_2, w_3\}$ by Proposition 1.5. The graph $\mathcal{M}(\Gamma)$ with the vertex set $\{w_1, w_2, w_3\}$ has the edges $\{w_1, w_2\}$ and $\{w_2, w_3\}$, hence it is connected. This implies by 3.6 that $\Gamma$ is left-connected.

3.8. In the remaining part of the section, we always assume that the set $F(\Omega)$ is known explicitly for a given two-sided cell $\Omega$ of $W$. For any $x \in W$, denote by $\Gamma_x$ the left cell of $W$ containing $x$. Take a distinguished subset $F_0(\Omega)$ of $F(\Omega)$ such that any $w \in F_0(\Omega)$ is a shortest element in the left cell $\Gamma_w$ and that for any $w \in F(\Omega)$, there is some $w' \in F_0(\Omega)$ and some $x \in W$ with $w' \cdot x$ a shortest element in the left cell $\Gamma_w$. In particular, when all the elements of $F(\Omega)$ have the same length (hence each $w \in F(\Omega)$ is a shortest element in the left cell $\Gamma_w$, see 4.6–4.7), we can take $F_0(\Omega)$ to be any maximal distinguished subset of $F(\Omega)$.

**Lemma 3.9.** Assume that the set $F_0(\Omega)$ has been chosen for a two-sided cell $\Omega$ of $W$. Then any left cell $\Gamma$ in $\Omega$ contains a shortest element $w$ which has an expression of the form $w = z \cdot y$ for some $z \in F_0(\Omega)$ and $y \in W$.

**Proof.** Let $\Gamma, \Gamma'$ be two left cells of $W$ in $\Omega$ and let $x \in \Gamma$. We see by 1.3 (9) that there exists a sequence $x_0 = x, x_1, ..., x_r$ in $\Omega$ with some $r \geq 0$ such that $x_r \in \Gamma'$ and that $x_{i-1} \rightarrow x_i$, $\mathcal{R}(x_{i-1}) \nsubseteq \mathcal{R}(x_i)$ for every $1 \leq i \leq r$. We claim that for any $x' \in \Gamma$, there exists a sequence $x'_0 = x', x'_1, ..., x'_r$ such that $x'_{i-1} \rightarrow x'_i$
and \( x_i' \sim x_i \) for every \( 1 \leq i \leq r \). To show the claim, it is enough to consider the case where \( r = 1 \) and \( x' \sim x \) with \( \mathcal{L}(x') \nsubseteq \mathcal{L}(x) \). Take \( s \in \mathcal{R}(x_1) \setminus \mathcal{R}(x) \) and \( t \in \mathcal{L}(x) \setminus \mathcal{L}(x') \). Then we have \( a_{x_1} \neq b_x \) in the expressions \( C_x C_s = \sum z a_z C_z \) and \( C_t C_{x'} = \sum y b_y C_y \), where \( a_z, b_y \in A \). By the positivity of the coefficients of \( a_z, b_y \) in \( u \) (see [9, Subsection 3.1]), we see that \( c_{x_1} \neq 0 \) in the expression \( C_t C_{x'} C_s = \sum_v c_v C_v \) (\( c_v \in A \)). By the multiplication associativity of \( \mathcal{H} \), this implies that there exists some \( x'_1 \in W \) with \( d_{x'_1} \neq f_{x_1} \) in the expressions \( C_{x'} C_s = \sum_y d_y' C_y' \) and \( C_t C_{x'_1} = \sum_{v'} f_{v'} C_{v'} \), where \( d_{y'}, f_{v'} \in A \). By 1.3 (1), we get \( a(x') = a(x) = a(x_1) \geq a(x'_1) \geq a(x') \) and hence \( a(x'_1) = a(x_1) \). So \( x'_1 \sim x_1 \) by 1.3 (2) and the fact \( x_1 \leq x'_1 \). The claim is proved.

Now we are ready to show our result. Take a shortest element \( w' \) in \( \Gamma \). Then \( w' \in E(\Omega) \). There exist some \( z \in F(\Omega) \) and \( y \in W \) with \( w' = z \cdot y \) by Lemma 3.3 (2). By the construction of the set \( F_0(\Omega) \), there exist some \( z' \in F_0(\Omega) \) and \( y' \in W \) such that \( z' \cdot y' \sim z \) and \( \ell(z' \cdot y') \leq \ell(z) \). Let \( y = s_1 s_2 \cdots s_r \) be a reduced expression of \( y \) with \( s_i \in S \) and let \( z_i = z s_1 s_2 \cdots s_i \) for any \( 0 \leq i \leq r \). Then the sequence \( z_0 = z, z_1, \ldots, z_r = w' \) is in \( \Omega \) with \( z_{i-1} \sim z_i \) and \( s_i \in \mathcal{R}(z_i) \setminus \mathcal{R}(z_{i-1}) \). By the above claim, there exists some sequence \( x_0 = z' \cdot y', x_1, \ldots, x_r \) in \( \Omega \) such that \( x_{i-1} \sim x_i \) and \( x_i \sim z_i \) for any \( 1 \leq i \leq r \). By 1.3 (7), we have \( \ell(x_r) \leq \ell(x_0) + r \leq \ell(z) + r = \ell(z \cdot y) = \ell(w') \). Since \( w' \) is shortest in \( \Gamma \) and \( w' \sim x_r \), this forces \( \ell(x_r) = \ell(w') \), hence \( \ell(z' \cdot y') = \ell(z) \) and \( x_i = x_{i-1} \cdot s_i \) for any \( 1 \leq i \leq r \), in particular, \( x_r = z' \cdot y' \cdot y \), which is a required element \( w \) in the lemma. □

3.10. For any left cell \( \Gamma \) in a two-sided cell \( \Omega \) of \( W \), let \( n(\Gamma) \) be the length of a shortest element in \( \Gamma \). Then \( n(\Gamma) \) is also the smallest number of \( \ell(z \cdot y) \) as \( z \cdot y \) ranges over all such expressions that \( z \in F_0(\Omega) \) and \( y \in W \) and \( z \cdot y \in \Gamma \) by Lemma 3.9. Let \( n(\Omega) \) be the length of a shortest element in \( \Omega \).

By modifying Algorithm 3.4, we get the following algorithm such that the resulting set forms an l.c.r. set of \( W \) in \( \Omega \) (see Theorem 3.12):

Algorithm 3.11.
3.11.1 Let \( X_0 = \{ w \in F_0(\Omega) \mid \ell(w) = n(\Omega) \} \).

For \( k \geq 0 \), suppose that the set \( X_k \) has been found.

3.11.2 If \( X_k = \emptyset \), then our algorithm terminates;

3.11.3 If \( X_k \neq \emptyset \), then find the set

\[
X'_{k+1} = \{ xs \mid x \in X_k, s \in S \setminus R(x), xs \in E(\Omega) \} \cup \{ w \in F_0(\Omega) \mid \ell(w) = n(\Omega) + k + 1 \}
\]

and then take a maximal subset \( X_{k+1} \) in \( X'_{k+1} \) such that \( \bigcup_{i=0}^{k+1} X_i \) is distinguished whenever \( X'_{k+1} \neq \emptyset \).

**Theorem 3.12.** Let \( E_0(\Omega) := \bigcup_{k \geq 0} X_k \).

1. \( E_0(\Omega) \subseteq E(\Omega) \);
2. The set \( E_0(\Omega) \) forms an l.c.r. set of \( W \) in \( \Omega \);
3. Any \( w \in E_0(\Omega) \) satisfies \( \ell(w) = n(\Gamma) \).

**Proof.** Assertion (1) and the distinguishedness of \( E_0(\Omega) \) follows by the construction of the set \( E_0(\Omega) \). So for the assertions (2)–(3), it is enough to prove that \( \Gamma \cap E_0(\Omega) \) contains an element \( w \) with \( \ell(w) = n(\Gamma) \) for any left cell \( \Gamma \) of \( W \) in \( \Omega \).

By Lemma 3.9, there exists some \( w' \in \Gamma \) with \( \ell(w') = n(\Gamma) \) (hence \( w' \in E(\Omega) \)) and \( w' = x \cdot y \) for some \( x \in F_0(\Omega) \) and \( y \in W \). We want to find some \( w \in E_0(\Omega) \cap \Gamma \) with \( \ell(w) = n(\Gamma) \). Applying induction on \( n(\Gamma) \geq n(\Omega) \) (see 3.10). If \( n(\Gamma) = n(\Omega) \) then there exists some \( w \in X_0 \cap \Gamma \subseteq E_0 \cap \Gamma \) by the construction of the set \( F_0(\Omega) \) and Algorithm 3.11. Now assume \( n(\Gamma) > n(\Omega) \). Let \( k = n(\Gamma) - n(\Omega) \). If \( w' \in F_0(\Omega) \) then we can find some \( w \in X_k \cap \Gamma \subseteq E_0 \cap \Gamma \) by Algorithm 3.11. If \( w' = x \cdot y \notin F_0(\Omega) \), i.e., \( \ell(y) > 0 \), take any \( s \in R(y) \), then \( z := w' s \in E(\Omega) \).

We claim that \( z \) is a shortest element in the left cell \( \Gamma_z \). For, otherwise, there would exist some \( z' \in \Gamma_z \) with \( \ell(z') < \ell(z) \). By 1.3 (6), there is some \( w'' \in \Gamma \) with \( w'' \sim z' \) by the facts that \( w' \sim z \) (by (1.1.2)) and \( z \sim z' \) and \( R(w') \notin R(z) \). Since \( s \in R(w') \setminus R(z) \), we have \( s \in R(w'') \setminus R(z') \) by 1.3 (4). Hence \( \ell(w'') \leq \ell(z') + 1 \leq \ell(z) < \ell(w') \) by 1.3 (7), contradicting the assumption of \( \ell(w') = n(\Gamma) \). The claim is proved.

Since \( \ell(z) < \ell(w') \), we have \( n(\Gamma_z) < n(\Gamma) \). By inductive hypothesis, there
exists some \( z_0 \in E_0(\Omega) \cap \Gamma_z \) with \( \ell(z_0) = n(\Gamma_z) = \ell(z) \). By the same argument as above with \( z_0 \) in the place of \( z' \), there exists some \( w_0 \in \Gamma \) with \( w_0 - z_0 \) and \( s \in R(w_0) \setminus R(z_0) \) and \( \ell(w) \leq \ell(z_0) + 1 = \ell(z) + 1 = \ell(w') \). By the assumption of \( \ell(w') = n(\Gamma) \), we have \( \ell(w_0) = \ell(z_0) + 1 \) and \( n(\Gamma) = n(\Gamma_z) + 1 \). Hence \( w_0 = z_0 \cdot s \in X'_k \) by 1.3 (7). By the construction of the set \( X_k \) in Algorithm 3.11 and the fact \( n(\Gamma) = \ell(w_0) \), there must exist some element in the set \( X_k \cap \Gamma \) (hence in \( E_0(\Omega) \cap \Gamma \)). So our result follows by induction. □

Remark 3.13. (1) By Theorem 3.5, there is some \( t_0 \leq t \) with \( E_0(\Omega) = \bigcup_{k=0}^{t_0} X_k \), where \( t \) is given as in Theorem 3.5 (1).

(2) In the case where all the elements in \( F_0(\Omega) \) have the same length, we can take \( X_0 = F_0(\Omega) \). This is so for the most cases we have dealt with in applying Algorithm 3.11 so far in practice.

§4. Some applications of Algorithm 3.11.

In this section, we shall discuss some technical tricks in applying Algorithm 3.11.

Let us first observe an example.

Example 4.1. Let \( W = \tilde{C}_4 \) be with \( S = \{s_0, s_1, s_2, s_3, s_4\} \) its Coxeter generator set, where \( o(s_0s_1) = o(s_3s_4) = 4 \) and \( o(s_1s_2) = o(s_2s_3) = 3 \). In the subsequent discussion, we denote \( s_i \) by \( i, 0 \leq i \leq 4 \), for simplifying the notation.

The set \( W_{(5)} \) is a single two-sided cell of \( W \) by Theorem 1.9. Let \( x_1 = 01013 \), \( x_2 = 01014 \), \( x_3 = 1210124 \), \( y_1 = 34341 \), \( y_2 = 34340 \), \( y_3 = 3234320 \). Then \( F(W_{(5)}) = \{x_i, y_j \mid 1 \leq i, j \leq 3\} \). Since \( x_3 \sim x_221 \) and \( y_3 \sim y_223 \) with \( \ell(x_3) = \ell(x_221) \) and \( \ell(y_3) = \ell(y_223) \), we can take \( F_0(W_{(5)}) = \{x_1, x_2, y_1, y_2\} \) by 3.8.

By applying Algorithm 3.11, we get \( X_0 = F_0(W_{(5)}) \).

\[
X_1 = X'_1 = \{x_12, x_22, y_12, y_22\}.
\]

\[
X_2 = X'_2 = \{x_121, x_123, x_221, x_223, y_123, y_121, y_223, y_221\}.
\]

\[
X'_3 = \{x_1210, x_1213, x_1234, x_2210, x_2213, x_2234, y_1234, y_1231, y_1210, y_2234, y_2231, y_2210\}
\]

and \( X_3 = \{x_1210, x_1213, x_1234, x_2213, x_2234, y_1234, y_1231, y_1210, y_2231, y_2210\} \) since
A new algorithm for finding an l.c.r. set

15

\[ x_2210 \sim x_22 \text{ and } y_2234 \sim y_22. \]

\[ X_4' = \{ x_12101, x_12310, x_12134, x_122134, x_221345, x_221134, x_22134, x_12134, x_122134, y_12343, y_122134, y_22310, y_12101, y_22341, y_223212, y_223210\} \text{ and } X_4 = \{ x_12101, x_12310, x_12134, x_122134, x_22134, x_22134, y_12343, y_122134, y_12310, y_12101, y_22312, y_223210\} \text{ since } x_22103 \sim x_223 \text{ and } y_22341 \sim y_2211.

\[ X_5' = \{ x_123104, x_121343, x_221032, x_221034, x_221324, y_121340, y_123101, y_223412, y_223410, y_223120\} \text{ and } X_5 = \{ x_123104, x_121343, x_221034, x_221324, y_121340, y_123101, y_223410, y_223120\} \text{ since } x_221032 \sim x_2213 \text{ and } y_223412 \sim y_1211.

\[ X_6' = \{ x_1231043, x_1213432, x_2210324, x_2213243, y_1213401, y_1231012, y_2234120, y_2231201\} \text{ and } X_6 = \{ x_1231043, x_1213432, x_2213243, y_1213401, y_1231012, y_2231201\} \text{ since } x_2210324 \sim x_22134 \text{ and } y_2234120 \sim y_1211.

\[ X_7' = \{ x_12310432, x_22132434, y_12134012, y_22312010\} \text{ and } X_7 = \{ x_12310432, y_12134012\} \text{ since } x_22132434 \sim y_12343 \text{ and } y_22312010 \sim x_1211.

\[ X_8 = X_8' = \{ x_123104321, y_121340123\} \text{ and } X_9 = X_9' = \{ x_1231043210, y_1213401234\}.

Since \( X_{10} = X_{10'} = \emptyset \), we see by Theorem 3.12 that \( E_0 = \bigcup_{i=0}^{9} X_i \) forms an l.c.r. set of \( W \) in \( W_{(5)} \) with \(|X| = 56\).

4.2. The most technical part in applying Algorithm 3.11 is to determine whether or not the element \( xs \) is in \( E(\Omega) \) for any given \( x \in X_k \) and \( s \in S \setminus R(x) \). That is, to check the equation \( a(xs) = a(x) \) and the inequality \( a(rxs) < a(xs) \) for any \( r \in L(x) \).

4.3. Checking the equation \( a(xs) = a(x) \) amounts to determining the value \( a(xs) \). The relation \( a(xs) \geq a(x) \) holds in general by 1.3 (5).

It is helpful if one could find out all the graphs \( \mathcal{M}(x) \) and \( \mathcal{M}(xs) \) for any \( x \in X_k \) and any \( s \in S \setminus R(x) \).

These graphs could be worked out efficiently by some computer programme. In the case when the graph \( \mathcal{M}(x) \) is larger or even infinite, one need only to work out a local part \( \mathcal{M} \) of \( \mathcal{M}(x) \) around the vertex \( x \). It depends for the actual size of \( \mathcal{M} \). Usually, we take \( \mathcal{M} \) to be a connected subgraph with its vertex set \( M \subseteq M(x) \) satisfying that
Call a subgraph $\mathbb{M}$ of $\mathbb{M}(x)$ representative if the vertex set $M$ of $\mathbb{M}$ satisfies condition $(\ast)$.

Checking a subgraph $\mathbb{M}$ being representative in $\mathbb{M}(x)$ is an easy matter: one need only to check if there always exists some $z_0 \in M$ satisfying $z_0 \sim_{L} z$ for any $y \in M$ and any $z \in M(x)$ with $\{y, z\}$ an edge of $\mathbb{M}(x)$.

For any $x \in W$, the following method is efficient for finding the value $a(x)$ in the case where a direct computation for $a(x)$ is difficult (for example, when $\ell(x)$ is larger). One may try to find a sequence $x_0 = x, x_1, ..., x_r$ in $W$ such that for every $1 \leq i \leq r$, the element $x_i s_i$ is in $M(x_{i-1})$ with $\{x_i, x_i s_i\}$ a right primitive pair for some $s_i \in S$ and that the computation for the value $a(x_r)$ is much easier than that for $a(x)$ (for example, this is the case when $w_J \in M(x_r)$ for some $J \subset S$). In this case, we have $a(x) = a(x_r)$ by repeatedly applying Proposition 1.7.

In practice, we often choose such a sequence $x_0 = x, x_1, ..., x_r$ with $\ell(x_r)$ much smaller than $\ell(x_0)$ since the value $a(z)$ can be computed relatively easier when $\ell(z)$ is getting smaller in general.

When $W$ is a finite Weyl group, one can easily get the value $a(x)$ from the value $a(w_0 x)$ by Theorem 1.9 and by the knowledge of the special unipotent classes of the corresponding reductive algebraic group, where $w_0$ is the longest element of $W$ (see [7, Subsection 3.3]).

4.4. For any $x \in X_k$ and any $s \in S \setminus R(x)$ with $a(x s) = a(x)$, checking the inequality $a(r x s) < a(x s)$ for any $r \in L(x s)$ amounts to checking if we always have $y = 1$ in any expression of the form $x s = y \cdot w \cdot z$ with $w \in F(\Omega)$ and $y, z \in W$. The latter can proceed efficiently in terms of alcoves forms of elements once the set $F(\Omega)$ is given explicitly.

4.5. To find $X_{k+1}$ from the set $\bigcup_{i=0}^{k} X_i \cup X'_{k+1}$, we need to determine whether or not two concerning elements $x, y$, with at least one of them in $X'_{k+1}$, are in the same left cell of $W$. 
By Propositions 1.5 and 1.7, this can proceed either by comparing their right
generalized \( \tau \)-invariants or with the aid of right primitive pairs.

Suppose that we have got all the graphs \( M(x) \) (or their representative subgraphs)
with \( x \) ranging over \( \bigcup_{i=0}^{k} X_i \cup X'_{k+1} \). These data will help us in determining if
two elements (say \( x, y \)) so obtained are in the same left cell: We have \( x \sim_L y \) only if
\( x, y \) have the same right generalized \( \tau \)-invariants; while 1.3 (6) provides a complete
invariant for the relation \( \sim_L \).

4.6. The most interesting case to apply our algorithm is when \( F(\Omega) = \{ w_J \in \Omega \mid J \subseteq S \} \neq \emptyset \). In this case, \( F(\Omega) \) is distinguished and all the elements in \( F(\Omega) \) have
the same length, hence \( F_0(\Omega) = F(\Omega) \) by 3.8. The followings are some known cases
(not being exhausted) for \( F(\Omega) \) of such a form:

1. \( \Omega \) is the lowest two-sided cell of \( W \) under the partial order \( \leq_{LR} \) (see [16, Theorem 1.1]).

2. \( \Omega \) consists of fully-commutative elements (e.g., the case when the Coxeter
graph of \( W \) contains no subgraph of type \( D_4, \tilde{B}_3 \) or \( \tilde{F}_4 \), and \( \Omega \) contains a fully
-commutative element) (see [24, Theorem 3.4 and Subsection 3.5]).

3. \( W \) is of simply-laced type and \( a(\Omega) \leq 6 \) (see [25, Theorem B]).

4. \( W \) is of type \( \tilde{A}_{n-1} \) (\( n > 1 \)) and \( \Omega \) corresponds to a partition \( \lambda = (\lambda_1, ..., \lambda_r, 1, ..., 1) \)
of \( n \) with \( \lambda_r + 1 \geq \lambda_1 \geq ... \geq \lambda_r > 1 \) (see [20, Theorems 3.1]).

5. \( W \) is of type \( \tilde{C}_l \) (\( l > 1 \)) and \( a(\Omega) = (l - 1)^2 + 1 \).

6. \( W \) is of type \( \tilde{B}_l \) (\( l > 2 \)) and \( a(\Omega) = l(l - 1) \).

4.7. We can describe the set \( F(\Omega) \) for some two-sided cell \( \Omega \) of \( W \) even when
\( F(\Omega) \) does not consist of elements of the form \( w_J, J \subseteq S \). For example, when
\( W = \tilde{D}_4 \), the set \( W(7) = \{ z \in W \mid a(z) = 7 \} \) forms a single two-sided cell but
contains no element of the form \( w_J, J \subset S \). Let \( s_0, s_1, s_2, s_3, s_4 \) be the Coxeter
generator set of \( W \) with \( s_2 \) corresponding to the branching node of its Coxeter
graph. Then \( F(W(7)) = \{ s_is_2s_ks_is_2s_is_2s_is_2s_i \mid i, j, k \in \{ 0, 1, 3, 4 \} \ \text{distinct} \} \) (see [5, Theorem 4.6]).
It is desirable to find the sets $F(\Omega)$ for more two-sided cells $\Omega$ of $W$ in order to apply Algorithm 3.11.

Some more technical tricks are needed in applying Algorithm 3.11. For example, when the set $W_{(i)} = \bigcup_{j=1}^{r} \Omega_j$ for some $i \in \mathbb{N}$ is a union of two-sided cells $\Omega_j$ with some $r > 1$, sometimes we know the set $F(W_{(i)}) := \{ x \in W_{(i)} \mid a(tx), a(xs) < i, \forall t \in L(x), s \in R(x) \}$ but not the sets $F(\Omega_j)$ individually. Let us explain it by examples.

**Examples 4.8.** Let $W = \widetilde{C}_4$ with $S = \{0, 1, 2, 3, 4\}$ be as in Example 4.1.

(a) The set $W_{(3)}$ is a union of two two-sided cells (say $\Omega_{3,1}$ and $\Omega_{3,2}$) of $W$ by Theorem 1.9. We have $F(W_{(3)}) = \{121, 232, 024\}$ and $F_0(\Omega_{3,i}) = F(\Omega_{3,i})$. At moment, we don’t know what the set $F_0(\Omega_{3,i})$ is for any $i = 1, 2$. So we have to assume $X_0 = \{121, 232, 024\}$ in applying Algorithm 3.11 to find an l.c.r. set for each of the $\Omega_{3,i}, i = 1, 2$. We get

$X_1' = \{1213, 1210, 2324, 2321, 0241, 0243\}$ and $X_1 = \{1213, 1210, 2324, 0241, 0243\}$ since $2321 \sim_{L} 1213$.

$X_2' = \{12134, 12130, 12101, 23241, 23243, 02413, 02410, 02434\}$ and $X_2 = \{12134, 12130, 12101, 23243, 02413, 02410, 02434\}$ since $23241 \sim_{L} 12134$.

$X_3 = X_3' = \{121343, 121340, 121301, 024132, 024103, 024341\}$.

$X_4 = X_4' = \{1213432, 1213430, 1213012, 0241324, 0241320, 0241034\}$.

$X_5 = X_5' = \{12134320, 12130142, 12130143, 02413243, 02413201\}$.

$X_6 = X_6' = \{121343201, 121301423, 121301432, 024132434, 024132010\}$.

$X_7 = X_7' = \{1213432010, 1213014234\}$ and $X_8 = X_8' = \emptyset$.

Call a subset $K$ of $W$ right-connected, if, for any pair $x, y \in K$, there is a sequence $x_0 = x, x_1, ..., x_r = y$ in $K$ with some $r \geq 0$ such that $x_i^{-1}x_{i-1} \in S$ for every $1 \leq i \leq r$.

By 1.3 (2), we see that for any $i \geq 0$ with $W_{(i)} \neq \emptyset$, any non-empty right-connected subset of $W_{(i)}$ is contained in a right cell of $W$ and hence also in a two-sided cell of $W$. 
Assume $121 \in \Omega_{3,1}$. Let

$$E_1 = \{121, 232, 1213, 1210, 2324, 12134, 12130, 12101, 23243, 121343, 121340, 121301, 1213432, 1213430, 1213014, 1213012, 12134320, 12130142, 121343201, 121301423, 121301432, 1213432010, 1213014234\}$$

and

$$E_2 = \{024, 0241, 0243, 02413, 02410, 02434, 024132, 024103, 024341, 0241324, 0241320, 0241034, 02413243, 02413201, 024132434, 024132010\}$$

Then $121 \in E_1$ and $E := \bigcup_{i=1}^{2} E_0(\Omega_{3,i}) = \bigcup_{k=0}^{7} X_k = E_1 \cup E_2$. We see that $E_2$ is a maximal right-connected subset of the set $E$. Also, $E' := E_1 \cup \{2321\}$ is a union of two right-connected subsets with $1213 \sim_{L} 2321$ such that $1213$ and $2321$ belong to different right-connected subsets of $E'$. This implies that $E_0(\Omega_{3,1}) = E_1$ and $E_0(\Omega_{4,2}) = E_2$ by 1.3 (2) and by the fact that $W_{(3)} = \bigcup_{i=1}^{2} \Omega_{3,i}$.

(b) The set $W_{(4)}$ is a union of two two-sided cells (say $\Omega_{4,1}$ and $\Omega_{4,2}$) of $W$ by Theorem 1.9. Let $x_1 = 0101, x_2 = 1214, x_3 = 121012, y_1 = 3434, y_2 = 2320, y_3 = 232432$. Then $F(W_{(4)}) = \{x_i, y_j \mid 1 \leq i, j \leq 3\}$. Since $x_3 \sim_{L} x_1, 21$ and $y_3 \sim_{L} y_1, 23$ with $\ell(x_1, 21) = \ell(x_3)$ and $\ell(y_1, 23) = \ell(y_3)$, we can take $\bigcup_{i=1}^{2} F_0(\Omega_{4,i}) = \{x_1, x_2, y_1, y_2\}$ by 3.8. Again, we don’t know what the set $F_0(\Omega_{4,i})$ is for any $i = 1, 2$ at moment.

We assume $X_0 = \{x_1, x_2, y_1, y_2\}$ in applying Algorithm 3.11. Then

$$X_1 = X'_1 = \{x_1, x_2, x_23, x_20, y_1, y_2, y_21, y_24\}.$$

$$X_2 = X'_2 = \{x_1, 21, x_1, 23, x_230, x_232, x_234, x_201, y_1, 21, y_1, 20, y_2, 12, y_2, 14, y_243\}.$$

$$X'_3 = \{x_1, 210, x_1, 213, x_1, 234, x_2301, x_2324, x_2304, y_1, 234, y_1, 213, y_1, 210, y_2, 143, y_2, 102, y_2, 104\}$$

and

$$X_3 = \{x_1, 213, x_1, 234, x_2301, x_2324, x_2304, y_1, 213, y_1, 210, y_2, 143, y_2, 102, y_2, 104\}$$

since $x_1210 \sim_{L} x_12$ and $y_1234 \sim_{L} y_12$.

$$X'_4 = \{x_1, 210, 23, x_1, 2134, x_1, 2343, x_23012, x_23014, x_23243, y_1, 2101, y_1, 2103, y_1, 2132 \}$$

and

$$X_4 = \{x_1, 2132, x_1, 234, x_23012, x_23014, x_23243, y_1, 2101, y_1, 2103, y_1, 2132 \}$$

and

$$X'_5 = \{x_1, 210, 34, x_1, 2134, x_1, 2343, x_23014, y_1, 21012, y_1, 21013, y_1, 21032, y_1, 21034, y_2, 10432\}$$

and

$$X_5 = \{x_1, 2132, x_1, 2134, x_1, 2343, x_23014, y_1, 21012, y_1, 21013, y_1, 21032, y_1, 21034, y_2, 10432\}$$

since $x_121034 \sim_{L} x_1234$ and $y_121034 \sim_{L} y_1210$. 
$X'_6 = \{x_1210343, x_1213243, x_1213432, x_1234321, x_2301243, y_1210123, y_1210132, y_1210134, y_1210321, y_2104321\}$ and $X_6 = \{x_1213243, x_1213432, x_1234321, x_2301243, y_1210123, y_1210132, y_1210321, y_2104321\}$ since $x_1210343 \sim x_12343$ and $y_1210134 \sim y_12101.$

$X'_7 = \{x_12103432, x_12134321, x_2343210, x_23012434, y_12341012, y_12310123, y_12101234, y_21432010\}$ and $X_7 = \{x_12343210, x_23012434, y_12101234, y_21432010\}$ since $x_12103432 \sim x_123432$ and $x_12134321 \sim y_12132$ and $y_12341012 \sim y_121012$ and $y_12310123 \sim x_12132.$

$X_8 = X'_8 = \emptyset.$

Assume $x_1 \in \Omega_{4,1}.$ Then by 1.3 (2), we have $E_0(\Omega_{4,1}) = E_{11} \cup E_{12}$ and $E_0(\Omega_{4,2}) = E_{21} \cup E_{22},$ where

$E_{11} = \{x_1, x_12, x_121, x_123, x_1234, x_12341, x_12134, x_12342, x_121343, x_123432, x_1213432, x_1234321, x_1234320\},$

$E_{12} = \{y_1, y_12, y_121, y_123, y_1234, y_12341, y_12134, y_12342, y_121343, y_123432, y_1213432, y_1234321, y_1234320\},$

$E_{21} = \{x_2, x_23, x_24, x_234, x_2341, x_2342, x_2343, x_23432, x_234321, x_234320\},$

and $E_{22} = \{x_1213423, x_1213432\}$ forms a right primitive pair.

(i) Each of $E_{ij}, i, j = 1, 2,$ is a maximal right-connected set in $\bigcup_{i,j=1}^2 E_{ij};$

(ii) $y_212 \sim y_2124 \sim y_22$ and $y_212 \in E_{22}$ and $x_2 \in E_{21};$

(iii) $\{x_1213423, x_1213432\}$ forms a right primitive pair.

(iv) $x_1213423 \in E_{11}$ and $y_1 \in E_{12}$ and $x_12134232 \sim 234232 \sim 2342324 \sim 234234 \sim y_1;$

(v) $W_{(4)}$ is a union of two two-sided cells of $W.$

4.9. Assume that $W$ is an irreducible finite or affine Coxeter group of simply-laced type. We see by [25, Lemma 6.1] that if $w \in W$ satisfies $a(w) \geq 6$ and $a(tw), a(ws) < a(w)$ for any $t \in J := \mathcal{L}(w)$ and $s \in I := \mathcal{R}(w),$ then $\ell(w_J), \ell(w_I) \geq 6.$ This fact will help us to find the set $F(W_{(7)}).$ Actually, all the elements of the form $w_J$ with $J \subseteq S$ and $\ell(w_J) = 7$ should be in $F(W_{(7)});$ while all the other
elements $w$ of $F(W_{(7)})$ should satisfy $\ell(w_J) = \ell(w_I) = 6$ and $a(tw), a(ws) < a(w) = 7$ for any $t \in J := L(w)$ and $s \in I := R(w)$. The set $F(W_{(k)}), k > 7$, can be described similarly but involving more cases.

**Example 4.10.** Let $W = \tilde{E}_6$ be with $S = \{s_i \mid 0 \leq i \leq 6\}$ its Coxeter generator set, where $o(s_1s_3) = o(s_3s_4) = o(s_4s_2) = o(s_2s_6) = o(s_4s_5) = o(s_5s_6) = 3$. Then the set $W_{(7)}$ is a single two-sided cell of $\tilde{E}_6$ by Theorem 1.9. Denote $s_i$ simply by $i, 0 \leq i \leq 6$. By the facts mentioned in 4.9, we get

$$F(W_{(7)}) = \{w_{1346}, w_{1340}, w_{0246}, w_{0241}, w_{4561}, w_{4560}, w_{2346}, w_{2451}, w_{3450}, w_{13562}, w_{13560}, w_{13025}, w_{13026}, w_{02561}, w_{02563}, w_{243 \cdot 543}, w_{243 \cdot 542}, w_{345 \cdot 243}, 2031 \cdot w_{342}, 5631 \cdot w_{345}, 5620 \cdot w_{245}, w_{245 \cdot 345}, w_{345 \cdot 245}, w_{245 \cdot 342}, w_{342 \cdot 1302}, w_{345 \cdot 1365}, w_{245 \cdot 0265}\}.$$ 

The set $F_0(W_{(7)})$ is obtained from $F(W_{(7)})$ by removing the last nine elements since

$$w_{243 \cdot 543} \sim_L w_{245 \cdot 345} \sim_L 5631 \cdot w_{345} \quad \text{and} \quad w_{243 \cdot 542} \sim_L w_{345 \cdot 245} \sim 5620 \cdot w_{245} \quad \text{and}$$

$$w_{345 \cdot 243} \sim_L w_{245 \cdot 342} \sim 2031 \cdot w_{342} \quad \text{and} \quad w_{342 \cdot 1302} \sim_L w_{1340} \cdot 20 \quad \text{and} \quad w_{345 \cdot 1365} \sim_L w_{4561} \cdot 31 \quad \text{and} \quad w_{245 \cdot 0265} \sim_L w_{0246} \cdot 56.$$ 

**Remark 4.11.** In each of Examples 4.1, 4.8 and 4.10, the related set $F(W_{(k)})$ is given at the beginning. As the set of all two-sided minimal elements $w$ of $W_{(k)}$ (i.e. $w \in W_{(k)}$) but $sw, wt \notin W_{(k)}$ for any $s \in L(w)$ and $t \in R(w)$), $F(W_{(k)})$ can be found easily since it was described explicitly for all the sets $W_{(k)}, k \in \mathbb{N}$, of the group $\tilde{C}_4$ and for the set $W_{(7)}$ of $\tilde{E}_6$ (see [23], [27]). In general, without knowing the set $W_{(k)}$ in advance, the set $F(W_{(k)}), k \in \mathbb{N}$, of any Weyl or affine Weyl group $W$ can be found recurrently as follows. By Theorem 1.9 and the knowledge of unipotent conjugacy classes of reductive algebraic groups (see [2]), we can get the set $E(W) := \{i \in \mathbb{N} \mid W_{(i)} \neq \emptyset\}$. For any $k \in E(W)$, suppose that the sets $F(W_{(h_i)}), h < k$, have been found already. Then the set $W_{<k} := \{w \in W \mid a(w) < k\} = \bigcup_{h<k} W_{(h_i)}$ can be described explicitly by Algorithm 3.11 together with some other techniques. Find the set $E_{\geq k}$ of all two-sided minimal elements of $W \setminus W_{<k}$, which is finite by Theorem 1.9 and by the fact $E_{\geq k} \subseteq \bigcup_{\Omega \in \text{Cell}(W)} F(\Omega)$. One can determine the set $F(W_{(k)}) = \{w \in E_{\geq k} \mid a(w) = k\}$ by computing the $a$-values of
elements in $E_{\geq k}$.

References