FULLY COMMUTATIVE ELEMENTS AND KAZHDAN-LUSZTIG CELLS IN THE FINITE AND AFFINE COXETER GROUPS, II

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Abstract. Let $W$ be an irreducible finite or affine Coxeter group and let $W_c$ be the set of fully commutative elements in $W$. We prove that the set $W_c$ is closed under the Kazhdan-Lusztig’s preorder $\succeq_{LR}$ if and only if $W_c$ is a union of two-sided cells of $W$.

Introduction.

Let $W = (W, S)$ be a Coxeter group with $S$ the distinguished generator set. For any $J = \{s_1, ..., s_r\} \subseteq S$, denote by $w_J$ or $w_{s_1s_2...s_r}$ the longest element in the subgroup $W_J$ of $W$ generated by $J$. The fully commutative elements of $W$ were defined by Stembridge: $w \in W$ is fully commutative, if any two reduced expressions of $w$ can be transformed from each other by only applying the relations $st = ts$ with $s, t \in S$ and $o(st) = 2$ ($o(st)$ being the order of $st$), or equivalently, $w$ has no reduced expression of the form $w = xw_{st}y$ with $o(st) > 2$ for some $s \neq t$ in $S$ (see [17, Proposition 2.1]). The fully commutative elements were studied extensively by a number of people (see [2], [4], [6], [7], [16], [17]). Let $W_c$ be the set of all the fully commutative elements in $W$. 

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In the present paper, we only consider (and always assume) the case where $W$ is an irreducible finite or affine Coxeter group unless otherwise specified. The paper is a continuation of my previous paper [16]; the latter proved that the set $W_c$ is a union of two-sided cells (in the sense of Kazhdan-Lusztig, see [8]) if and only if $W$ has a non-branching Coxeter graph and is not $\tilde{F}_4$. The aim of this paper is to give a necessary and sufficient condition for the set $W_c$ being closed under the Kazhdan–Lusztig preorder $\geq_{LR}$ (see Theorem 2.1). We use the result of [16] mentioned above and the following key observation: If $W$ has a non-branching Coxeter graph and is not $\tilde{F}_4$, then for any $w \notin W_c$, there exists some $y \in M(w)$ (see 1.5 for the notation) such that $L(y)$ is not fully commutative (see 1.1). Then we get our result by comparing the generalized $\tau$-invariants on the elements in the set $W_c$ and in its complement $W \setminus W_c$ (see [12, Section 4]).

In [7, Section 3.1], Green and Losonczy proved that an irreducible finite Coxeter group $W$ contains no subgraph of type $D_4$ in its Coxeter graph if and only if the set $W_c$ is closed under $\geq_{LR}$ and is a union of two-sided cells. They gave a conceptual (resp., a computer) proof for $W = B_m, A_n, m \geq 2, n \geq 1$ (resp., $W = F_4, H_3, H_4$) and referred the proof for the other cases to the papers [3], [5]. Then in [6, Theorem 3.4], Green proved that $W$ is a union of two-sided cells closed under $\geq_{LR}$ for $W = A_n, n \geq 1$. The results [6, Theorem 3.4] and [7, Section 3.1] on $A_n, A_n, n \geq 1$, may also be obtained from my earlier results [11, Theorem 17.4], [13, Theorem 3.1] and [14, Section 2.9] by [17, Theorem 2.1].

In the proof of our main result (i.e., Theorem 2.1), we use the right cell graphs, rather than a computer, in dealing with the cases of $W = \tilde{G}_2, F_4, H_3, H_4$ (see Appendix and the proof of Lemma 2.2).

The contents of the paper are organized as follows. We collect some notations, terminology and known results concerning Kazhdan–Lusztig cells of a Coxeter group $W$ in Section 1. Then we prove our main result in Section 2. In Appendix we list some right cell graphs in $W \setminus W_c$ for $W = \tilde{G}_2, F_4, H_4, H_3$, which are used in the proof of Lemma 2.2.

§1. Some results on Coxeter groups.

Let $(W, S)$ be a Coxeter system. In the Introduction we defined the set $W_c$ of all the
fully commutative elements of $W$. In this section, we collect some notations, terminology and known results for later use.

1.1. Let $\leq$ be the Bruhat–Chevalley order and $\ell(w)$ the length function on $W$. Call a subset $J$ of $S$ fully commutative if the element $w_J$ is so.

For $w, x, y \in W$, we use the notation $w = x \cdot y$ to mean $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$. If $w = x \cdot y \in W_c$ then $x, y \in W_c$. In particular, if $w \in W_c$ has an expression $w = x \cdot w_J \cdot y$ with $x, y \in W$ and $J \subseteq S$, then $J$ is fully commutative.

1.2. Let $\preceq_L$ (resp., $\preceq_R$, $\preceq_{LR}$) be the preorder on $W$ defined as in [8], and let $\sim_L$ (resp., $\sim_R$, $\sim_{LR}$) be the equivalence relation on $W$ determined by $\preceq_L$ (resp., $\preceq_R$, $\preceq_{LR}$). The corresponding equivalence classes are called left (resp., right, two-sided) cells of $W$. The preorder $\preceq_L$ (resp., $\preceq_R$, $\preceq_{LR}$) on $W$ induces a partial order on the set of left (resp., right, two-sided) cells of $W$.

1.3. For any $w \in W$, let $L(w) = \{ s \in S \mid sw < w \}$ and $R(w) = \{ s \in S \mid ws < w \}$.

Assume $m = o(st) > 2$ for some $s, t \in S$. A sequence of elements

$$ sy, tsy, stsy, \ldots $$

$m-1$ terms

is called a left $\{s, t\}$-string if $y \in W$ satisfies $L(y) \cap \{s, t\} = \emptyset$.

We say that $z$ is obtained from $w$ by a left $\{s, t\}$-star operation, if $z, w$ are two neighboring terms in a left $\{s, t\}$-string. Clearly, a resulting element $z$ of a left $\{s, t\}$-star operation on $w$, when it exists, need not be unique unless $w$ is a terminal term of the left $\{s, t\}$-string containing it.

The following result follows directly from the definition of the relation $\sim_L$ on $W$.

**Lemma.** If $x, y \in W$ can be obtained from each other by successively applying left star operations, then $x \sim_L y$.

1.4. By the notation $x \mapright{L} y$ in $W$, we mean that either $x < y$ or $y < x$ holds and that $\max\{ \deg P_{x,y}, \deg P_{y,x} \} = \frac{1}{2}(|\ell(x) - \ell(y)| - 1)$, where $P_{x,y}$ is the celebrated Kazhdan–Lusztig polynomial associated to $x, y \in W$ (see [8, Theorem 1.1]).
(a) The relation $x \leq_L y$ (resp., $x \leq_R y$) implies $R(x) \supseteq R(y)$ (resp., $L(x) \supseteq L(y)$). In particular, the relation $x \sim_L y$ (resp., $x \sim_R y$) implies $R(x) = R(y)$ (resp., $L(x) = L(y)$) (see [8, Proposition 2.4]). Hence it makes sense to write $L(\Gamma)$ (resp., $R(\Gamma)$) for any right (resp., left) cell $\Gamma$ of $W$, where $L(\Gamma) = L(z)$ (resp., $R(\Gamma) = R(z)$) for any $z \in \Gamma$.

(b) If $x, y \in W$ with $x \rightarrow y$ are in some left $\{s, t\}$-strings (not necessarily in the same left string; see 1.3) for some $s, t \in S$ with $st \neq ts$, then there exist some $x', y' \in W$ which are obtained from $x, y$ respectively by a left $\{s, t\}$-star operation and satisfy $x' \rightarrow y'$ (see [9, Section 10.4]).

(c) $x \sim_{LR} x^{-1}$ for any $x \in W$ (see [10, Corollary 1.9 (a) and Theorem 1.10] and [1, Corollary 3.2]).

1.5. For any $w \in W$, let $M(w)$ be the set of all the elements $y$ satisfying: there exists a sequence of elements $z_0 = w, z_1, ..., z_t = y$ in $W$ with $t \geq 0$ such that $z_i$ is obtained from $z_{i-1}$ by a left star operation for every $1 \leq i \leq t$. We see by Lemma 1.3 that all the elements in $M(w)$ are in the same left cell of $W$.

§2. The condition for $W_c$ being closed under the preorder $\geq_{LR}$.

In this section, assume that $W$ is an irreducible finite or affine Coxeter group. In [16, Theorem 3.4 and Sections 3.5–3.7], we showed that the set $W_c$ is a union of two-sided cells of $W$ if and only if $W$ has a non-branching Coxeter graph and is not $\tilde{F}_4$. We understand that this result was already known in the case where $W$ is any irreducible finite Coxeter group (see [7]).

A subset $K$ of $W$ is closed under the preorder $\geq_{LR}$ if the conditions $x \in K, y \in W$ and $y \geq_{LR} x$ together imply $y \in K$.

In the present section, we want to give a necessary and sufficient condition for the set $W_c$ to be closed under $\geq_{LR}$.

**Theorem 2.1.** Let $W$ be an irreducible finite or affine Coxeter group. Then $W_c$ is closed under $\geq_{LR}$ if and only if $W_c$ is a union of two-sided cells of $W$.

To prove Theorem 2.1, we need prove some lemmas.
**Lemma 2.2.** If $W$ is an irreducible finite or affine Coxeter group such that $W_c$ is a union of two-sided cells of $W$, then for any $w \in W \setminus W_c$, there exists some $y \in M(w)$ (see 1.5) such that $L(y)$ is not fully commutative (see 1.1).

**Proof.** By [16, Theorem 3.4 and 3.5–3.7], we know that $W_c$ is a union of two-sided cells of $W$ if and only if $W$ has a non-branching Coxeter graph and is not $\widetilde{F}_4$, i.e., $W$ is one of the following groups: $A_n$, $\widetilde{A}_n$, $I_2(m)$, $\widetilde{C}_l$, $B_l$, $F_4$, $H_3$, $H_4$, $\widetilde{G}_2$, where $n \geq 1$, $m \geq 5$ and $l \geq 2$. The result follows by [11, Theorems 17.4, 17.6 and Propositions 9.3.7, 16.2.4] for the groups $\widetilde{A}_n$ and $A_n$, and by [16, Corollary 3.3] for the groups $\widetilde{C}_l$. By the fact that $B_l$ is a standard parabolic subgroup of $\widetilde{C}_l$, we can show the result for the groups $B_l$ by the same argument as that for [16, Corollary 3.3]. Then the result for the groups $F_4$, $H_3$, $H_4$ and $\widetilde{G}_2$ can be checked directly from their right cell graphs (see Appendix). Finally, the result for the groups $I_2(m)$ is obvious. □

**Remark 2.3.** It is necessary for the assumption that $W_c$ is a union of two-sided cells of $W$ in Lemma 2.2. There is a counter-example when such a condition is removed. Let $W = \widetilde{F}_4$ and $S = \{s_0, s_1, s_2, s_3, s_4\}$ be with $o(s_0s_1) = o(s_1s_2) = o(s_3s_4) = 3$ and $o(s_2s_3) = 4$. Then the element $w = s_4s_2s_3s_2s_0s_1s_0$ is not fully commutative. However, $L(y)$ is fully commutative for any element $y$ in $M(w)$ (see [12, Section 5.4]).

By Lemma 2.2, we can prove the following

**Lemma 2.4.** When it is a union of two-sided cells of $W$, the set $W_c$ is closed under the preorder $\geq$. 

**Proof.** Suppose not. Then there exist some $x \in W_c$ and some $w \in W \setminus W_c$ with $x \leq w$. We may assume $x—w$ and $L(x) \not\subseteq L(w)$ without loss of generality. So $R(x) \supseteq R(w)$ by 1.4 (a). Hence $L(x^{-1}) \supseteq L(w^{-1})$. By Lemma 2.2, there exists an element $y$ in $M(w^{-1})$ with $L(y)$ not fully commutative. Then there exists a sequence of elements $w_0 = w^{-1}, w_1, ..., w_r = y$ in $M(w^{-1})$ such that $w_i$ is obtained from $w_{i-1}$ by a left $\{s_i, t_i\}$-star operation for every $1 \leq i \leq r$ and some $s_i, t_i \in S$ with $s_it_i \neq t_is_i$. We may assume $r$ minimal with this property. Hence the $L(w_i)$’s, $0 \leq i < r$, are all fully commutative. Since $w_1$ is obtained from $w^{-1}$ by a left $\{s_1, t_1\}$-star operation, we have $|\{s_1, t_1\} \cap L(w^{-1})| = 1$. 

Since $\mathcal{L}(x^{-1})$ is fully commutative and $\mathcal{L}(x^{-1}) \supseteq \mathcal{L}(w^{-1})$, we have $|\{s_1, t_1\} \cap \mathcal{L}(x^{-1})| = 1$ also. So we can apply a left $\{s_1, t_1\}$-star operation on $x^{-1}$ to obtain some element $x_1$ in $M(x^{-1})$ with $x_1 = w_1$ by 1.4 (b). Since $\mathcal{R}(x_1) = \mathcal{R}(x^{-1}) = \mathcal{L}(x) \not\subseteq \mathcal{L}(w) = \mathcal{R}(w^{-1}) = \mathcal{R}(w_1)$, we have $x_1 \ll w_1$ and hence $\mathcal{L}(x_1) \supseteq \mathcal{L}(w_1)$ by 1.4 (a). When $r > 1$, we can apply a left $\{s_2, t_2\}$-star operation on $x_1$ to obtain some element $x_2$ with $x_2 = w_2$ by the same reason as that for getting $x_1$ from $x^{-1}$. Continuing this process, we get a sequence of elements $x_0 = x^{-1}, x_1, \ldots, x_r$ in $M(x^{-1})$ such that $x_i$ is obtained from $x_{i-1}$ by a left $\{s_i, t_i\}$-star operation and $x_i = w_i$ for $1 \leq i \leq r$. By the assumption that $W_c$ is a union of two-sided cells of $W$ and by the facts that $x_r \sim x^{-1} \sim x$ (by 1.4 (c)) and $x \in W_c$, we have $x_r \in W_c$ and hence the set $\mathcal{L}(x_r)$ is fully commutative. Since $\mathcal{L}(w_r)$ is not fully commutative, we have $\mathcal{L}(w_r) \not\subseteq \mathcal{L}(x_r)$. Since $x_r \sim w_r$, this implies $w_r \ll x_r$ and hence $x \sim w_r \sim w^{-1} \sim x_r \sim x^{-1} \sim x$ by 1.4 (c). We get $x \sim w$, contradicting the assumption that $W_c$ is a union of two-sided cells of $W$. So our result follows. \[\square\]

2.5. Proof of Theorem 2.1. The implication “$\Rightarrow$” is just Lemma 2.4. For the implication “$\Leftarrow$”, we need only show that $x \not\sim y$ for any $x \in W_c$ and any $y \in W \setminus W_c$. Suppose not. Then there exist some $x \in W_c$ and some $y \in W \setminus W_c$ with $x \sim y$ (and hence $y \gg x$). But this would imply $y \in W_c$ by the assumption that $W_c$ is closed under $\gg$, a contradiction. So Theorem 2.1 follows. \[\square\]

Appendix.

A right cell graph associated to an element $x \in W$ (written $\mathcal{M}_R(x)$) is by definition a graph whose vertex set $V(x)$ consists of all the right cells $\Gamma$ of $W$ with $\Gamma \cap M(x) \neq \emptyset$ (each right cell is represented by a box). Two vertices $\Gamma, \Gamma'$ of $\mathcal{M}_R(x)$ are joined by an edge, if there are some $y \in M(x) \cap \Gamma$ and $z \in M(x) \cap \Gamma'$ such that $y, z$ are two neighboring terms of a left string. Each vertex $\Gamma$ of $\mathcal{M}_R(x)$ is labelled by the set $\mathcal{L}(\Gamma)$ (see 1.4 (a)).

It is easily seen that the set of the subsets of $S$ occurring as the labels of the vertices in $\mathcal{M}_R(x)$ is equal to the set $\{I \subseteq S \mid I = \mathcal{L}(y)$ for some $y \in M(x)\}$.

Two right cell graphs $\mathcal{M}_R(x)$ and $\mathcal{M}_R(y)$ are isomorphic if there exists a bijection $\phi : V(x) \to V(y)$ such that $\mathcal{L}(\Gamma) = \mathcal{L}(\phi(\Gamma))$ for any $\Gamma \in V(x)$ and such that any pair
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\( \Gamma, \Gamma' \in V(x) \) are joined by an edge if and only if \( \phi(\Gamma), \phi(\Gamma') \) are so.

Note that the definition of a right cell graph imitates that of a left cell graph, the latter was given in my previous paper [15, Subsection 2.11].

We work out all the right cell graphs in \( W \setminus W_c \) (resp., a representative set of the isomorphism classes of those graphs) for the groups \( W = \tilde{G}_2, F_4 \) (resp., \( H_4, H_3 \)) according to the results in [9], [18], [1].

(1) \( W = \tilde{G}_2 \) with \( S = \{s_0, s_1, s_2\} \) satisfying \( o(s_0s_2) = 3 \) and \( o(s_1s_2) = 6 \):

Here and later the boldfaced numbers in a box \( \Gamma \) represent the elements in \( L(\Gamma) \). The box of \( M_R(x) \) with inside numbers underlined represents the right cell \( \Gamma_x \) containing \( x \). For example, the box \( \boxed{02} \) in \( M_R(s_0s_2s_1s_2s_0w_{12}) \) represents the right cell \( \Gamma = \Gamma_{s_0s_2s_1s_2s_0w_{12}} \) with \( L(\Gamma) = \{s_0, s_2\} \); while two boxes \( \boxed{01} \) in \( M_R(w_{12}) \) represent respectively two right cells \( \Gamma, \Gamma' \in V(w_{12}) \) with \( L(\Gamma) = L(\Gamma') = \{s_0, s_1\} \). The notation \( w_{i_1i_2...} \) stands for the element \( w_{i_1i_2...} \) (see the first paragraph in Introduction).

(2) \( W = F_4 \) with \( S = \{s_1, s_2, s_3, s_4\} \) satisfying \( o(s_1s_2) = o(s_3s_4) = 3 \) and \( o(s_2s_3) = 4 \).
(3) $W = H_4$ with $S = \{s_1, s_2, s_3, s_4\}$ satisfying $o(s_1s_2) = o(s_2s_3) = 3$ and $o(s_3s_4) = 5$. 
(4) $W = H_3$ with $S = \{s_1, s_2, s_3\}$ satisfying $o(s_1 s_2) = 3$ and $o(s_2 s_3) = 5$.

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