Triple covers on smooth algebraic varieties

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Abstract

It is well known that any triple cover can be constructed by a minimal cubic equation. Based on this fact, we compute the normalization, the rank two trace-free sheaf, the branch locus and the singular locus of a triple cover. We give also a canonical resolution of the singularities of a triple cover surface, which provides us the formulas to compute the global invariants. The classical criterion for cubic extensions to be Galois is presented for triple covers. Finally, we establish a relationship between Miranda's triple cover data and the minimal cubic equations.

Introduction

Double covers are well understood and play an important role in the classification problem of algebraic surfaces (see [**3**] and [**7**]). Recently, many authors started to establish a similar theory for triple covers ([**8**], [**4**], [**1**], [**2**], [**10**], [**18**], [**19**]). However, such an investigation becomes much harder, for here one has to deal with the very difficult non-Galois situation.

Recall that a *triple cover* $\pi : Y \to X$ is a surjective finite morphism of degree 3 between two varieties X and Y over an algebraically closed field k. Generally, one can always assume that X is smooth and $H^0(X, \mathcal{O}_X) = k$.

In fact, the study of finite covers of degree n in algebraic geometry is essentially equivalent to solving algebraic equations of degree n over commutative rings. Therefore, the classical method of solving algebraic equations should provide us an effective way to deal with finite covers.

The purpose of the present paper is to introduce the classical method of solving cubic equations to the study of triple covers. Then we can easily resolve the singularities of a triple cover, and compute its branch locus as well as its numerical invariants. Our method is based on the computation of the normalization of a cubic extension over a Noetherian unique factorization ring ([**13**]).

Our first step is to give good construction data. Namely, we prove that every triple cover can be constructed by a (minimal) cubic equation from which we can see clearly the branch locus. On a smooth algebraic variety, we know that, up

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to tensoring a non zero constant, a non zero global section a of a line bundle is determined uniquely by its divisor A. Thus we can factorize a as the product of *prime* sections whose divisors are reduced and irreducible. Then for any triple cover $\pi : Y \to X$, there is a line bundle L on X and two global sections $a_1 a_2^2 b_1 a_0$ and $a_1 a_2^2 b_1^2 b_0$ of L^2 and L^3, respectively, such that Y is the normalization of the subvariety of L defined by

$$z^3 + a_1 a_2^2 b_1 a_0 z + a_1 a_2^2 b_1^2 b_0 = 0,$$

where a_1, a_2 and b_1 are square-free and coprime. Moreover

$$a := 4a_1 a_2^2 a_0^3 \quad \text{and} \quad b := 27 b_1 b_0^2$$

are also coprime. Since Y is irreducible, the constant term and b are non zero. For a non-Galois triple cover, the coefficient of z is also non zero, and thus $a \neq 0$. In fact, L can be the dual of any maximal invertible subsheaf of the trace free sheaf of π. Conversely, it is obvious that for any coprime pair (a, b) of sections of a line bundle, we can uniquely construct such a cubic equation by using the factors of a and b according to the above formulas. In fact, up to equivalence, this correspondence is one to one. Thus (a, b) can also be viewed as the data of π.

If $c = a + b$ has a factorization $c = c_1 c_0^2$ such that c_1 is square-free, then the branch locus of π is defined by

$$a_1^2 a_2^2 b_1 c_1 = 0$$

and the divisor of $a_1 a_2$ (resp. $b_1 c_1$) is the branch locus over which π is totally (resp. simply) ramified. Furthermore, the divisor of $b_1 c_1$ is always even. Note that in number theory $b_1 c_1$ is called the *fundamental discriminant* of the cubic extension (triple cover).

In Sect. 2, we show that a ramified triple cover is Galois if and only if it is totally ramified over its branch locus, i.e., b_1 and c_1 are non zero constants. Thus any non-Galois triple cover will become Galois under the canonical base change of degree 2 over X branched along the divisor of $b_1 c_1$ (Corollary 2.3). Based on Miranda's [**8**] local computation on the locus over which Y is singular, we shall compute in Sect. 3 this locus directly from the global data a_1, a_2, b_1 and c_1. In Sect. 5, we are focus on computing the codimension two locus over which π is totally ramified. In Sect. 4, we shall give a very simple method to resolve the singularities of the triple cover surface Y over a smooth surface X. In this case, Y has only a finite number of singular points. Because for the Galois case, the resolution is well-known (see Sect. 1.4 or [**1**], [**11**]), we can assume that (a, b) is the data of π. Let $\sigma : X_1 \to X$ be the blowing-up of X at a singular point of the reduced branch locus defined by $a_1 a_2 b_1 c_1 = 0$, let

$$a^{(1)} = \frac{\sigma^* a}{\gcd(\sigma^* a, \sigma^* b)}, \quad b^{(1)} = \frac{\sigma^* b}{\gcd(\sigma^* a, \sigma^* b)}.$$

Then we get new triple cover data $(a^{(1)}, b^{(1)})$ on X_1. We can prove that after a finite number of such blowing-ups, the induced new triple cover data $(a^{(k)}, b^{(k)})$ on X_k admits a smooth branch locus, hence its triple cover surface Y_k is smooth. Y_k is a resolution of Y. This process can be done without noting the triple cover. In fact, what we need to know is the graphs of the curves of a, b and $c = a + b$. Hence any triple cover is birationally equivalent to a smooth triple cover with a smooth branch locus. This is not the case for covers of degree at least 4.

Due to this resolution, we give in Sect. 6 the computation formulas of the invariants of the resolution surface Y_k in terms of the data (a, b).

The result of this paper has been applied to give a characterization of the branch curve of a generic triple cover $\pi : Y \to \mathbb{P}^2$ ([**14**] and [**17**]), which is a classical problem. As another application, we showed that any rank two vector bundle on an algebraic surface can be constructed uniquely by a curve with cusps. In a later paper, we shall describe the relations between Hilbert's algebraic invariant theory and triple covers.

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1. Basic properties of triple covers

1.1. Construction of triple covers. Let X be a smooth algebraic variety defined over k, and let $\pi : Y \to X$ be a normal triple cover. We always assume that $\operatorname{char} k \neq 2, 3$ and $H^0(X, \mathcal{O}_X) = k$, i.e., the regular functions on X are constants.

Throughout this paper, we use little letters $(a, b, c, ..., \delta, ...)$ to denote the global regular sections of invertible sheaves on X, and use capital letters $(A, B, C, ..., \Delta, ...)$ to denote the divisors of the corresponding sections. A section p is called a prime if its divisor P is a reduced and irreducible divisor. So we can talk about the factorization of sections. We denote by a_p the highest power of p in a, and by $[n/m]$ the maximal integer $\leq n/m$.

Definition 1.1. A pair of sections (s, t) are called *triple cover data* if $t \neq 0$ and there exists an invertible sheaf $\mathcal{L}$ such that s and t are global sections of $\mathcal{L}^2$ and $\mathcal{L}^3$ respectively. If $s = 0$, then $(t, \mathcal{L})$ is called *Galois triple cover data.* (s, t) is said to be *minimal* if there is no reduced and irreducible divisor P in X such that $\mu_P(S) \geq 2$ and $\mu_P(T) \geq 3$, where $\mu_P(S)$ and $\mu_P(T)$ denote the multiplicities of P in S and T respectively.

Now we recall the well known construction of triple covers of a variety X.

Let $(s, t, \mathcal{L})$ be as in Definition 1.1. Denote by $V(\mathcal{L}) = \operatorname{Spec} S(\mathcal{L})$ the associated line bundle of $\mathcal{L}$, where $S(\mathcal{L})$ is the symmetric $\mathcal{O}_X$-algebra of $\mathcal{L}$. Let z be the global coordinate in the fibers of $V(\mathcal{L})$. Then z is a global section of $p^*\mathcal{L}$, where p is the bundle projection of $V(\mathcal{L})$. Thus we obtain a polynomial section of $p^*\mathcal{L}^3$

$$p(z) = z^3 + sz + t,$$

where s and t are viewed as sections of $p^*\mathcal{L}^2$ and $p^*\mathcal{L}^3$ respectively. Then the zero set of $p(z)$ defines a subscheme Σ of $V(\mathcal{L})$. Let Y be the normalization of Σ. Then the composition of the normalization with the bundle projection defines a finite morphism $f : Y \to X$ of degree 3. f will be called the *triple cover determined by* (s, t).

In the above construction, we do not need to assume that (s, t) is minimal (or equivalently, the three sections (a, b, c) do not need to be coprime (see Sect. 1.2)). In fact, if λ is a non-zero global section, then the data $(\lambda^2 s, \lambda^3 t)$ and (s, t) determine the same triple cover (cf. [**13**]).

We always assume that Y is integral, i.e., $p(z)$ is irreducible over the function field $K(X)$ of X.

In Sect. 7 we shall prove that every triple cover $\pi : Y \to X$ can be constructed by this method.

1.2. Equivalent triple cover data and j-invariant. If $s=0$, then the triple cover determined by (s,t) is Galois, everything is known (see Sect. 1.4, or [**8**], [**11**], [**12**]). When $s\neq 0$, (s,t) is triple cover data if and only if

$$3S\equiv 2T,$$

where $\equiv$ is the linear equivalence of divisors. Two minimal pairs (s',t') and (s,t) are said to be equivalent if there is a global section λ_0 without zeros such that $s'=\lambda_0^2 s$ and $t'=\lambda_0^3 t$. Note that if $H^0(X,\mathcal{O}_X)=k$, then λ_0 is a non-zero constant.

Proposition 1.2. *The map $\Phi(s,t)=(a,b,c)$ defined by*

$$a=\frac{4s^3}{\gcd(s^3,t^2)},\quad b=\frac{27t^2}{\gcd(s^3,t^2)},\quad c=\frac{4s^3+27t^2}{\gcd(s^3,t^2)} \tag{1}$$

gives a one-to-one correspondence between the following two sets (up to equivalence):

$$\left\{\begin{array}{c}\textit{Minimal triple cover data}\\ (s,t)\ \textit{with}\ s\neq 0\end{array}\right\}\overset{\Phi}{\longleftrightarrow}\left\{\begin{array}{c}\textit{Coprime triples}\ (a,b,c)\ \textit{with}\\ a+b=c\end{array}\right\},$$

where a, b and c are three non-zero sections of an invertible sheaf. Two such triples (a',b',c') and (a,b,c) are said to be equivalent if there is a global section λ_0 without zeros such that $a'=\lambda_0 a$, $b'=\lambda_0 b$ and $c'=\lambda_0 c$.

Proof. For a given (s,t) (not necessarily minimal), we define

$$\varepsilon_p=3s_p-2t_p,\qquad \lambda_p=\min\left\{\left[\frac{s_p}{2}\right],\ \left[\frac{t_p}{3}\right]\right\}.$$

Let $\lambda=\prod_p p^{\lambda_p}$, and let

$$a_1=\prod_{\substack{\varepsilon_p>0\\ \varepsilon_p\equiv 1\,(3)}}p,\ \ a_2=\prod_{\substack{\varepsilon_p>0\\ \varepsilon_p\equiv 2\,(3)}}p,\ \ b_1=\prod_{\substack{\varepsilon_p<0\\ \varepsilon_p\equiv 1\,(2)}}p,\ \ c_1=\prod_{\substack{\varepsilon_p=0\\ \delta_p\equiv 1\,(2)}}p. \tag{2}$$

(Note that we always let $\prod_{p\in\text{empty set}}p=1$.) Then we have the factorizations

$$s=a_1b_1a_2^2\lambda^2a_0,\quad t=a_1b_1^2a_2^2\lambda^3b_0,\quad (a_0,b_0)=1. \tag{3}$$

Let

$$\delta=4s^3+27t^2=a_1^2b_1^3a_2^4\lambda^6(4a_1a_2^2a_0^3+27b_1b_0^2).$$

By definition, we have

$$4a_1a_2^2a_0^3+27b_1b_0^2=c_1{c_0}^2 \tag{4}$$

where $c_0$has the following factorization (up to a unit),

$$c_0=\prod_{\varepsilon_p=0}p^{[\frac{\delta_p}{2}]-t_p}. \tag{5}$$

Hence

$$a=4a_1a_2^2a_0^3,\quad b=27b_1b_0^2,\quad c=c_1{c_0}^2. \tag{6}$$

Hence (a,b,c) are coprime sections satisfying $a+b=c$.

Conversely, for a given coprime triple (a,b,c) with $a+b=c$, we define

$$a_1=\prod_{a_p\equiv 1\,(3)}p,\quad a_2=\prod_{a_p\equiv 2\,(3)}p,\quad b_1=\prod_{b_p\equiv 1\,(2)}p,\quad c_1=\prod_{c_p\equiv 1\,(2)}p, \tag{7}$$

$$a_0=2^{-\frac{2}{3}}\prod_p p^{[\frac{a_p}{3}]},\quad b_0=3^{-\frac{3}{2}}\prod_p p^{[\frac{b_p}{2}]},\quad c_0=\prod_p p^{[\frac{c_p}{2}]},$$

where the definition of c_0 is up to a unit. Therefore we get a minimal pair (s,t) by (3) (here $\lambda = 1$). This gives a well defined map $\Psi(a,b,c) = (s,t)$.

Now one can prove easily that $\Psi \circ \Phi = \mathrm{Id}$ and $\Phi \circ \Psi = \mathrm{Id}$. Hence Φ gives a one-to-one correspondence between the two sets. Therefore the triple cover data (s,t) is equivalent to (a,b,c). □

In fact, for a given (s,t), the equivalent triple cover data a, b and c are obtained from the equality $4s^3+27t^2 = \delta$ by eliminating the common factors from both sides. For the triple cover determined by $z^3+sz+t = 0$, we can associate it with an elliptic curve $y^2 = z^3 + sz + t$ over the function field of X. Then we have the standard j-invariant $1728\frac{4s^3}{4s^3+27t^2}$. Because we are working over $\mathbb{C}$ or k with $\operatorname{char} k \neq 2, 3$, the constant 1728 is of no use. For simplicity, for the triple cover data (s,t), we can define the j-invariant as,

$$j(s,t) = \frac{4s^3}{4s^3 + 27t^2}.$$

This invariant can be defined for the equivalent data (a,b,c) under the above correspondence Φ,

$$j(a,b,c) = \frac{a}{c}.$$

Obviously, the j-invariant is a rational function on X. If $j \neq 0$, i.e., $s \neq 0$, then it characterizes the triple cover data. From the above lemma, we see easily that if the j-invariants of (s,t) and (s',t') are equal, then (s,t) and (s',t') are equivalent. But different j may determine the same triple cover. For example, if j, j' and j'' satisfy the following equalities,

$$j' = \frac{j}{j-1}, \quad j'' = \frac{j^2}{4j-4},$$

then the triple covers of j, j' and j'' are isomorphic (cf. [**16**]). Note that if $j = j(a,b,c)$, then $j' = j(a,-c,-b)$ and $j'' = j(-a^2,(a+2b)^2,4bc)$.

1.3. Trace-free sheaf and branch locus. If $\mathcal{E}_\pi$ is the trace-free subsheaf of $\pi_*(\mathcal{O}_Y)$, then we have

$$\pi_*(\mathcal{O}_Y) = \mathcal{O}_X \oplus \mathcal{E}_\pi. \tag{8}$$

We denote by $\mathcal{F}$ the syzygy sheaf of $f = (f_1, f_2, f_3) := (2a_0a_2/3, b_0, c_0)$, i.e., $\mathcal{F}$ is the kernel of the following morphism f,

$$0 \to \mathcal{F} \to \mathcal{O}(-F_1) \oplus \mathcal{O}(-F_2) \oplus \mathcal{O}(-F_3) \xrightarrow{f} \mathcal{I}_Z \to 0, \tag{9}$$

where $f(u,v,w) = f_1u+f_2v+f_3w$ and $\mathcal{I}_Z$ is the ideal of $\mathcal{O}_X$ generated by f_1, f_2, f_3.

Theorem 1.3. *Let $\pi : Y \to X$ be a triple cover with normal Y, and let $\mathcal{E}_\pi$ be the trace-free subsheaf of $\pi_*(\mathcal{O}_Y)$. Then*

(1) *π is determined by some minimal triple cover data (s,t). If $\mathcal{L}$ is the invertible sheaf in Definition 1.1, then we can assume that $\mathcal{L}^{-1}$ is a maximal subsheaf of $\mathcal{E}_\pi$.*
(2) *The branch locus of π is $2A_1 + 2A_2 + B_1 + C_1$, and the divisor over which π is totally ramified is $A_1 + A_2$.*
(3) *$B_1 + C_1 \equiv 2\eta$ is an even divisor, where $\eta = 3L - A_1 - B_1 - 2A_2 - C_0$.*
(4) *$\mathcal{E}_\pi \cong \mathcal{F}(D)$, where $D = 4L - 2A_1 - 3A_2 - 2B_1 - \eta$ and L is the divisor of $\mathcal{L}$.*

(5) $c_1(\mathcal{E}_\pi) = -A_1 - A_2 - \frac{1}{2}(B_1 + C_1)$.
(6) $A_2 + B_1 + C_0$ *is the image in* X *of the non-normal locus of the variety* Σ *defined by* $z^3 + sz + t = 0$ *in Sect. 1.2.*

(See [**13**] and see Sect. 7 for 1.).

1.4. Galois triple covers. In this section, we recall some basic facts on Galois triple covers. The Galois triple cover data $(t, \mathcal{L})$ can be viewed as a special case of the general triple cover data (s, t). Then the Galois triple cover $\pi : Y \to X$ determined by $(t, \mathcal{L})$ is defined as the normalization of the triple cover defined by $z^3 + t = 0$. Assume that t has the factorization

$$t = a_1 a_2^2 \lambda^3. \tag{10}$$

In fact, the above factorization of t is unique. Then Y is defined locally by

$$z^2 = a_1 w, \quad zw = a_1 a_2, \quad w^2 = a_2 z.$$

Hence we have

(1) The branch locus of π is $A_1 + A_2$;
(2) Y is smooth iff $A_1 + A_2$ is smooth;
(3) $\pi_*(\mathcal{O}_Y) = \mathcal{O}_X \oplus \mathcal{O}_X(\Lambda) \otimes \mathcal{L}^{-1} \oplus \mathcal{O}_X(2\Lambda + A_2) \otimes \mathcal{L}^{-2}$. Note that Λ is the divisor of λ.

For the convenience of the readers, we give a brief description of the canonical resolution of the singularity of Y. In order to resolve the singularity of Y (lying over the singular points of $A_1 + A_2$), we only need to resolve the singularity of the branch locus. Let $\sigma_1 : X_1 \to X$ be the blowing-up at a singular point p_0 of the branch locus. The new triple cover data is $(t_1, \mathcal{L}_1) = (\sigma_1^*(t), \sigma_1^*(\mathcal{L}))$. We get a new branch locus $A_1' + A_2'$. If it is smooth, we stop. Otherwise, we repeat the same procedure for a singular point of $A_1' + A_2'$. We claim that after a finite number of steps, the new branch locus is smooth. It is well known that after a finite number of steps, the branch locus has at worst nodes as its singularity. We assume that $A_1 + A_2$ is a curve with normal crossing. So the singular points p have only two types:

(A) $p \in A_1 \cap A_2$,
(B) p is asingular point of A_1 or A_2.

In case (A), the singular point can be resolved by blowing up X at p. In case (B), after blowing-up at p, the new branch locus has two singular points of type (A), and they can be resolved by two additional blowing-ups at them.

Therefore, after a finite number of blowing-ups, the normalization of the pull back triple cover surface is smooth. This is the canonical resolution of the singular points of Y.

2. When is a triple cover Galois

In this section, we shall give a criterion for a triple cover to be Galois.

Theorem 2.1. *Let* $\pi : Y \to X$ *be a ramified triple cover over a factorial variety* X *with* $H^0(X, \mathcal{O}_X) = k$. *Then* π *is Galois if and only if* π *is totally ramified over its branch locus.*

Proof. If π is Galois, then it is trivial to see that π has the desired ramification.

We assume that π is totally ramified over its branch locus. Let (s,t) be its minimal triple cover data. If $s=0$, then π is Galois. So we can assume that $s\neq 0$. π has no non-total ramification, i.e., b_1 and c_1 have no zero points. Thus $b_1=c_1=1$ as $H^0(X,\mathcal{O}_X)=k$.

We first consider the case when $X=\operatorname{Spec}(R)$ is affine, $Y=\operatorname{Spec}(\widetilde{R[\alpha]})$ is the normalization of the triple cover defined by $\alpha^3+s\alpha+t=0$. Let

$$z=\frac{(\sqrt{3}f_3/9-f_2)\alpha+f_1\beta}{f_3},$$

where $\beta=(3\alpha^2+2s)/3b_1a_2$. Since $(\sqrt{3}f_3/9-f_2,f_1,\sqrt{3}f_1/9)$ is a syzygy of (f_1,f_2,f_3), z is in the normalization of $R[\alpha]$ (cf. [**13**]).

We shall prove that z satisfies

$$z^3=\frac{4}{3^6}a_1a_2^2(c_0-3\sqrt{3}b_0). \tag{11}$$

Because $R[z]$ is also a cubic extension, the normalization of $R[z]$ equals that of $R[\alpha]$.

The idea of the proof of (11) is simple, but the computation is complicated. We can use Maple to compute it. Note that z^3 is a polynomial of α of degree 6. Firstly, we substitute $\alpha^3=-s\alpha-t$ into z^3 such that z^3 is a quadratic polynomial of α. Secondly, by using (4), we substitute

$$b_0^2=-\frac{4}{27}a_1a_2^2a_0^3+\frac{1}{27}c_0^2$$

into z^3 such that z^3 is linear in b_0. Then we get (11). The following is the Maple computation.

```
> s := a[1]*b[1]*a[2]^2*a[0];   t := a[1]*b[1]^2*a[2]^2*b[0];
  f[1] := 2*a[2]*a[0]/3;   f[2] := b[0];   f[3] := c[0];
  beta := (3*alpha^2+2*s)/(3*b[1]*a[2]);
  z := (((1/9)*(3^(1/2))*f[3]-f[2])*alpha+f[1]*beta)/f[3];
  b[1] := 1;   c[1] := 1;
  F := expand(z^3) :
  G := alpha^3+s*alpha+t;
  H := 4*a[0]^3*a[1]*a[2]^2+27*b[1]*b[0]^2-c[1]*c[0]^2;
  Frem := rem(F, G, alpha) :
  `result` := factor(rem(Frem, H, b[0]));
```

Note that from (4),

$$4a_1a_2^2a_0^3=(c_0-3\sqrt{3}b_0)(c_0+3\sqrt{3}b_0). \tag{12}$$

Because the two factors on the right hand side are coprime, we can see that the power of a prime in $a_1a_2^2(c_0-3\sqrt{3}b_0)$ is not divided by 3 iff p is a divisor of a_1a_2. So the normal Galois triple cover determined by (11) is totally ramified over A_1+A_2. Now we let $U=\cup_iU_i$ be an affine open cover of X. Let σ_i be the automorphism of order 3 of $\pi^{-1}(U_i)$ which determines the triple cover over U_i. Then on $U_i\cap U_j$, there is an integer $n_{ij}=1$ or -1 such that $\sigma_i|_{U_i\cap U_j}=\sigma_j^{n_{ij}}|_{U_i\cap U_j}$. We fixed i_0. Then we can assume that $n_{i_0j}=1$ for any j, because if $n_{i_0j}=-1$, we can replace

$\sigma_{i_0 j}$ by its inverse. Now we see that $n_{ij} = 1$ for all i, j, because σ_i and σ_j coincide on $U_{i_0} \cap U_i \cap U_j$. Thus $\{\sigma_i\}$ determines an automorphism σ of X of order 3, and the triple cover π is the quotient of X under the action of the group $\{1, \sigma, \sigma^2\}$. □

Remark 2.2. The condition $H^0(X, \mathcal{O}_X) = k$ is used to imply that

$$b_1 = c_1 = 1. \tag{13}$$

In fact, locally, b_1 and c_1 have square roots on $U = X \setminus (B_1 + C_1)$, hence we have the same factorization as (12). This means that π is locally Galois over U. The triple cover of (s, t) is Galois iff there is a rational function h on X such that $j(s, t) = 1 - h^2$.

Corollary 2.3. *In the following commutative diagram, $\varphi : X' \to X$ is a double cover ramified over $B_1 + C_1$, and Y' is the normalization of the pull back $X' \times_X Y$.*

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi'} & Y \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow[\varphi]{} & X \end{array}$$

Then π' is a cyclic triple cover ramified over $\varphi^*(A_1 + A_2)$ and some subscheme of codimension at least 2.

Proof. Note that $B_1 + C_1$ is an even divisor, so φ exists. Since $X' \times_X Y$ is a double cover of Y ramified over $\pi^*(B_1 + C_1) = 2(\widehat{B}_1 + \widehat{C}_1) + \widehat{B}'_1 + \widehat{C}'_1$, φ' is a double cover ramified over $\widehat{B}'_1 + \widehat{C}'_1$. Hence π' is a triple cover whose non-total ramification locus D' has codimension ≥ 2. In fact, the data of π' is just the pull back of (4) by φ. On the other hand, $\varphi^*(b_1)$ and $\varphi^*(c_1)$ are square of some sections. So for π', the condition (13) is satisfied. Then we have the same factorization as in (12) for π'. Let D'' be the non-factorial locus on X'. Then we know that the codimension of D'' is ≥ 2. Thus π is Galois over $X' \setminus (D' + D'')$. The automorphism of ${\pi'}^{-1}(U)$ of order 3 can be extended to an automorphism of X' such that π' is the quotient map of this automorphism of order 3. Hence π' is a cyclic triple cover. □

Corollary 2.4. *Let $Y \to \mathbb{P}^2$ be a generic triple cover. Then Y' is smooth and π' is a cyclic triple cover ramified over the singular points* (*cusps*) *of X'.*

3. When is a triple cover smooth

In this section, we shall give a criterion for a triple cover to be smooth. Our criterion is directly from the global data A_1, A_2, B_1, C_1. We first recall Miranda's local analysis. Let $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ be an affine flat triple cover over a nonsingular variety $\widetilde{X} = \operatorname{Spec}(R)$. Then $\widetilde{Y} = \operatorname{Spec}(R[z, w]/I)$, here I is the ideal generated by the following 3 equations:

$$\begin{aligned} z^2 &= \widetilde{a}z + \widetilde{b}w + 2\widetilde{A}, \\ zw &= -\widetilde{d}z - \widetilde{a}w - \widetilde{B}, \\ w^2 &= \widetilde{c}z + \widetilde{d}w + 2\widetilde{C}, \end{aligned} \tag{14}$$

where $\widetilde{A} = \widetilde{a}^2 - \widetilde{b}\widetilde{d}$, $\widetilde{B} = \widetilde{a}\widetilde{d} - \widetilde{b}\widetilde{c}$, $\widetilde{C} = \widetilde{d}^2 - \widetilde{a}\widetilde{c}$, and $\widetilde{\cdot}$ denotes the notation used in [**8**]. Let $m \subset R$ be the ideal of a point p in $\widetilde{X}$. Then $\widetilde{Y}$ is singular over p if and only if

(i) $\widetilde{a},\ \widetilde{b},\ \widetilde{c},\ \widetilde{d} \in m$,
(ii) $\widetilde{d} \notin m,\ \widetilde{a},\ \widetilde{c} \in m,\ \widetilde{b} \in m^2$,
(iii) $\widetilde{a} \notin m,\ \widetilde{b},\ \widetilde{d} \in m,\ \widetilde{c} \in m^2$,
(iv) $\widetilde{b} \notin m,\ \widetilde{A} \in m,\ \widetilde{b}\widetilde{B} - 2\widetilde{a}\widetilde{A} \in m^2$,
(v) $\widetilde{c} \notin m,\ \widetilde{C} \in m,\ \widetilde{c}\widetilde{B} - 2\widetilde{d}\widetilde{C} \in m^2$,
(vi) $\widetilde{b} \notin m,\ \widetilde{A} \notin m,\ \widetilde{D} \in m^2$,
(vii) $\widetilde{c} \notin m,\ \widetilde{C} \notin m,\ \widetilde{D} \in m^2$,

where $\widetilde{D} = \widetilde{B}^2 - 4\widetilde{A}\widetilde{C}$ defines the branch locus of $\widetilde{\pi}$. Note that the locus over which $\widetilde{\pi}$ is totally ramified is defined by the ideal $(\widetilde{A}, \widetilde{B}, \widetilde{C})$. In cases (i), (iv) and (v), $\widetilde{\pi}$ is totally ramified over p, in the other cases, $\widetilde{\pi}$ is simply ramified over p. In the last 4 cases, $\widetilde{Y}$ is locally defined by a cubic equation $z^3 + gz + h = 0$, then $\widetilde{Y}$ is singular over p if and only if

(I) $g \in m,\ h \in m^2$,
(II) $g \notin m, 4g^3 + 27h^2 \in m^2$.

Lemma 3.1. *In case* (i)*, let $\widetilde{p}$ be the inverse image of p in $\widetilde{Y}$. Then $\widetilde{Y}$ has multiplicity* 3 *at $\widetilde{p}$.*

Proof. Let $m_{\widetilde{p}} \subset \mathcal{O}_{\widetilde{p}}$ be the maximal ideal of $\widetilde{Y}$ at $\widetilde{p}$, and let $n = \dim X$. Then there is a polynomial $P(\ell) = \frac{\mu}{n!}\ell^n + k_1\ell^{n-1} \cdots + k_0$ of ℓ such that for $\ell >> 0$, $P(\ell) = \text{length}(\mathcal{O}_{\widetilde{p}}/m_{\widetilde{p}}^{\ell})$. By definition, μ is the multiplicity of $\widetilde{Y}$ at $\widetilde{p}$. From the equations (3.1) and the condition (i), it is easy to prove that $\mathcal{O}_{\widetilde{p}} = \mathcal{O}_p[z, w]/I$, $m_{\widetilde{p}} = (m + \mathcal{O}_p z + \mathcal{O}_p w)/I$ and

$$m_{\widetilde{p}}^{\ell-1}/m_{\widetilde{p}}^{\ell} = m^{\ell-1}/m^{\ell} + (m^{\ell-2}/m^{\ell-1})z + (m^{\ell-2}/m^{\ell-1})w.$$

We know that

$$\text{length}(m^{\ell}/m^{\ell+1}) = \frac{(\ell+1)(\ell+2)\cdots(\ell+n-1)}{(n-1)!}.$$

Note that $P(\ell) - P(\ell-1) = \text{length}(m_{\widetilde{p}}^{\ell-1}/m_{\widetilde{p}}^{\ell})$, and the leading term of $P(\ell) - P(\ell-1)$ is $\frac{\mu}{(n-1)!}\ell^{n-1}$. On the other hand,

$$\begin{aligned} \text{length}(m_{\widetilde{p}}^{\ell-1}/m_{\widetilde{p}}^{\ell}) &= \text{length}(m^{\ell-1}/m^{\ell}) + 2\,\text{length}(m^{\ell-2}/m^{\ell-1}) \\ &= \frac{3}{(n-1)!}\ell^{n-1} + \text{ terms of lower degree }, \end{aligned}$$

hence $\mu = 3$. □

Theorem 3.2. *Let $\pi : Y \to X$ be a flat triple cover over a smooth variety X of dimension n. Then Y is smooth if and only if $A_1 + A_2$ is smooth, $A_1 + A_2$ and $B_1 + C_1$ have no common points, and all of the singular points of $B_1 + C_1$ are cusps (i.e., locally defined by $x_1^2 + f(x_1, \cdots, x_n)^3 = 0,\ f(0, \cdots, 0) = 0$) where π is totally ramified.*

Proof. This is a local problem. We assume that p is a singular point of $D_{\text{red}} := A_1 + A_2 + B_1 + C_1$. Because π is flat over p, Y is locally defined by the equations (14), where $R = \mathcal{O}_{X,p}$. It is obvious that π is totally ramified over p, otherwise we know that Y is singular over p as it is locally a double cover. Thus

$$\widetilde{A}(p) = \widetilde{B}(p) = \widetilde{C}(p) = 0. \tag{15}$$

If $\widetilde{b}(p) = \widetilde{c}(p) = 0$, then from (15) we have $\widetilde{a}(p) = \widetilde{d}(p) = 0$. So Y is singular over p, a contradiction. Thus at least one of $\widetilde{b}$ and $\widetilde{c}$ is not vanishing at p. Then we can assume that the defining equation of π near p is

$$z^3 + gz + h = 0. \tag{16}$$

The surface defined by (3.3) is normal and smooth. For the data (g, h), $a'_2 = b'_1 = 1$ and c'_0 is a nonzero constant. So

$$g = a'_1 a'_0, \quad h = a'_1 b'_0.$$

Firstly, we claim that $g(p) = h(p) = 0$. Indeed, if $g(p) \neq 0$, then Y is singular over p because the branch locus $4g^3 + 27h^2 = 0$ is singular at p by assumption. Similarly, we can see that the hypersurface $h = 0$ is smooth at p.

Secondly, we claim that $a'_1(p) \neq 0$. Indeed, if $a'_1(p) = 0$, then $b'_0(p) \neq 0$ and $a'_1 = 0$ is smooth because the hypersurface of $h = a'_1 b'_0$ is smooth at p. On the other hand, the reduced branch locus D_{red} defined by $c'_1 a'_1 = 0$ is singular at p, where $c'_1 = 4a'_1 {a'_0}^3 + 27{b'_0}^2$, so $c'_1(p) = 0$ which implies $b'_0(p) = 0$, a contradiction. Hence a'_1 is invertible near p, i.e., $\gcd(g, h) = 1$ and π is totally ramified over p.

Therefore (D_{red}, p) is a cusp defined by $4g^3 + 27h^2 = 0$ lying on $(B_1 + C_1) \setminus (A_1 + A_2)$.

Conversely, we need to prove that Y is smooth over p. Assume that Y is singular over p. The branch locus $D = 2A_1 + 2A_2 + B_1 + C_1$, defined by $\widetilde{B}^2 - 4\widetilde{A}\widetilde{C} = 0$, has a double point at p, so at least one of $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$ is nonzero at p, (i) can not occur. (ii) and (iii) can not occur because in this case π is not totally ramified over p. So $\widetilde{b}$ or $\widetilde{c}$ is not vanishing at p, which means π is locally defined by (16). There are only two cases (I) and (II) as above. Case (II) can not occur since π is not totally ramified over p in this case. In case (I), because the variety defined by (16) is normal and $A_1 + A_2$ does not pass through p, we can see that $\gcd(g, h) = 1$ near p. By assumption, the branch locus defined by $4g^3 + 27h^2 = 0$ has a cusp at p, hence the hypersurface $h = 0$ is smooth at p. Then we know that X is smooth over p, a contradiction. This completes the proof. □

Remark 3.3. In Sect. 5, we shall discuss when π is totally ramified over a cusp.

4. Canonical resolution of the singularities

Theorem 4.1. *Let $\pi : Y \to X$ be a triple cover of a smooth surface X. Assume that Y is normal. Then there are a finite number blowing-ups $\sigma : \widetilde{X} \to X$ of X such that the following induced triple cover $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ has a smooth branch locus.*

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\widetilde{\sigma}} & Y \\ {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widetilde{X} & \xrightarrow[\sigma]{} & X \end{array}$$

where $\widetilde{Y}$ is the normalization of $\widetilde{X} \times_X Y$. So $\widetilde{Y}$ is a resolution of the singularities of Y.

Proof. Let $\pi : Y \to X$ be a triple cover determined by the data (s, t).

If $s = 0$, then the triple cover is Galois. In this case, we have given the canonical resolution of the singularities (see Sect. 1.2).

Now assume that $s \neq 0$. We denote by (a, b, c) the triple cover data corresponding to (s, t), so

$$a + b = c. \tag{17}$$

Consider the branch curve $A_1 + A_2 + B_1 + C_1$. If p is a singular point of the branch locus, then we blow up X at p, $\sigma_1 : X_1 \to X$. We have

$$\sigma_1^*(a) + \sigma_1^*(b) = \sigma_1^*(c). \tag{18}$$

Let a', b', c' be the corresponding sections obtained from (18) by eliminating the common factors from both sides. Then we have

$$a' + b' = c'. \tag{19}$$

From (1), we see that the three new sections (a', b', c') are the triple cover data corresponding to $(\sigma_1^*(s), \sigma_1^*(t))$. So the branch locus of the triple cover determined by (a', b', c') is contained in the total transform of $A_1 + A_2 + B_1 + C_1$.

By the well known embedded resolution of the singularities of a curve in a surface (see [**5**], p.391), there are a finite number of blowing-ups $\sigma' : \widetilde{X} \to X$ such that the curve ${\sigma'}^*(A + B + C)$ has at worst nodes as its singularities. So we can always assume that $A + B + C$ is normal crossing. From

(17) we see that any two of the curves A, B and C have no common intersection points. Hence the curves $A_1 + A_2 \subset A$, $B_1 \subset B$ and $C_1 \subset C$ are disjoint. Now we need to resolve the singularities of the three curves.

Let p be a singular point of $A_1 + A_2$, or B_1, or C_1, and let $\sigma : \widetilde{X} \to X$ be the blowing-up of X at p.

I) If $p \in A_1 \cap A_2$, then the local equation of A at p is $x^{3n+1}y^{3m+2} = 0$. We know that the multiplicity of the exceptional curve E in $\sigma^*(A)$ is $3(m + n + 1)$. So E is not in the branch locus of $\widetilde{\pi}$.

II) If p is a singular point of A_1 or A_2, then the local equation of A at p is $x^{3n+\varepsilon}y^{3m+\varepsilon} = 0$ ($\varepsilon = 1$ or 2). Then there are two new singular points of type I) on E, which can be resolved by two additional blowing-ups as in I).

III) If p is a singular point of B_1 (resp. C_1), then the local equation of B (resp. C) at p is $x^{2n+1}y^{2m+1} = 0$. So the multiplicity of E in $\sigma^*(B)$ (resp. $\sigma^*(C)$) is $2(m + n + 1)$. Hence E is not contained in the branch locus of $\widetilde{\pi}$. The singular point has been resolved. Therefore, after a finite number of blowing-ups, the branch locus of the induced triple cover $\widetilde{\pi}$ is smooth. □

Remark 4.2. The j-invariant $j = \frac{a}{c}$ of the triple cover data is a rational function on X, which defines a rational map from X to $\mathbb{P}^1$. The first step of the above proof is to resolve the "singularities" of j such that it induces a morphism to $\mathbb{P}^1$. The second step is just the canonical resolution of the singularities of Galois triple covers and double covers. $\sigma^*(j)$ is the j-invariant of the pull back triple cover data on $\widetilde{X}$. Note that the canonical resolution does not exist for covers of degree larger than 3.

Example 4.3. Consider the normalization $\widetilde{\Sigma}$ of the local surface Σ defined by

$$z^3 + x^2yz + y^4 = 0$$

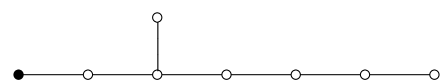
at $o = (0,0,0)$ in $\mathbb{C}^3$. The inverse image of o on $\widetilde{\Sigma}$ is a singular point. We can use the above method to resolve this singularity when we view $\widetilde{\Sigma}$ as a triple cover of $\mathbb{C}^2$. Note that in this case,

$$a = 4x^6, \quad b = 27y^5, \quad c = 4x^6 + 27y^5.$$

σ consists of 6 blowing-ups. We denote by E_i the strict transformation of the exceptional curve of the i-th blowing-up. In the induced triple cover data (a', b', c') after the i-th blowing-up, E_i is contained in at most one of A', B' and C'. If E_i is in A' (resp. B' or C') with multiplicity k_i, we shall assign E_i a "multiplicity" k_iA (resp. k_iB or k_iC). Then we see that the assigned multiplicities of $E_1, \cdots, E_6$ are respectively

$$A,\ 4B,\ 3B,\ 2B,\ B,\ 0,$$

where 0 means E_6 is not in A, B, C. Thus E_2, E_4 and E_6 are not in the branch locus, E_1 is in A_1, E_3 and E_5 are in B_1. The self-intersection numbers of $E_1, \cdots, E_6$ are respectively $-6, -1, -2, -2, -2, -2$. After blowing down two (-1)-curves in the exceptional curves, we get the minimal resolution. The dual graph of the exceptional curves of the minimal resolution is



where $\bullet$ is a (-3)-curve and $\circ$ is a (-2)-curve. Thus the singular point is a rational triple point. Please note that outside of the totally ramified locus, the triple cover is factorized as a double cover and a one-to-one cover.

Remark 4.4. T. Ashikaga gives in [**2**] a resolution of the singularities of some special triple covers, i.e., the triple cover itself is defined by a cubic equation and without normalization. In our language, it is the special case when $A_2 = B_1 = C_0 = 0$. In this case, all the singular points are hypersurface singularities.

5. Codimension 2 totally ramified locus

Let $\pi : Y \to X$ be a triple cover with data as above. X is a smooth variety of dimension $n \geq 2$. We denote by D_2 the locus over which there is a total ramification, and by $U \subset X$ the Zariski open subset of the flat points of π. Then we know that $X \setminus U$ has codimension at least 3 since X is nonsingular. On U, D_2 is defined locally by the ideal $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ which are the 3 maximal minors of the matrix

$$\begin{pmatrix} \widetilde{a} & \widetilde{c} & \widetilde{d} \\ \widetilde{b} & \widetilde{d} & \widetilde{a} \end{pmatrix}.$$

Hence the codimension of D_2 at any point on U is ≤ 2. Obviously, π is totally ramified over $X \setminus U$ because normal double covers over nonsingular varieties are always flat. So we pay our attention to the flat case. We have known that the codimension 1 part of D_2 is $A_1 + A_2$. In this section, we shall try to find the codimension 2 part ${D_2}'$ of D_2.

Theorem 5.1. *On $X \setminus A_0 \cap B_0 \cap C_0$, the codimension 2 part of D_2 is $A_0 \cap (B_1 + C_1)$. In particular, the non-divisorial part of D_2 is contained in A_0.*

Proof. Note that π is flat over $X \setminus Z$ ($Z = F_1 \cap F_2 \cap F_3$). So we have (14) locally. Then the totally ramified branch locus is defined by $\widetilde{A} = \widetilde{B} = \widetilde{C} = 0$. It is enough to consider the points $x \notin A_1 + A_2$, hence $x \in B_1 + C_1$ if π is totally ramified over x. If $x \in X \setminus F_1$, we choose a local base $z = \alpha$, $w = (-f_2\alpha + f_1\beta)/f_3$ of $\mathcal{E}_\pi$. By the computations in the proof of Theorem 3.5 of [**13**], we have

$$\begin{aligned} (\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}) &= \left(\frac{3b_1 f_2}{2a_0}, \frac{3b_1 f_3}{2a_0}, \frac{c_1 f_2}{18a_0}, \frac{f_3 f c_1}{18a_0} \right), \\ (\widetilde{A}, \widetilde{B}, \widetilde{C}) &= \left(-\frac{1}{3} a_1 b_1 a_2^2 a_0, \, 0, \, \frac{1}{81} a_1 a_2^2 c_1 a_0 \right), \end{aligned}$$

From (4), we can see that on $X \setminus F_1$ the only total ramification locus is $A_1 + A_2$. If $x \in X \setminus F_2$, then we choose a base $z = \beta$, $w = (-f_2\alpha + f_1\beta)/f_3$. Then we have

$$\begin{aligned} (\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}) &= \left(-\frac{1}{3} a_1 a_2 a_0, \; a_1 f_3, \; \frac{1}{27} a_2 c_1, \; 0 \right), \\ (\widetilde{A}, \widetilde{B}, \widetilde{C}) &= \left(\frac{1}{9} a_1^2 a_2^2 a_0^2, \; -\frac{1}{27} a_1 f_3 a_2 c_1, \; \frac{1}{81} a_1 a_2^2 c_1 a_0 \right). \end{aligned}$$

D_2' is equal to $A_0 \cap (B_1 + C_1)$ on $X \setminus F_2$. If $x \in X \setminus F_3$, we choose a base $z = \alpha$, $w = \beta$. Then we have

$$\begin{aligned} (\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}) &= \left(0, \; b_1 a_2, \; -a_1 f_2, \; \frac{1}{3} a_1 a_2 a_0 \right), \\ (\widetilde{A}, \widetilde{B}, \widetilde{C}) &= \left(-\frac{1}{3} a_1 b_1 a_2^2 a_0, \; a_1 b_1 a_2 f_2, \; \frac{1}{9} a_1^2 a_2^2 a_0^2 \right). \end{aligned}$$

So D_2' is also equal to $A_0 \cap (B_1 + C_1)$ on $X \setminus F_3$.

Now note that $F_1 = A_2 + A_0$, $F_2 = B_0$ and $F_3 = C_0$. Hence on $X \setminus A_0 \cap B_0 \cap C_0$, the theorem is truce. □

It is not easy to give a similar criterion for the points in $A_0 \cap B_0 \cap C_0$. In the surface case, we can overcome this difficulty by using the canonical resolution given in Sect. 4, because there is no total ramification over the smooth part of $(B_1 + C_1) \setminus (A_1 + A_2)$ (otherwise the branch locus defined by $\widetilde{B}^2 - 4\widetilde{A}\widetilde{C} = 0$ is singular as $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ are vanishing at the total ramification locus).

Now we consider the double cover base change φ as in Corollary 2.3.

$$\begin{array}{ccccc} \widetilde{Y} & \xrightarrow{\widetilde{\varphi}'} & Y' & \xrightarrow{\varphi'} & Y \\ \widetilde{\pi}\downarrow & & \pi'\downarrow & & \downarrow\pi \\ \widetilde{X} & \xrightarrow[\widetilde{\varphi}]{} & X' & \xrightarrow[\varphi]{} & X \end{array}$$

Assume that X is a smooth surface. Let $p \in B_1 + C_1$. Then π is totally ramified over p if and only if π' is totally ramified over $p' = \varphi^{-1}(p)$. If p is a singular point of $B_1 + C_1$, then p' is a singular point of X'. Let $\widetilde{\varphi} : \widetilde{X} \to X'$ be the canonical resolution of X' at p', and let $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ be the pull back of π'. Then it is easy to see that π' is totally ramified over p' if and only if $\widetilde{\pi}^{-1}(\widetilde{\varphi}^{-1}(p'))$ is connected. This can be verified from the data (a, b, c).

Corollary 5.2. *With the assumptions as in Theorem 3.2. Assume that X is a smooth surface and (a, b, c) is the triple cover data of π. Let p be an ordinary cusp of $B_1 + C_1$. If p is not on $A_1 + A_2$, then π is totally ramified over p if and only if*

$$\varepsilon_p := \mu_p(a) - \mu_p(b) > 0, \quad \varepsilon_p \not\equiv 0 \pmod 3, \tag{20}$$

where $\mu_p(a)$ denotes the multiplicity of a at p.

Proof. We know that p' is a rational double point of type A_2. Hence the exceptional curve of $\widetilde{\varphi}$ over p' is a curve of type A_2, i.e., two (-2)-curves E_1 and E_2 meeting at one point. Because $\widetilde{X}$ is smooth and $\widetilde{\pi}$ is cyclic, $\widetilde{\pi}$ has no isolated branch points. Note that $\widetilde{\pi}$ is unramified near E_1+E_2, and that E_1+E_2 is simply connected. Hence $\widetilde{\pi}^{-1}(E_1+E_2)$ is connected if and only if at least one of E_1 and E_2 is in the branch locus. Now we consider the canonical resolution of p',

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\varphi}} & X' \\ {\scriptstyle \widetilde{\sigma}_1}\downarrow & & {\scriptstyle \varphi}\downarrow \\ X_1 & \xrightarrow[\sigma_1]{} & X \end{array}$$

where σ_1 is the blowing-up of X at p with exceptional curve E. Then E_1+E_2 is the pull back of E. The above condition is equivalent to that E is contained in the total ramification locus of the new triple cover data $(\sigma_1^*(a),\sigma_1^*(b),\sigma_1^*(c))$, which is just (20). □

Let p be a cusp of B_1+C_1 defined by $x^2+y^n=0$. Consider the canonical resolution of p':

$$X_r \overset{\sigma_r}{\to} X_{r-1} \overset{\sigma_{r-1}}{\to} \cdots \to X_1 \overset{\sigma_1}{\to} X_0 = X$$

We denote by $p_0=p, p_1, \cdots, p_{r-1}$ the infinitely near points of (B_1+C_1,p) and by $(a(i),b(i),c(i))$ the pull back triple cover data on X_i. Then we have

Corollary 5.3. *π is totally ramified over p if and only if there is an i such that the data $(a(i),b(i),c(i))$ satisfies (20) at p_i.*

Proof. The proof is similar. □

6. Invariants of a triple cover

In this section, we compute the global invariants of a triple cover $\pi: Y\to X$ of a smooth surface X. Let

$$D_1 = B_1+C_1, \quad D_2 = A_1+A_2.$$

Then

$$\pi^*(D_2) = 3\widehat{D}_2, \quad \pi^*(D_1) = 2\widehat{D}_1+\widehat{D}_1'.$$

Lemma 6.1. *Assume that the branch locus of the triple cover π is smooth. If Γ_i is a reduced and irreducible curve in D_i, then $\pi^*(\Gamma_2)=3\widehat{\Gamma}_2$, $\pi^*(\Gamma_1)=2\widehat{\Gamma}_1+\widehat{\Gamma}_1'$. We have*

$$\widehat{\Gamma}_2^2 = \frac{1}{3}\Gamma_2^2, \quad \widehat{\Gamma}_1^2=\frac{1}{2}\Gamma_1^2, \quad \widehat{\Gamma}_1'^2=\Gamma_1^2, \quad \widehat{\Gamma}_1\widehat{\Gamma}_1'=0.$$

Hence

$$K_Y = \pi^*(K_X)+\widehat{D}_1+2\widehat{D}_2.$$

The Chern classes of $\mathcal{E}_\pi$ are denoted by c_1, c_2. By Theorem 1.3, we have (see [**13**] or [**8**])

$$c_1 = -A_1-A_2-\frac{1}{2}B_1-\frac{1}{2}C_1, \tag{21}$$

In order to give a simple formula for c_2, we will use the following equality (see Theorem 1.3, (4.)),

$$c_1^2 - 4c_2 = c_1(\mathcal{F})^2 - 4c_2(\mathcal{F}),$$

as the discriminant $c_1^2 - 4c_2$ is invariant up to tensoring a line bundle. Then

$$c_1(\mathcal{F})^2 - 4c_2(\mathcal{F}) = \Gamma^2 - 4\deg\Delta(F),$$

where $\Gamma = F_2 + F_3 - F_1$ and $\Delta(F) = F_2F_3 - Z(F)$ is the residue subscheme of $Z(F)$ in F_2F_3 whose degree is $F_2F_3 - \deg Z(F)$ (see [**15**]). Hence

$$c_2 = \frac{1}{4}c_1^2 - \frac{1}{4}\Gamma^2 + \deg\Delta(F). \tag{22}$$

By (1.8) and Riemann-Roch Theorem, we have

$$\chi(\mathcal{O}_Y) = 3\chi(\mathcal{O}_X) + \frac{1}{2}(c_1^2 - c_1K_X) - c_2. \tag{23}$$

Proposition 6.2. *Assume that the branch locus of $\pi : Y \to X$ is smooth. Then*

$$\begin{aligned}
\chi(\mathcal{O}_Y) &= 3\chi(\mathcal{O}_X) + \frac{1}{8}D_1^2 + \frac{1}{4}D_1K_X + \frac{5}{18}D_2^2 + \frac{1}{2}D_2K_X,\\
K_Y^2 &= 3K_X^2 + \frac{1}{2}D_1^2 + 2D_1K_X + \frac{4}{3}D_2^2 + 4D_2K_X,\\
\chi_{top}(Y) &= 3\chi_{top}(X) + D_1^2 + D_1K_X + 2(D_2^2 + D_2K_X).
\end{aligned}$$

Proof. By Hurwitz Formula, $K_Y = \pi^*(K_X) + \widehat{D}_1 + 2\widehat{D}_2$. From Lemma 6.1, we have

$$\begin{aligned}
K_Y^2 &= (\pi^*(K_X))^2 + 2\pi^*(K_X)(\widehat{D}_1 + 2\widehat{D}_2) + (\widehat{D}_1 + 2\widehat{D}_2)^2\\
&= 3K_X^2 + 2K_X(\pi_*(\widehat{D}_1 + 2\widehat{D}_2) + \widehat{D}_1^2 + 4\widehat{D}_2^2\\
&= 3K_X^2 + 2K_X(D_1 + 2D_2) + \frac{1}{2}D_1^2 + \frac{4}{3}D_2^2.
\end{aligned}$$

Thus we get the second formula.

For the third formula, we use $\chi_{\mathrm{top}}(C) = -C^2 - CK_X$,

$$\begin{aligned}
\chi_{\mathrm{top}}(Y) &= \chi_{\mathrm{top}}(Y \setminus (\widehat{D}_1 + \widehat{D}_1' + \widehat{D}_2)) + \chi_{\mathrm{top}}(\widehat{D}_1 + \widehat{D}_1' + \widehat{D}_2)\\
&= 3\chi_{\mathrm{top}}(X \setminus (D_1 + D_2)) + 2\chi_{\mathrm{top}}(D_1) + \chi_{\mathrm{top}}(D_2)\\
&= 3\chi_{\mathrm{top}}(X) - \chi_{\mathrm{top}}(D_1) - 2\chi_{\mathrm{top}}(D_2)\\
&= 3\chi_{\mathrm{top}}(X) + D_1^2 + D_1K_X + 2(D_2^2 + D_2K_X).
\end{aligned}$$

We can get the first formula by using Noether equality $\chi(\mathcal{O}_Y) = \frac{1}{12}(K_Y^2 + \chi_{\mathrm{top}}(Y))$ and the second and third formulas. □

By the canonical resolution of the singularities of a triple cover, we have the following commutative diagram.

$$\begin{array}{ccccccccccc}
Y_k & \xrightarrow{\tau_k} & Y_{k-1} & \xrightarrow{\tau_{k-1}} & \cdots & \longrightarrow & Y_2 & \xrightarrow{\tau_2} & Y_1 & \xrightarrow{\tau_1} & Y\\
\downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_{k-1}} & & & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi}\\
X_k & \xrightarrow[\sigma_k]{} & X_{k-1} & \xrightarrow[\sigma_{k-1}]{} & \cdots & \longrightarrow & X_2 & \xrightarrow[\sigma_2]{} & X_1 & \xrightarrow[\sigma_1]{} & X
\end{array}$$

σ_{i+1} is the blowing-up of the X_i at a singular point p_i of the branch locus of π_i. Y_{i+1} is the normalization of $X_{i+1}\times_{X_i} Y_i$. $Y_0=Y, X_0=X, \pi_0=\pi$. We know after a finite number of blowing-ups, π_k has a smooth branch locus. So Y_k is smooth. We denote by $\widetilde{\pi}:\widetilde{Y}\to\widetilde{S}$ the last triple cover, and by $a_j^{(i)}$, $b_j^{(i)}$ $c_j^{(i)}$ the corresponding data of π_i.

We denote by $\mu_p(C)$ the multiplicity of C at p, and put

$$(24)\qquad \begin{cases} u_i' = \left[\dfrac{\mu_{p_i}(B_1^{(i)})-d_i}{2}\right], \\ v_i' = \left[\dfrac{\mu_{p_i}(C_1^{(i)})-d_i}{2}\right], \end{cases} \quad \begin{cases} u_i = \left[\dfrac{\mu_{p_i}(A_1^{(i)})+2\mu_{p_i}(A_2^{(i)})-d_i}{3}\right], \\ v_i = \left[\dfrac{2\mu_{p_i}(A_1^{(i)})+\mu_{p_i}(A_2^{(i)})+d_i}{3}\right], \end{cases}$$

where $d_i=\min\{\mu_{p_i}(A^{(i)}),\mu_{p_i}(B^{(i)}),\mu_{p_i}(C^{(i)})\}$. ¿From $a^{(i)}+b^{(i)}=c^{(i)}$, we can see that at most one of the multiplicities $\mu_{p_i}(A^{(i)})$, $\mu_{p_i}(B^{(i)})$ and $\mu_{p_i}(C^{(i)})$ is bigger than d_i.

Let

$$m_i=u_i'+v_i'+d_i,\qquad n_i=u_i+v_i.$$

Then $m_i=\left[\mu_{p_i}(D_1^{(i)})/2\right]$, and

$$n_i=\begin{cases} \mu_{p_i}(A_1^{(i)})+\mu_{p_i}(A_2^{(i)}), & \text{if } d_i\equiv\mu_{p_i}(A^{(i)}) \pmod 3; \\ \mu_{p_i}(A_1^{(i)})+\mu_{p_i}(A_2^{(i)})-1, & \text{otherwise.}\end{cases}$$

Note that u_i', v_i', n_i and u_i are not necessarily non-negative, but m_i and v_i are non-negative and $n_i\geq -1$.

Let E_i be the exceptional curve of σ_i, let $\mathcal{E}_i$ be the total transform of E_i in $\widetilde{X}=X_k$, and let σ be the composition of the blowing-ups. Then we have

$$(25)\qquad \begin{cases} \widetilde{B}_1 &=& \sigma^*(B_1)-\sum_{i=0}^{k-1}(2u_i'+d_i)\mathcal{E}_{i+1}, \\ \widetilde{C}_1 &=& \sigma^*(C_1)-\sum_{i=0}^{k-1}(2v_i'+d_i)\mathcal{E}_{i+1}, \\ \widetilde{A}_1 &=& \sigma^*(A_1)-\sum_{i=0}^{k-1}(2v_i-u_i-d_i)\mathcal{E}_{i+1}, \\ \widetilde{A}_2 &=& \sigma^*(A_2)-\sum_{i=0}^{k-1}(2u_i-v_i+d_i)\mathcal{E}_{i+1}, \\ \widetilde{A}_0 &=& \sigma^*(A_0)+\sum_{i=0}^{k-1}u_i\mathcal{E}_{i+1}, \\ \widetilde{B}_0 &=& \sigma^*(B_0)+\sum_{i=0}^{k-1}u_i'\mathcal{E}_{i+1}, \\ \widetilde{C}_0 &=& \sigma^*(C_0)+\sum_{i=0}^{k-1}v_i'\mathcal{E}_{i+1}. \end{cases}$$

In particular, we get

$$(26)\qquad \begin{cases} \widetilde{D}_1 &=& \sigma^*(D_1)-2\sum_{i=0}^{k-1}m_i\mathcal{E}_{i+1}, \\ \widetilde{D}_2 &=& \sigma^*(D_2)-\sum_{i=0}^{k-1}n_i\mathcal{E}_{i+1}, \\ K_{\widetilde{X}} &=& \sigma^*(K_X)+\sum_{i=0}^{k-1}\mathcal{E}_{i+1}. \end{cases}$$

Note that

$$\mathcal{E}_i^2=-1,\quad \mathcal{E}_i\mathcal{E}_j=0\ \ (i\neq j),\quad \sigma^*(\cdot)\mathcal{E}_i=0.$$

Hence we have the following formulas.

Theorem 6.3.

$$\begin{aligned}\chi(\mathcal{O}_{\widetilde{Y}}) &= 3\chi(\mathcal{O}_X) + \frac{1}{8}D_1^2 + \frac{1}{4}D_1K_X + \frac{5}{18}D_2^2 + \frac{1}{2}D_2K_X \\ &\quad -\sum_{i=0}^{k-1}\frac{m_i(m_i-1)}{2} - \sum_{i=0}^{k-1}\frac{n_i(5n_i-9)}{18}, \\ K_{\widetilde{Y}}^2 &= 3K_X^2 + \frac{1}{2}D_1^2 + 2D_1K_X + \frac{4}{3}D_2^2 + 4D_2K_X \\ &\quad -\sum_{i=0}^{k-1}2(m_i-1)^2 - \sum_{i=0}^{k-1}\frac{4n_i(n_i-3)}{3} - k.\end{aligned}$$

The form of the formulas is not unique, because $\widetilde{A}_1\widetilde{A}_2 = 0$, A_1A_2 can be calculated from the local data of the singularities. We shall give a new formula for $\chi(\mathcal{O}_X)$ such that it is easy to be used to control the singularity, and the error term is the invariant of the resolution.

By the definition of the geometric genus of a singularity, if we denote by $p_g(\tau)$ the sum of the geometric genera of the singular points of Y, then we have $\chi(\mathcal{O}_{\widetilde{Y}}) = \chi(\mathcal{O}_Y) - p_g(\tau)$, where $\tau : \widetilde{Y} \to Y$ is the canonical resolution. In this case, we can use (23) to compute $\chi(\mathcal{O}_Y)$ and $\chi(\mathcal{O}_{\widetilde{Y}})$. Then we have

$$\chi(\mathcal{O}_{\widetilde{Y}}) = 3\chi(\mathcal{O}_X) + \frac{1}{2}(c_1^2 - c_1K_X) - c_2 - p_g(\tau). \tag{27}$$

Hence

$$p_g(\tau) = -\frac{1}{2}((\widetilde{c}_1^2 - c_1^2) - (\widetilde{c}_1K_{\widetilde{X}} - c_1K_X)) + (\widetilde{c}_2 - c_2).$$

Now we use (21) and (26) to compute c_1, and use (22) and (25) to compute c_2. Note that $\Gamma = B_0 + C_0 - A_0 - A_2$. Then we can get easily

$$\begin{cases} \widetilde{c}_1^2 - c_1^2 = -\sum_{i=0}^{k-1}(m_i+n_i)^2, \\ \widetilde{c}_1K_{\widetilde{X}} - c_1K_X = -\sum_{i=0}^{k-1}(m_i+n_i), \\ \widetilde{c}_2 - c_2 = -\sum_{i=0}^{k-1}v_i^2 + \sum_{i=0}^{k-1}(m_i+n_i)v_i + \deg\widetilde{\Delta} - \deg\Delta, \end{cases} \tag{28}$$

where Δ (resp. $\widetilde{\Delta}$) is the residue subscheme of $Z(F)$ (resp. $Z(\widetilde{F})$) in B_0C_0 (resp. $\widetilde{B}_0\widetilde{C}_0$). Thus we have the following.

Theorem 6.4. *Let $w_i = m_i + n_i$. Then the sum of the geometric genera of the singularities on Y is*

$$p_g(\tau) = \sum_{i=0}^{k-1}\left(\frac{1}{2}w_i^2 - w_iv_i + v_i^2 - \frac{1}{2}w_i\right) + \deg\widetilde{\Delta} - \deg\Delta.$$

We shall also give a new formula for K^2 in terms of the Chern classes of $\mathcal{E}_\pi$. Assume that X is smooth. Then Y can be embedded in $\mathbb{P}(\mathcal{E}_\pi)$ such that the triple $\pi : Y \to X$ is induced by the projection $p : \mathbb{P}(\mathcal{E}_\pi) \to X$. We have

$$\begin{aligned} K_{\mathbb{P}(\mathcal{E}_\pi)} &\equiv -2H + p^*(c_1) + p^*(K_X), \\ Y &\equiv 3H - 2p^*(c_1), \end{aligned}$$

where $H = \mathcal{O}_{\mathbb{P}(\mathcal{E}_\pi)}(1)$. Thus we get $K_Y = (K_{\mathbb{P}(\mathcal{E}_\pi)} + Y)|_Y$ and

$$K_Y^2 = 3K_X^2 + 2c_1^2 - 4c_1K_X - 3c_2. \tag{29}$$

Theorem 6.5. *Using* (28) *and* (29) *for* $\widetilde{Y} \to \widetilde{X}$, *we have*

$$\begin{aligned} K_{\widetilde{Y}}^2 &= 3K_X^2 + 2c_1^2 - 4c_1K_X - 3c_2 \\ &\quad - \sum_{i=0}^{k-1} \left(2w_i^2 + 3w_iv_i - 3v_i^2 - 4w_i + 3\right) - 3\left(\deg \widetilde{\Delta} - \deg \Delta\right). \end{aligned}$$

7. Construct cubic equations from Miranda's triple cover data

We shall show that any triple cover over a factorial variety X is determined by some (minimal) triple cover data (s, t). Two triple covers $\pi_1 : Y_1 \to X$ and $\pi_2 : Y_2 \to X$ are said to be isomorphic to each other if there is an isomorphism $\sigma : Y_1 \to Y_2$ such that $\pi_1 = \pi_2 \circ \sigma$.

Let U be an open subvariety of X. Then any triple cover $\pi : Y \to X$ induces a new triple cover $\pi_U : \pi^{-1}(U) \to U$.

Lemma 7.1. *Let A be a closed subset of X of codimension at least two, and let $U = X \setminus A$. If the induced triple covers π_{1U} and π_{2U} are isomorphic, so are π_1 and π_2.*

Proof. Since Y_i is isomorphic to $\operatorname{Spec} \pi_{i*}\mathcal{O}_{Y_i}$ over X, we only need to show that $\pi_{1*}\mathcal{O}_{Y_1}$ and $\pi_{2*}\mathcal{O}_{Y_2}$ are isomorphic $\mathcal{O}_X$-algebras. We know that they are both reflexive (cf. [**6**], §1). By hypothesis, $\pi_{1*}\mathcal{O}_{Y_1}|_U$ and $\pi_{2*}\mathcal{O}_{Y_2}|_U$ are isomorphic $\mathcal{O}_U$-algebras. Now we know $\pi_{1*}\mathcal{O}_{Y_1}$ is isomorphic to $\pi_{2*}\mathcal{O}_{Y_2}$ as $\mathcal{O}_X$-modules because there are both reflexive. Obviously, this isomorphism preserves the $\mathcal{O}_X$-algebra structures. This completes the proof. □

Theorem 7.2. *Let $\pi : Y \to X$ be a triple cover of a factorial variety. If* $\operatorname{char} k \neq 3$, *then π is determined by some minimal triple cover data (s, t).*

Proof. Since $\operatorname{char} k \neq 3$, the trace map $tr. : \pi_*\mathcal{O}_Y \to \mathcal{O}_X$ is nonzero, and the trace-free subsheaf $\mathcal{E}_0$ of $\pi_*\mathcal{O}_Y$ is a rank 2 reflexive sheaf. In this case the trace map splits $\pi_*\mathcal{O}_Y$ as $\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{E}_0$. We consider a maximal rank one subsheaf of $\mathcal{E}_0$ with a torsion-free quotient. Then this subsheaf is reflexive and hence invertible because X is factorial (cf. [**6**]). We denote by $\mathcal{L}^\vee$ this invertible subsheaf. Obviously, the quotient sheaf $\mathcal{E}_0/\mathcal{L}^\vee$ is isomorphic to $\mathcal{I}_A \otimes \mathcal{M}^\vee$, where $\mathcal{M}^\vee = (\mathcal{E}_0/\mathcal{L}^\vee)^{**}$ is an invertible sheaf and $\mathcal{I}_A$ is the ideal sheaf of a subscheme A of codimension at least two. Let $U = X \setminus A$, and let $\{\, U_i \mid i \in I \,\}$ be an affine open cover of U such that $\mathcal{L}^\vee$ and $\mathcal{E}_0$ are trivial over each U_i. Let z_i and $\{z_i,\ w_i\}$ be respectively the local bases of $\mathcal{L}^\vee$ and $\mathcal{E}_0$ on U_i. Then on $U_i \cap U_j$, we have

$$\begin{pmatrix} z_i \\ w_i \end{pmatrix} = \begin{pmatrix} \ell_{ij} & 0 \\ \delta_{ij} & m_{ij} \end{pmatrix} \begin{pmatrix} z_j \\ w_j \end{pmatrix}, \tag{30}$$

where ℓ_{ij} and m_{ij} are respectively the transition functions of $\mathcal{L}^\vee$ and $\mathcal{M}^\vee$. Since $\{1, z_i, w_i\}$ is a base of the $\mathcal{O}_{U_i}$-algebra $\pi_*\mathcal{O}_Y|_{U_i}$ as an $\mathcal{O}_{U_i}$-module. Hence we have

$$z_i^2 = \widetilde{b}_i w_i + \widetilde{a}_i z_i + 2\widetilde{A}_i, \tag{31}$$

z_i can be viewed as a rational function on Y, but not a rational function on X. Hence the minimal polynomial of z_i over the function field $K(X)$ is of degree 3. This means $\widetilde{b}_i$ is nonzero for each i.

Substituting z_i and w_i in (31) by (30), we have

$$z_j^2 = m_{ij}\ell_{ij}^{-2}\widetilde{b}_i w_j + (\widetilde{a}_i\ell_{ij}^{-1} + \delta_{ij}\ell_{ij}^{-2}\widetilde{b}_i)z_j + \ell_{ij}^{-2}2\widetilde{A}_i.$$

Thus, on $U_i \cap U_j$, we have

$$\widetilde{b}_j = m_{ij}\ell_{ij}^{-2}\widetilde{b}_i, \quad \widetilde{A}_j = \ell_{ij}^{-2}\widetilde{A}_i,$$

hence $\{\widetilde{b}_i\}$ (resp. $\{\widetilde{A}_i\}$) is a section $\widetilde{b}$ of $\mathcal{M}^{-1}\otimes\mathcal{L}^2$ (resp. $\mathcal{L}^2$) over U, here $\mathcal{L}$ and $\mathcal{M}$ are respectively the dual of $\mathcal{L}^\vee$ and $\mathcal{M}^\vee$. Let $\widetilde{\mathcal{B}}$ be the divisor of $\widetilde{b}$ on U.

Since z_i can be viewed as an endomorphism of $\pi_*\mathcal{O}_Y|_{U_i}$ over $\mathcal{O}_{U_i}$ by multiplication, by choice, the trace of z_i is zero. Hence the characteristic polynomial of z_i over $\mathcal{O}_{U_i}$ is of the form:

$$z_i^3 + s_i z_i + t_i = 0.$$

With the same proof as above, we see that $\{s_i\}$ and $\{t_i\}$ are respectively two sections of $\mathcal{L}^2$ and $\mathcal{L}^3$ over U. Because X is factorial and A has codimension at least two, these two sections can be extended respectively to two global sections $s\in H^0(\mathcal{L}^2)$ and $t\in H^0(\mathcal{L}^3)$.

Now we can construct a new triple cover $\pi' : Y' \to X$ by

$$z^3 + sz + t = 0.$$

Note that on $U_i \setminus \widetilde{\mathcal{B}}$, z_i generates $\pi_*\mathcal{O}_Y$ since $w_i \in \mathcal{O}_{U_i\setminus\widetilde{\mathcal{B}}}[z_i]$ by (31). In fact, $\pi_*\mathcal{O}_Y|_{U_i}$ is exactly the normalization of $\mathcal{O}_{U_i}[z_i]$ in the function field $K(Y)$, as w_i is integral over $\mathcal{O}_{U_i}[z_i]$. Hence $\pi'_*(\mathcal{O}_{Y'})|_U = \pi_*(\mathcal{O}_Y)|_U$. That is to say π'_U and π_U are isomorphic triple covers of U. By Lemma 7.1, π is nothing but π', which is what we wanted. □

Note that Miranda's triple cover data for flat triple cover $\pi : Y \to X$ is a rank 2 vector bundle $\mathcal{E}$ with a morphism $\Phi : S^3\mathcal{E} \to \wedge^2\mathcal{E}$. Giving the morphism Φ is equivalent to giving the local data $\widetilde{a}, \widetilde{b}, \widetilde{c}$, and $\widetilde{d}$ as in (14). From the above proof, we can see easily that if $\mathcal{L}^\vee$ is a maximal invertible subsheaf of $\mathcal{E}$ and if we choose a local base z, w of $\mathcal{E}$ (outside of a codimension 2 subsset) such that z is a local base of $\mathcal{L}^\vee$, then $\widetilde{b}\neq 0$, $\widetilde{b}\widetilde{d}-\widetilde{a}^2$ and $3\widetilde{a}\widetilde{b}\widetilde{d}-2\widetilde{a}^3-\widetilde{b}^2\widetilde{c}$ are respectively the global sections of $\mathcal{L}^2\otimes\mathcal{M}^{-1} = \mathcal{L}^3\otimes\det\mathcal{E}$, $\mathcal{L}^2$ and $\mathcal{L}^3$. In particular, the cubic equation defining $\pi : Y \to X$ is

$$z^3 + 3(\widetilde{b}\widetilde{d}-\widetilde{a}^2)z + (3\widetilde{a}\widetilde{b}\widetilde{d}-2\widetilde{a}^3-\widetilde{b}^2\widetilde{c}) = 0.$$

For a given triple cover $\pi : Y \to X$, our data $(s, t, \mathcal{L})$ is determined uniquely by its j-invariant $j = j(s,t)$, i.e., determined by a rational function j on X. Therefore, there is a one to one correspondence between the following two sets.

$$\{\, j\in K(X) \,|\, \pi_j = \pi \,\} \;\leftrightarrow\; \{\, \mathcal{L} \,|\, \mathcal{L}^\vee \subset \mathcal{E}_\pi \text{ is an invertible subsheaf } \}$$

Corollary 7.3. *Let $\mathcal{E}$ be the trace-free sheaf of a triple cover. Then for any invertible subsheaf $\mathcal{L}^\vee$ of $\mathcal{E}$, we have*

$$H^0(\mathcal{L}^3\otimes\det\mathcal{E})\neq 0, \quad H^0(\mathcal{L}^3)\neq 0.$$

If the triple cover is not Galois, then $H^0(\mathcal{L}^2)\neq 0$.

Proof. One can reduce the proof to the case where $\mathcal{L}^\vee$ is a maximal invertible subsheaf of $\mathcal{E}$. Then use the proof of Theorem 7.2. □

Corollary 7.4. *Let $\mathcal{E}$ be the trace-free sheaf of a triple cover determined by $(s, t, \mathcal{L})$. Let $\{z_i, w_i\}$ be the base of $\mathcal{E}$ such that $\{z_i\}$ generates the subsheaf $\mathcal{L}^\vee$ of $\mathcal{E}$. Then we have*

$$\widetilde{b} = a_2 b_1 c_0, \quad \widetilde{A} = -\frac{s}{3}.$$

Proof. In the proof of Theorem 7.2, we have seen that $\widetilde{b}$ and $\widetilde{A}$ are global sections of invertible sheaves. Thus we only need to prove the equalities on $U = X \setminus (F_1 \cap F_3)$. In the proof of Theorem 5.1, we computed $\widetilde{b}$ and $\widetilde{A}$ on $X \setminus F_1$ and $X \setminus F_3$, where $z = \alpha$ is part of the base. Obviously the equalities hold true on U. This proves the corollary. □

In what follows, we are going to find the relationships between rank 2 reflexive coherent sheaves and triple covers. We shall prove that up to tensoring a line bundle, any rank 2 reflexive sheaf is isomorphic to the trace-free sheaf of some triple cover.

Lemma 7.5. *Let A be a codimension ≥ 2 subscheme of a factorial variety X. Then any triple cover of $X \setminus A$ can be extended uniquely to a triple cover of X.*

Proof. Let $U = X \setminus A$, and let $\pi_U : V \to U$ be a triple cover. By Theorem 7.2, we can find an invertible sheaf $\mathcal{L}_U$ on U and sections $s \in H^0(U, \mathcal{L}_U^2)$, $t \in H^0(U, \mathcal{L}_U^3)$, such that π_U is defined by

$$z^3 + sz + t = 0.$$

Since X is factorial and A has codimension ≥ 2, $\mathrm{Pic}(X) \cong \mathrm{Pic}(U)$. Thus $\mathcal{L}_U$ is the restriction of an invertible sheaf $\mathcal{L}$ on X, and the sections s and t can be extended respectively to global sections of $\mathcal{L}^2$ and $\mathcal{L}^3$ on X. From these sections and $\mathcal{L}$, we can construct a triple cover of X, which is obviously the extension of π_U.

The uniqueness has been proved in Lemma 7.1. □

Theorem 7.6. *Assume that $\mathcal{E}$ is a rank two reflexive sheaf such that $S^3\mathcal{E} \otimes (\det \mathcal{E})^{-2}$ is generated by global sections. Then $\mathcal{E}$ is the trace-free sheaf of some triple cover $\pi : Y \to X$ with reduced and irreducible Y.*

Proof. Let A be the singularity locus of $\mathcal{E}$ and X. Then A has codimension at least 2 because X is factorial. Let $U = X \setminus A$. Then U is nonsingular and $\mathcal{E}' = \mathcal{E}_U$ is locally free.

In order to construct the desired triple cover of X, we consider the projective line bundle

$$p : P = \mathbb{P}(\mathcal{E}') \longrightarrow U.$$

We have an invertible sheaf $\mathcal{O}(1)$ on P.

By assumption, $\mathcal{M} = \mathcal{O}(3) \otimes p^*(\det \mathcal{E})^{-2}$ is a base point free invertible sheaf on P. Since P is nonsingular, by Bertini Theorem, we can find a generic nonsingular irreducible divisor V in the linear system $|\mathcal{M}|$ of $\mathcal{M}$. Let $\pi' : V \to U$ be the projection. We can assume that π' is surjective. We know that the restriction of $\mathcal{O}(V)$ to a generic fiber of p is $\mathcal{O}_{\mathbb{P}^1}(3)$, hence π' is a generically finite morphism of degree 3. Because V and U are nonsingular, we can find a closed subset A' of U of codimension ≥ 2 such that π' is a flat finite morphism over $U' = U \setminus A'$ (cf. [**5**], p.436, Ex.6.2]). Without loss of generality, we assume that $U = U'$, namely we assume that $\pi' : V \to U$ is a flat triple cover. By Lemma 7.5, π' can be extended uniquely to a triple cover $\pi : Y \to X$.

Since $\mathcal{O}_P(-V) \cong \mathcal{O}(-3) \otimes p^* \det \mathcal{E}^2$, by standard formulas (cf. [**5**], p.253, Ex. 8.4), we have

$$p_*(\mathcal{O}_P) = \mathcal{O}_U, \quad p_*\mathcal{O}(-V) = 0,$$
$$R^1 p_* \mathcal{O}(-V) \cong (\mathcal{E} \otimes \det \mathcal{E}^{-1})^*|_U.$$

From the exact sequence

$$0 \longrightarrow \mathcal{O}_P(-V) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_V \longrightarrow 0,$$

we obtain

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \pi'_* \mathcal{O}_V \longrightarrow (\mathcal{E} \otimes \det \mathcal{E}^{-1})^*|_U \longrightarrow 0.$$

If $\mathcal{E}_\pi$ is the trace-free sheaf of π, then $\mathcal{E}_\pi|_U$ is the trace-free sheaf of π'. Hence

$$\mathcal{E}_\pi|_U \cong (\mathcal{E} \otimes \det \mathcal{E}^{-1})^*|_U.$$

Because both sheaves are reflexive, we have

$$\mathcal{E}_\pi \cong \mathcal{E}^* \otimes \det \mathcal{E} \cong \mathcal{E}.$$

This completes the proof of the theorem. □

Corollary 7.6. *Tensoring an invertible sheaf, any rank* 2 *reflexive sheaf on a projective factorial variety is isomorphic to the trace-free sheaf of a triple cover.*

Proof. Let $\mathcal{E}$ be a rank two reflexive sheaf on X. Because X is projective, we can choose a very ample invertible sheaf $\mathcal{L}$ on X and a sufficiently large n such that $\mathcal{E}' = \mathcal{E} \otimes \mathcal{L}^{-n}$ satisfies the condition of the previous theorem. Then we know that $\mathcal{E}'$ is the trace-free sheaf of some triple cover. □

Note that when $\mathcal{E}$ is locally free, Miranda's triple cover data is a global section of $S^3\mathcal{E} \otimes (\det \mathcal{E})^{-2}$. Using the above technique, one can generalize Miranda's Theorem 1.1 in [**8**] to the general (non-flat) case: a triple cover of X is determined by a rank 2 reflexive sheaf $\mathcal{E}$ and a morphism $\Phi : S^3\mathcal{E} \to \wedge^2\mathcal{E}$, and conversely. In fact, the data can also be stated as a rank two reflexive sheaf $\mathcal{E}$ on X such that $\mathcal{O}_P(3) \otimes p^*(\det \mathcal{E})^{-2}$ has a non-zero section whose divisor is reduced and irreducible, where $p : P = P(\mathcal{E}) \to X$ is the projection. Because $\mathcal{E}$ is reflexive, the singular locus of P has codimension at least 3. Thus we can talk about the divisor of a section.

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Added in Proof. The following global condition should be added in the criterion of Theorem 2.1: "$-c_1(\pi_*(\mathcal{O}_Y))$ is linearly equivalent to the reduced branch divisor of π." In fact, this condition is equivalent to that "B_0 and C_0 are linearly equivalent" or " $\eta = -c_1(\mathcal{E}_\pi) - A_1 - A_2$ is linearly equivalent to zero" (see Theorem 1.3). The proof is unchanged and we only need to note that the element z constructed in the proof is global. Thus Theorem 2.1 is also true for unramified triple covers. Note that $2B_0$ is always linearly equivalent to $2C_0$. If $\mathrm{Pic}(X)$ contains no 2-torsion, then the above condition holds true automatically. The detail will appear in a joint paper with D.-Q. Zhang