

# CUSPS ON SOME ALGEBRAIC SURFACES AND PLANE CURVES

SHENG-LI TAN

To Prof. Makoto Namba on his 60th birthday

**ABSTRACT.** We determine the number of cusps in a 3-divisible set (introduced by W. Barth, analogous to even set of nodes) on plane curves and on some surfaces (space surfaces of small degree, K3 surfaces and Godeaux surface). As an application, we give an algebro-geometric characterization on the branch curves of generic triple coverings over  $\mathbb{P}^2$ , which is the first nontrivial case of a classic problem. We prove also that the branch curve determines uniquely the generic triple covering. As another application, we present a relationship between the existences of the following interesting surfaces: quintic surfaces with 20 cusps, Godeaux surfaces with 4 cusps, fake projective plane and surfaces whose canonical maps are of maximal degree.

## 0. Introduction

A cusp on an algebraic surface (resp. a curve) is a singularity defined by  $x^2 + y^3 = z^2$  (resp.  $x^2 + y^3 = 0$ ), i.e., a singular point of type  $A_2$ . In this paper, a normal surface (or curve) admits at most cusps as its singularities.

Let  $N = \{s_1, \dots, s_n\}$  be a set of cusps on a normal surface  $\Sigma$ . Let  $\sigma : Y \rightarrow \Sigma$  be the minimal resolution of the cusps. So  $\sigma^{-1}(s_i)$  consists of two  $(-2)$ -curves meeting at one point.

**Definition A.** ([Ba1]) If there exists a divisor  $\delta$  on  $Y$  such that the  $(-2)$ -curves  $B_i + C_i = \sigma^{-1}(s_i)$  (for a proper choice of  $B_i$  and  $C_i$ ) satisfy

$$\sum_{i=1}^n (B_i + 2C_i) \equiv 3\delta,$$

then we shall call  $(N, \delta)$  (denoted simply by  $N$ ) a *3-divisible set of cusps* on  $\Sigma$ .

Note that  $N$  is a 3-divisible set if and only if there is a cyclic triple covering  $\pi : X \rightarrow \Sigma$  ramified exactly over  $N$  such that  $X$  is nonsingular over the cusps. (See [Ba1] or [Ta2]).  $\pi : X \rightarrow \Sigma$  is called the *cyclic triple covering determined by  $(N, \delta)$*  and denoted also by  $\pi_N : X_N \rightarrow \Sigma$ .

Similar to even sets of nodes, 3-divisible sets of cusps appear naturally in the classification of surfaces. For example, if the canonical map of a surface of general type is a (cyclic) triple covering over a canonical surface  $\Sigma$ , then this triple covering is induced by a 3-divisible set of cusps on  $\Sigma$ .

V. V. Nikulin [Ni] showed that 16 singular points on a K3 surface form an even set of nodes. A. Beauville [Be1] used successfully binary coding theory to determine even sets

---

*Key words and phrases.* three divisible set of cusps, surface, plane curve, generic covering.

\* This work is partially supported by the Kort Foundation and the Emmy Noether Research Institute for Mathematics. This research is also supported by the 973 Project Foundation, the Doctoral Program Foundation of EMC and the Foundation for Shanghai Priority Academic Discipline.

of nodes and the maximal number of singularities on a quintic surface. This method is proved very powerful in solving the same problem on surfaces of higher degree.

Recently W. Barth starts to study cusps on K3 surfaces and obtain a very interesting result similar to Nikulin's. He proves that 9 cusps on a K3 surface are 3-divisible [Ba1]. In [Ba2], he gives a complete classification of such K3 surfaces.

In this paper, we shall generalize Beauville's method to ternary coding theory so that it can be used to study cusps on surfaces and plane curves. Precisely, we shall determine 3-divisible sets of cusps on space surfaces of small degrees, K3 surfaces, numerical Godeaux surfaces and plane curves of even degrees. Note that a numerical Godeaux surface is a surface of general type with  $p_g = q = 0$ , and  $K^2 = 1$ . Its fundamental group is trivial or a cyclic group of order  $\leq 5$  (see [BPV, p.237]).

**Theorem I.** *Let  $n$  be the number of cusps in a 3-divisible set  $N$  on a normal surface  $\Sigma$ , and let  $\pi : X \rightarrow \Sigma$  be the cyclic triple covering determined by  $N$ .*

- (1) *If  $\Sigma \subset \mathbb{P}^3$  is a cubic surface, then  $n = 3$  and  $X \cong \mathbb{P}^2$ . Furthermore,  $\Sigma$  possesses at most 3 cusps. If  $\Sigma$  admits 3 cusps, then they must be 3-divisible, and  $\Sigma$  is projectively equivalent to the surface  $w^3 = xyz$ .*
- (2) *If  $\Sigma$  is a K3 surface, then either*
  - (a)  *$n = 6$  and  $X$  is also a K3 surface, or*
  - (b)  *$n = 9$  and  $X$  is an Abelian surface.**9 cusps on  $\Sigma$  are 3-divisible [Ba1] and 8 cusps contain a 3-divisible set of 6 cusps.*
- (3) *If  $\Sigma \subset \mathbb{P}^3$  is a quintic surface, then*
  - (a)  *$n = 12$ ,  $p_g(X) = 6$ ,  $q(X) = 0$  and  $X$  is a canonical surface.*
  - (b)  *$n = 15$ ,  $p_g(X) = 4$ ,  $q(X) = 0$  and  $\pi$  is the canonical map of  $X$ .*
  - (c)  *$n = 18$ ,  $p_g(X) = 4$ ,  $q(X) = 2$  and  $\pi$  is the canonical map of  $X$ .**In fact,  $\Sigma$  has at most 20 cusps.*
- (4) *Let  $\Sigma$  be a Godeaux surface with fundamental group  $\mathbb{Z}_m$  ( $m \neq 3$ ), and let  $u : \Sigma_m \rightarrow \Sigma$  be the universal covering surface (so  $\Sigma_5$  is a quintic surface).*
  - (a) *3-divisible set on  $\Sigma$  consists of 3 cusps such that  $X$  is of general type with*

$$p_g(X) = 0, \quad q(X) = 0, \quad K_X^2 = 3.$$

- (b)  *$\Sigma$  has at most 4 cusps. 4 cusps on  $\Sigma$  contain two 3-divisible sets which induce a Galois covering  $\varphi : Y \rightarrow \Sigma$  of degree 9 such that  $Y$  is a smooth surface of general type with*

$$K_Y^2 = 9, \quad p_g(Y) = 0, \quad q(Y) = 0.$$

*So  $Y$  is a fake projective plane. Let  $Y_m = \Sigma_m \times_{\Sigma} Y$ .*

$$\begin{array}{ccc} Y_m & \xrightarrow[m:1]{\tilde{u}} & Y \\ \varphi_m \downarrow 9:1 & & 9:1 \downarrow \varphi \\ \Sigma_m & \xrightarrow[u]{m:1} & \Sigma \end{array}$$

*Then  $Y_m$  is a cyclic unramified covering of  $Y$  with group  $\mathbb{Z}_m$ . So  $Y_m$  is a surface of general type with*

$$p_g(Y_m) = m - 1, \quad q(Y_m) = 0, \quad K_{Y_m}^2 = 9m.$$

- (c) *The canonical map of  $Y_5$  is just  $\varphi_5 : Y_5 \rightarrow \Sigma_5 \subset \mathbb{P}^3$  whose degree is 9.*
- (d) *The canonical map  $\varphi_K : Y_4 \rightarrow \mathbb{P}^2$  of  $Y_4$  is of degree 36.*

Note that according to a well-known theorem of Beauville [Be2], if the image  $\Sigma$  of the canonical map  $\varphi_K$  of a surface  $X$  is also a surface, then there are two possibilities:

- (A)  $p_g(\Sigma)=0$ ;
- (B)  $\Sigma$  is a canonical surface with  $p_g(\Sigma) = p_g(X)$ .

In case (A),  $\deg \varphi_K \leq 36$ , with equality if and only if  $p_g(X) = 3$ ,  $q(X) = 0$ ,  $K_X^2 = 36$  and  $|K_X|$  has no base point.

In case (B),  $\deg \varphi_K \leq 9$ , with equality if and only if  $p_g(X) = 4$ ,  $q(X) = 0$ ,  $K_X^2 = 45$ ,  $|K_X|$  has no base point and  $\Sigma$  is a quintic surface in  $\mathbb{P}^3$ .

W. Barth told the author that he has constructed a quintic surface with 15 cusps which are 3-divisible, this quintic admits an automorphism of order 5 without fixed point, and so the quotient surface is a Godeaux surface with 3 cusps which are 3-divisible [Ba3]. He told the author also that it seems difficult to rule out the existence of a quintic with 20 cusps or Godeaux surface with 4 cusps. In [BaR], Barth and Rams give an example of quintic surface with 12 cusps which are 3-divisible.

Now we use 3-divisibility to characterize the branch curve  $B$  of a generic triple covering  $\pi : S \rightarrow \mathbb{P}^2$  (see Definition 7.1.1). In fact,  $B$  must be an irreducible curve of even degree with only cusps as its singularities,  $\pi$  is totally ramified exactly over the cusps and  $S$  is smooth.

**Definition B.** Let  $B$  be an irreducible plane curve in  $\mathbb{P}^2$  of even degree with  $n$  cusps. Let  $\pi_0 : \Sigma \rightarrow \mathbb{P}^2$  be the double covering ramified over  $B$ . Then  $\Sigma$  possesses  $n$  cusps lying over the cusps of  $B$ . The  $n$  cusps on  $B$  are called *3-divisible* if the  $n$  cusps on  $\Sigma$  are 3-divisible.

An old problem is to find the algebro-geometric conditions on a cuspidal curve such that it is the branch locus of a generic covering ([En], [Za1] or [Li, p.38], and see Definition 7.1.1). The aim of this classical problem is to reduce the classification of algebraic surfaces to that of cuspidal plane curves. In his famous paper [Za1], Zariski studied this problem in classical language. He showed that a sextic with 6 cusps is the branch locus of a generic covering if and only if the 6 cusps lie in a conic. In fact, the generic covering is a triple covering. In [Za2], he showed that an irreducible curve  $B_8$  of degree 8 possesses at most 15 cusps.

Another problem studied by Chisini [Ch1, Ch2] and Segre [Se] is the uniqueness of generic coverings (cf. [Fu, p.37]): *To what extent does the branch curve  $B$  determine the generic covering?* A sextic with 9 cusps is the branch curve of 4 generic coverings, one is of degree 3, the others are of degree 4. Up to base changes, this is the only known example for which there are more than one generic coverings ramified over a fixed curve (Example 6.1.5). Chisini conjectured that if we require the degree of the generic covering to be bigger than 4, then the branch curve determines uniquely the covering. This conjecture is solved by Kulikov [Ku] and Nemirovskij [Ne] for the case when degree  $\geq 12$ . So it is interesting to consider the case of small degree.

Our next purpose is to apply our method to the two problems for generic triple coverings, which is the first non-trivial case of the problems. Now we summarize our main results.

**Theorem II.** *Let  $B$  be an irreducible cuspidal curve of degree  $2d$  with  $n$  cusps.*

- (1) (Existence)  *$B$  is the branch curve of a generic triple covering totally ramified over the  $n$  cusps if and only if the  $n$  cusps are 3-divisible.*
- (2) (Uniqueness) *There is at most one generic triple covering ramified over  $B$  and totally ramified over the  $n$ -cusps.*

*Now we let  $\pi : S \rightarrow \mathbb{P}^2$  (resp.  $\pi_0 : \Sigma \rightarrow \mathbb{P}^2$ ) be a generic triple covering (resp. a*

(double covering) ramified over  $B$ , and let  $X$  be the normalization of  $\Sigma \times_{\mathbb{P}^2} S$ ,

$$\begin{array}{ccc} X & \longrightarrow & S \\ \pi_N \downarrow & & \downarrow \pi \\ \Sigma & \xrightarrow{\pi_0} & \mathbb{P}^2 \end{array}$$

Then  $X = X_N$  is a smooth surface determined by the cusps which form a 3-divisible set  $N$  on  $\Sigma$ ,

$$K_X^2 = 3K_\Sigma^2 = 6(d-3)^2, \quad \chi(\mathcal{O}_X) = 3\chi(\mathcal{O}_\Sigma) - \frac{2}{3}n, \quad (q(\Sigma) = 0).$$

- (3)  $n$  is divided by 3 and  $n \geq \frac{2}{3}d^2$ .  
(4) If  $\deg B = 4$ , then  $n = 3$  and  $X$  is rational. If  $B$  possesses 3 cusps, then they are 3-divisible, and  $B$  is projectively equivalent to the curve (see [BG])

$$x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x+y+z) = 0.$$

- (5) If  $\deg B = 6$ , then  $\Sigma$  is a K3 surface and  
(a)  $n = 6$ ,  $X$  is a K3 surface and  $B$  is defined by  $f_3^2 + f_2^3 = 0$  (Zariski [Za2]).  
(b)  $n = 9$ ,  $X$  is an Abelian surface.  
9 cusps on  $B$  are 3-divisible (Barth [Ba1]), and 8 cusps on  $B$  contain a 3-divisible set of type (a)  
(6) If  $\deg B = 8$ , then  $n = 12$ ,  $p_g(X) = 3$ ,  $q(X) = 0$ , and  $\pi_0 \circ \pi_N : X \rightarrow \mathbb{P}^2$  is the canonical map of  $X$ . Furthermore,  $B$  has at most 15 cusps ([Za2]) and 15 cusps must contain a 3-divisible set of 12 cusps.  
(7) If  $\deg B = 10$ , then  
(a)  $n = 18$ ,  $p_g(X) = 8$  and  $q(X) = 0$ .  
(b)  $n = 21$ ,  $p_g(X) = 6$ ,  $q(X) = 0$ , and the canonical map of  $X$  is

$$\varphi_2 \circ \pi_0 \circ \pi_N : X \rightarrow \mathbb{P}^5$$

where  $\varphi_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^5$  is the embedding defined by  $|\mathcal{O}_{\mathbb{P}^2}(2)|$ .

$B$  possesses at most 26 cusps (by Plücker's inequality). 25 cusps contain a 3-divisible set.

In Sect. 6, we shall present some examples of plane curves with a 3-divisible set of cusps.

For a sextic with 6 cusps, 3-divisibility of the 6 cusps is equivalent to Zariski's condition that they are on a conic. Like Zariski's classification of sextics with 6 cusps on a conic, we shall classify in [TaT] the plane curves of even degree  $\leq 10$  possessing a 3-divisible set of cusps.

In Sect. 7, we prove also that there is no generic covering of degree  $> 3$  ramified over a cuspidal curve  $B$  of degree 8 or 10 with a 3-divisible set of cusps (cf. Theorem 7.1.4). In general, if  $B$  is the branch locus of a generic triple covering  $\pi : S \rightarrow \mathbb{P}^2$ , and if there is another generic covering ramified over  $B$ , then its degree must be 4 or 5 and the number of cusps  $n \geq (\deg B)^2/4$  (this bound is sharp).

## 1. 3-torsion group of a Galois triple covering

### 1.1. Galois triple coverings of surfaces

Let  $X$  and  $Y$  be two smooth complex projective surfaces, and let  $\pi : X \rightarrow Y$  be a Galois triple covering. Miranda [Mi] proved that  $\pi$  is determined by the data  $(L, M, b, c)$ , where  $B = \text{div}(b) \in |2L - M|$ ,  $C = \text{div}(c) \in |2M - L|$ . There exists a group  $G = \langle \sigma \rangle$  of order 3 acting on  $X$  such that  $Y = X/G$ . In fact,  $X$  can be constructed as follows. Denote by  $V$  the vector bundle on  $Y$  corresponding to the locally free sheaf  $\mathcal{O}_Y(L) \oplus \mathcal{O}_Y(M)$  and let  $p : V \rightarrow Y$  be the bundle projection. Denote by  $z, w \in \Gamma(V, p^*(\mathcal{O}_Y(L) \oplus \mathcal{O}_Y(M)))$  the first and the second fiber coordinates respectively. Then we know that the common zero set of the sections

$$z^2 - p^*b \cdot w, \quad zw - p^*b \cdot p^*c, \quad w^2 - p^*c \cdot z$$

is a surface  $X$  in  $V$ , and  $p|_X : X \rightarrow Y$  is the Galois triple covering determined by the data  $(L, M, B, C)$ . Since  $X$  is smooth, by [Mi], we know that  $B + C$  is disjoint union of smooth curves. If we denote by  $\bar{B}$  and  $\bar{C}$  the ramification curves of  $\pi$  in  $X$  over  $B$  and  $C$  respectively, then we have

$$\pi^*B = 3\bar{B}, \quad \pi^*C = 3\bar{C}.$$

**Lemma 1.1.1.**

$$(1) \quad \pi^*L \equiv 2\bar{B} + \bar{C}, \quad \pi^*M \equiv 2\bar{C} + \bar{B}.$$

*Proof.* Let  $V_1 = \text{div}(p^*z)$  and  $V_2 = \text{div}(p^*w)$ . By some local computations, we have

$$(2) \quad V_1|_X = 2\bar{B} + \bar{C}, \quad V_2|_X = \bar{B} + 2\bar{C}.$$

On the other hand,  $p^*L = V_1$ ,  $p^*M = V_2$ , hence (1) is nothing but (2).  $\square$

### 1.2. On the 3-torsion group of a Galois triple covering

The purpose of this section is to compute the 3-torsion group of  $\pi : X \rightarrow Y$  from the branch locus  $B + C$ . We let  $B = \sum_{i \in I} B_i$  and  $C = \sum_{j \in J} C_j$ . Denote by  $\text{Tor}_3(X)$  the group of 3-torsions of  $\text{Pic}(X)$ , and  $\text{Tor}_3(X)^G$  the invariant subgroup of  $\text{Tor}_3(X)$  under the action of  $G$ . Define

$$\varphi : \mathbb{F}_3^I \oplus \mathbb{F}_3^J \rightarrow \text{Pic}(Y) \otimes \mathbb{F}_3, \quad \sum_{i \in I} \lambda_i e_i + \sum_{j \in J} \mu_j e'_j \mapsto \sum_{i \in I} \lambda_i B_i + \sum_{j \in J} \mu_j C_j.$$

Then we have

**Lemma 1.2.1.** *Let  $e = \sum_{i \in I} 2e_i + \sum_{j \in J} e'_j$ . Then*

$$\frac{\text{Tor}_3(X)^G}{\pi^* \text{Tor}_3(Y)} \cong \frac{\text{Ker}(\varphi)}{\mathbb{F}_3 e}.$$

*Proof.* Similar to the proof of (Lemma 2, [Be1]).  $\square$

**Lemma 1.2.2.** *With the same assumptions as above. For any 3-torsion  $\eta \in \text{Tor}_3(X)$ , we have*

$$\sigma^{*2}\eta + \sigma^*\eta + \eta \equiv \pi^*(\pi_*\eta) \in \pi^* \text{Tor}_3(Y).$$

*Proof.* We only need to note that the direct image divisor  $\pi_*\eta$  is a 3-torsion in  $\text{Pic}(Y)$ .  $\square$

**Corollary 1.2.3.**

$$\dim_{\mathbb{F}_3} \text{Tor}_3(X)^G \geq \frac{1}{2} (\dim_{\mathbb{F}_3} \text{Tor}_3(X) - \dim_{\mathbb{F}_3} \text{Tor}_3(Y)).$$

*Proof.* Define  $\beta : \text{Tor}_3(X) \rightarrow \text{Tor}_3(X)$  by  $\beta(\eta) = \sigma^* \eta - \eta$ . Then we have

$$\begin{aligned} 0 \rightarrow \text{Tor}_3(X)^G \rightarrow \text{Tor}_3(X) \xrightarrow{\beta} \text{Im}(\beta) \rightarrow 0, \\ 0 \rightarrow \text{Tor}_3(X)^G \cap \text{Im}(\beta) \rightarrow \text{Im}(\beta) \xrightarrow{\beta} \text{Im}(\beta^2) \rightarrow 0. \end{aligned}$$

By Lemma 1.2.2,  $\text{Im}(\beta^2) \subset \pi^* \text{Tor}_3(Y)$ . Because  $\pi$  is not unramified, no 3-torsion will be killed under  $\pi^*$ . Hence  $\text{Tor}_3(Y) \cong \pi^* \text{Tor}_3(Y)$ . Then it is easy to get the inequality.  $\square$

**2. Some lemmas on codes over  $\mathbb{F}_3$** 

In this section, we shall present some lemmas on codes over  $\mathbb{F}_3$ , which will be used in the computation of the invariants of a 3-divisible set of cusps.

A code of length  $n$  over  $\mathbb{F}_q$  is a linear subspace  $V$  of  $\mathbb{F}_q^n$ . If  $x$  is a vector in  $\mathbb{F}_q^n$ , then the number of non-zero coordinates of  $x$  is called the *weight* of  $x$ , denoted by  $w(x)$ . The minimal weight of  $V$  is  $d = \min\{w(x) \mid x \in V, x \neq 0\}$ . We shall say that a code is of type  $[n, k, d]$  if it is a  $k$ -dimensional code of length  $n$  with minimum weight  $d$ .

The following lemma can be found in [Lin, p.58].

**Lemma 2.1.1. (Griesmer's lemma).** *For an  $[n, k, d]$  code  $V$  over  $\mathbb{F}_q$ , there exists an  $[n - d, k - 1, d']$  code, called the residual code of  $V$ , with  $d' \geq d/q$*

*Proof.* (Cf. [Bel]). Let  $a \neq 0$  be an element of  $V$  with the minimal weight  $d$ . If  $a = \sum_{i \in A} \varepsilon_i e_i$ ,  $\varepsilon_i = \pm 1$  for  $i \in A$ , then we denote by  $E_1$  (resp.  $E_0$ ) the subspace of  $\mathbb{F}_q^n$  generated by those  $e_i$  for  $i \in A$  (resp.  $i \notin A$ ). Hence we have  $\mathbb{F}_q^n = E_0 \oplus E_1$ . Denote by  $p_0$  the projection of  $\mathbb{F}_q^n$  to  $E_0$ . Then we have  $\text{Ker}(p_0|_V) = E_1 \cap V = \mathbb{F}_q a$ , as  $a$  is of minimal weight. Thus  $p_0(V)$  is an  $[n - d, k - 1, d']$  code in  $E_0$  with a canonical base  $\{e_i \mid i \in A\}$ .

Now let  $x$  be an element in  $V - \mathbb{F}_q a$ , and let  $x = x_0 + x_1$  be the decomposition with  $x_0 \in E_0$  and  $x_1 \in E_1$ . Note that for any  $\lambda \in \mathbb{F}_q$ ,

$$w(x + \lambda a) = w(x_0) + w(x_1 + \lambda a)$$

and  $\sum_{\lambda \in \mathbb{F}_q} w(x_1 + \lambda a) = (q - 1)w(a)$ . Hence

$$(3) \quad \sum_{\lambda \in \mathbb{F}_q} w(x + \lambda a) = (q - 1)w(a) + qw(x_0).$$

On the other hand, by assumption,  $x + \lambda a$  is nonzero for any  $\lambda$ , thus  $w(x + \lambda a) \geq w(a)$ . Therefore, (3) implies that  $qw(x_0) \geq w(a) = d$ , i.e.,  $d' \geq d/q$ .  $\square$

**Lemma 2.1.2.** *Let  $V \subset \mathbb{F}_3^n$  be an  $[n, k, d]$  code. We have*

- (i) *If  $d \geq \frac{2}{3}(n + 1)$ , then  $n \geq 3^k - 1$ .*
- (ii) *If  $d \geq \frac{2}{3}n$ , then  $n \geq 3^{k-1}$ .*

*Proof.* We shall prove (i) by induction on  $k$ . For  $k = 1$ , there is nothing to prove. Now, suppose  $V \subset \mathbb{F}_3^n$  is an  $[n, k, d]$  code with  $k > 1$ . By Griesmer's Lemma, there exists an  $[n - d, k - 1, d']$  code  $V'$  with  $d' \geq d/3 \geq 2(n - d + 1)/3$ . By induction hypothesis, we have

$$3^{k-1} - 1 \leq n - d \leq n - \frac{2}{3}(n + 1),$$

hence  $n \geq 3^k - 1$ .

The proof of (ii) is similar.  $\square$

**Lemma 2.1.3.** *Let  $V$  be an  $[n, k, d]$  code over  $\mathbb{F}_3$ . Assume that for any non zero element  $x \in V$ ,  $w(x) = d$ . Then  $d$  is divided by  $3^{k-1}$ .*

*Proof.* (3) implies that the residual code  $V'$  of  $V$  is of type  $[n - d, k - 1, d']$  with  $d' = d/3$ , and  $V'$  satisfies also the conditions of the lemma. Hence the lemma can be proved easily by using induction on  $k$ .  $\square$

**Corollary 2.1.4.** (Griesmer bound). *For an  $[n, k, d]$  code, we have*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{3^i} \right\rceil,$$

where  $\lceil r \rceil$  is the minimal integer  $\geq r$ .

### 3. Some general facts on 3-divisible sets of cusps

#### 3.1. On the invariants

Let  $\Sigma$  be a normal surface, and  $N = \{s_1, \dots, s_n\}$  a 3-divisible set of cusps on  $\Sigma$ . Let  $\sigma : Y \rightarrow \Sigma$  be the minimal resolution of the singularities of  $\Sigma$ . With the notations as in Definition A, we can see that  $N$  determines a Galois triple covering data  $(B, C, L, M)$ , where

$$B = \sum_{i=1}^n B_i, \quad C = \sum_{i=1}^n C_i, \quad L = B + C - \delta, \quad M = \delta.$$

By Lemma 2.2.2 in [Ta2], we can construct a cyclic triple covering

$$\pi_N : X_N \rightarrow \Sigma$$

ramified exactly over  $N$  such that  $X_N$  is nonsingular at the ramification points.  $\pi_N$  is called the *triple covering of  $\Sigma$  determined by  $(N, \delta)$* .

Obviously, the converse of the statement above holds (cf. [Ta1]), i.e., we have

**Lemma 3.1.1.**  *$N = \{s_1, \dots, s_n\}$  is a 3-divisible set if and only if there is a cyclic triple covering  $\pi : X \rightarrow \Sigma$  ramified exactly over  $N$  such that  $X$  is nonsingular at the ramification points.*

**Definition 3.1.2.** The *geometric genus*  $p_g(N)$  and the *irregularity*  $q(N)$  of a 3-divisible set  $N$  is defined respectively by

$$p_g(N) = h^2(-L) + h^2(-M), \quad q(N) = h^1(-L) + h^1(-M).$$

By Lemma 2.2.3 of [Ta2], we have

$$(4) \quad \begin{aligned} K_{X_N}^2 &= 3K_\Sigma^2, \quad \chi(\mathcal{O}_{X_N}) = 3\chi(\mathcal{O}_\Sigma) - \frac{2}{3}n, \\ p_g(X_N) &= p_g(\Sigma) + p_g(N), \quad q(X_N) = q(\Sigma) + q(N). \end{aligned}$$

Note that  $n = \#N$  is divided by 3 if  $N$  is 3-divisible. From these formulas, we get

$$(5) \quad p_g(N) - q(N) = 2\chi(\mathcal{O}_\Sigma) - \frac{2}{3}n.$$

When  $K_\Sigma$  is base point free,  $X_N$  is minimal because  $K_{X_N} = \pi_N^*(K_\Sigma)$  is also base point free.

**Theorem 3.1.3.** (cf. [De]) *Let  $X$  be a minimal surface of general type with invariants  $K^2$ ,  $p_g$  and  $q$ . Then we have*

- 1) *Noether's inequality (modified by O. Debarre):  $K^2 \geq 2p_g + q - 4$ .*
- 2) *Castelnuovo's inequality (modified by F. Jongmans):  $K^2 \geq 3p_g + q - 7$ .*
- 3) *If  $q > 0$ , then  $K^2 \geq 2p_g$ .*
- 4) *Miyaoka-Yau inequality:  $K^2 \leq 9(p_g - q + 1)$ .*

If  $\Sigma$  has a positive Kodaira dimension, then so does  $X_N$ . Then by Miyaoka-Yau inequality for  $X_N$ , we obtain

$$(6) \quad n \leq \frac{1}{2}(9\chi(\mathcal{O}_\Sigma) - K_\Sigma^2).$$

Note that if  $n$  is the maximal number of cusps on  $\Sigma$  (the  $n$  cusps need not to be 3-divisible), (6) holds too (cf. [Miy]).

If  $\Sigma$  is a canonical surface, and  $p_g(N) = p_g(X_N) - p_g(\Sigma) > 0$ , then  $X_N$  is also a canonical surface. By Castelnuovo's inequality (Theorem 3.1.3), we have

$$(7) \quad n \geq \frac{9}{2}\chi(\mathcal{O}_\Sigma) - \frac{3}{2}K_\Sigma^2 + 2q(\Sigma) + 2q(N) - 5.$$

### 3.2. Canonical constructions of codes

Now, let  $\sigma : Y \rightarrow \Sigma$  be the minimal resolution of the singularities  $N$  with  $B_i + C_i = \sigma^{-1}(s_i)$ , and let  $\eta : \bar{Y} \rightarrow Y$  be the blowing-up of the  $n$  points  $B_i \cap C_i$  for all  $i$ . Denote by  $\bar{B}_i, \bar{C}_i$  the strict transforms of  $B_i$  and  $C_i$  in  $\bar{Y}$  respectively. We have a new relation

$$\sum \bar{B}_i + 2 \sum \bar{C}_i \equiv 3\bar{\delta}.$$

Hence there is a new triple covering  $\bar{\pi} : \bar{X} \rightarrow \bar{Y}$  over  $\bar{Y}$  ramified over  $\sum \bar{B}_i + \sum \bar{C}_i$ .  $\bar{X}$  is nonsingular and birational to  $X$ .

Now we consider the homomorphism  $\varphi$  defined in Sect. 1.2.

$$\begin{aligned} \varphi : \mathbb{F}_3^n \oplus \mathbb{F}_3^n &\rightarrow \text{Pic}(\bar{Y}) \otimes \mathbb{F}_3 \\ (b, c) &= \left( \sum_{i=1}^n b_i e_i, \sum_{i=1}^n c_i e_i \right) \mapsto \sum_{i=1}^n (b_i \bar{B}_i + c_i \bar{C}_i), \end{aligned}$$

where  $\{e_1, \dots, e_n\}$  is the canonical base of  $\mathbb{F}_3^n$ . In fact,  $\varphi$  induces the following homomorphism

$$\varphi' : \mathbb{F}_3^n \oplus \mathbb{F}_3^n \rightarrow H^2(\bar{Y}, \mathbb{Z}_3).$$

It is easy to see that  $(b, c) \in \text{Ker}(\varphi)$  or  $\text{Ker}(\varphi')$  only if  $b_i + c_i = 0$  in  $\mathbb{F}_3$ . If we denote by  $\bar{\mathbb{F}}_3^n$  the subspace of  $\mathbb{F}_3^n \oplus \mathbb{F}_3^n$  with a canonical base

$$\bar{e}_1 = (e_1, -e_1), \dots, \bar{e}_n = (e_n, -e_n),$$

then we have linear maps

$$(8) \quad \bar{\varphi} : \bar{\mathbb{F}}_3^n \rightarrow \text{Pic}(\bar{Y}) \otimes \mathbb{F}_3,$$

$$(9) \quad \bar{\varphi}' : \bar{\mathbb{F}}_3^n \rightarrow H^2(\bar{Y}, \mathbb{Z}_3),$$

hence we have two codes of length  $n$ :

$$K = \text{Ker } \varphi = \text{Ker } \bar{\varphi} \subset \bar{\mathbb{F}}_3^n, \quad K' = \text{Ker } \varphi' = \text{Ker } \bar{\varphi}' \subset \bar{\mathbb{F}}_3^n.$$

In fact,  $K \subset K'$ . When  $Y$  is simply connected,  $K = K'$ .

**Lemma 3.2.1.** *Assume that  $\text{Tor}_3(Y) = 0$ . Then we have*

$$\begin{aligned} k &= \dim_{\mathbb{F}_3} K \geq q(N) + 1. \\ k' &= \dim_{\mathbb{F}_3} K' \geq \frac{3}{2}n - \frac{1}{2}b_2(Y). \end{aligned}$$

*Proof.* The first inequality follows from Lemma 1.2.1 and Corollary 1.2.3 for  $\bar{\pi} : \bar{X} \rightarrow \bar{Y}$ . Note that the torsion group is a birational invariant and that  $\text{Tor}_3(\bar{Y}) \cong \bar{\pi}^* \text{Tor}_3(\bar{Y})$  because  $\bar{\pi}$  is not unramified. Hence

$$k = \dim_{\mathbb{F}_3} K \geq \frac{1}{2} \dim_{\mathbb{F}_3} \text{Tor}_3(X) + 1 - \frac{3}{2} \dim_{\mathbb{F}_3} \text{Tor}_3(Y).$$

Then we use  $\dim_{\mathbb{F}_3} \text{Tor}_3(X) \geq 2q(N)$ .

For the second formula, we know that there is a non-degenerated intersection form on  $H^2(\bar{Y}, \mathbb{Z}_3)$ , which is zero on the subspace  $\text{Im } \varphi'$ . Hence

$$2n - \dim_{\mathbb{F}_3} K' = \dim_{\mathbb{F}_3} \text{Im } \varphi' \leq \frac{1}{2} b_2(\bar{Y}).$$

Then we have the second inequality because  $b_2(\bar{Y}) = b_2(Y) + n$ .  $\square$

**Corollary 3.2.2.** *Let  $N$  be a 3-divisible set of  $n$  cusps on a normal surface  $\Sigma$ . Let  $d$  be the minimal number of the cusps in a 3-divisible set on  $\Sigma$ . If  $n < \frac{4}{3}d$ , then  $N$  is regular, i.e.,  $q(N) = 0$ . In particular, the minimal 3-divisible set of  $d$  cusps is regular.*

*Proof.* From Lemma 3.2.1, there is a code  $K$  of type  $[n, k, d]$  such that  $k \geq q(N) + 1$ . If  $q(N) > 0$ , then  $k \geq 2$ . We know that the residue code of  $K$  is of type  $[n - d, k - 1, d']$  with  $d' \geq d/3$ . By assumption,  $d' > n - d$ , thus the residue code does not exist, a contradiction. Hence  $q(N) = 0$ .  $\square$

## 4. 3-divisible sets of cusps on space surfaces

### 4.1. 3-divisible sets of cusps on a cubic surface

In this section, all surfaces  $\Sigma$  and  $Y$  are simply connected, hence  $K = K'$  and  $\text{Tor}_3(Y) = 0$ .

**Theorem 4.1.1.** *Let  $\Sigma \subset \mathbb{P}^3$  be a normal cubic surface. Then we have*

- 1)  $\Sigma$  has at most 3 cusps.
- 2) 3 cusps of  $\Sigma$  must form a 3-divisible set  $N$  with

$$p_g(N) = q(N) = 0 \quad \text{and} \quad X_N \cong \mathbb{P}^2.$$

*Proof.* Notice first that  $b_2(Y) = 7$  and 1) follows from 2). As in Sect. 3.2, for 3 cusps of  $\Sigma$ , we have a  $[3, k', d]$  code  $K'$  with  $k' \geq 1$ . We know  $d$  is divided by 3, hence there exists a nonzero  $x \in K$  with  $w(x) = 3$ , i.e. the 3 cusps form a 3-divisible set  $N$ . By (4), we obtain  $\chi(\mathcal{O}_{X_N}) = 1$ ,  $K_{X_N}^2 = 9$ . On the other hand, it is obvious that  $X_N$  is not of general type. By Enriques-Kodaira classification of surfaces [BPV, p.188], we know  $X_N \cong \mathbb{P}^2$ . Hence  $p_g(N) = q(N) = 0$ .  $\square$

Note that any cubic surface with 3 cusps are projectively equivalent to the surface defined by  $w^3 = xyz$ .

### 4.2. 3-divisible sets of cusps on K3 surfaces

Let  $\Sigma$  be a normal K3 surface with only cusps as its singularities, i.e., the minimal resolution of  $\Sigma$  is a K3 surface. For example, normal surfaces of degree 4 in  $\mathbb{P}^3$  with only cusps as its singularities are K3 surfaces. The following theorem implies that there are at most 9 cusps on a K3 surface, which can also be obtained by (6).

**Theorem 4.2.1.** *Let  $N$  be a 3-divisible set of  $n$  cusps on a K3 surface  $\Sigma$ . Then*

- 1)  $n = 6$ ,  $p_g(N) = 0$ ,  $q(N) = 0$ , and  $X_N$  is also a K3 surface, or
- 2)  $n = 9$ ,  $p_g(N) = 0$ ,  $q(N) = 2$ , and  $X_N$  is an Abelian surface.

*Furthermore, 9 cusps of a K3 surface form a 3-divisible set of type 2), and 8 cusps contain a 3-divisible set of type 1).*

*Proof.* Since  $Y$  is a simply connected surface with  $b_2(Y) = 22$ ,  $\chi(\mathcal{O}_Y) = 2$  and  $K_Y \equiv 0$ , by (4) we have  $\chi(\mathcal{O}_{X_N}) = 6 - \frac{2}{3}n$ , and  $K_{X_N} \equiv 0$ . By the classification of surfaces, we have  $\chi(\mathcal{O}_{X_N}) \geq 0$ , hence  $n = 6$  or  $9$ . Thus 1) and 2) follow at once.

Now we assume that  $\Sigma$  has 9 cusps, as in the proof of Theorem 4.1.1, we have a  $[9, k', d]$  code  $K'$  with  $d \geq 6$  and  $k' \geq 3$ . Hence  $w(x) = 6$  or  $9$  for all nonzero  $x \in K'$ . By Lemma 2.1.3, we can see that  $K'$  contains at least one element with weight 9. This proves that the 9 cusps are 3-divisible.

The last statement for 8 cusps can be proved similarly.  $\square$

We refer to [Ba1, Ba2] for the examples and more detailed discussions.

#### 4.3. 3-divisible sets of cusps on a quintic surface

**Theorem 4.3.1.** *Let  $N$  be a 3-divisible set of  $n$  cusps on a normal quintic surface  $\Sigma \subset \mathbb{P}^3$ . Then we have the following three cases.*

- 1)  $n = 12$ ,  $p_g(N) = 2$ ,  $q(N) = 0$ ;
- 2)  $n = 15$ ,  $p_g(N) = 0$ ,  $q(N) = 0$ ;
- 3)  $n = 18$ ,  $p_g(N) = 0$ ,  $q(N) = 2$ .

*In the first case,  $X_N$  is a canonical surface. In the last two cases,  $\pi_N$  is the canonical map of  $X_N$ .*

*Proof.* We know  $K_\Sigma^2 = 5$ ,  $\chi(\mathcal{O}_\Sigma) = 5$ . Hence  $K_{X_N}^2 = 15$ ,  $\chi(\mathcal{O}_{X_N}) = 15 - \frac{2}{3}n$ .

If  $n < 12$ , then  $n \leq 9$ .  $\chi(\mathcal{O}_{X_N}) \geq 9$ , hence  $p_g(X_N) \geq 8 > p_g(\Sigma)$ , which implies that  $X_N$  is a canonical surface. By Castelnuovo's inequality,  $K_{X_N}^2 \geq 3p_g(X_N) - 7 \geq 17$ , a contradiction. Hence  $n \geq 12$ .

By Miyaoka-Yau inequality (6), we obtain  $n \leq 20$ , hence  $n = 12, 15$  or  $18$ .

If  $n = 12$  or  $15$ , then by Corollary 3.2.2,  $q(N) = 0$ . From (5), we can compute the geometric genus.

For the case  $n = 18$ , we have  $\chi(\mathcal{O}_{X_N}) = 3$ ,  $p_g(X_N) \geq 4$ , hence  $q(N) = q(X_N) \geq 2$ . If  $q(N) \geq 3$ , then we have a code  $K$  of type  $[18, k, 12]$  with  $k \geq 4$ . By Lemma 2.1.4,  $n \geq 19$ , a contradiction. Thus  $q(N) = 2$ .

The statements on canonical maps follow from  $p_g(N)$ .  $\square$

## 5. 3-divisible sets of cusps on numerical Godeaux surfaces

A numerical Godeaux surface  $\Sigma$  is a surface of general type with

$$K_\Sigma^2 = 1, \quad p_g(\Sigma) = q(\Sigma) = 0.$$

Assume that  $\Sigma$  admits only cusps as its singularities, and denote by  $Z$  the minimal resolution of these cusps. Then the fundamental group  $\pi_1(Z)$  is a cyclic group of order  $\leq 5$ . In fact  $\pi_1(\Sigma) = \pi_1(Z)$ . By (6),  $\Sigma$  has at most 4 cusps. This claim can also be deduced from the following theorem.

**Theorem 5.1.1.** *Let  $\Sigma$  be a normal Godeaux surface without 3-torsion, i.e.,  $\text{Tor}_3(Z) = 0$ . Assume that  $\Sigma$  has only cusps as its singularities. Let  $N$  be a 3-divisible set of  $n$  cusps on  $\Sigma$ .*

- 1) *We have*

$$n = 3, \quad p_g(N) = q(N) = 0,$$

and  $X_N$  is a minimal surface of general type with

$$K^2 = 3, \quad p_g = q = 0, \quad \text{Tor}_3(X_N) = 0.$$

- 2) If  $\Sigma$  has 4 cusps, then they contain two 3-divisible sets  $N$  and  $N'$  with different supports. The 3 cusps of  $X_N$ , induced by the cusp not in  $N$ , form a new 3-divisible set  $N_1$  with

$$p_g(N_1) = q(N_1) = 0.$$

Hence the surface  $X_{N_1}$  is a fake projective plane, i.e., a surface of general type with

$$K^2 = 9, \quad p_g = q = 0.$$

*Proof.* Since  $X_N$  is a surface of general type with  $\chi(\mathcal{O}_{X_N}) = 3 - \frac{2}{3}n > 0$ , we have  $n = 3$ . From Corollary 3.2.2, we have  $q(N) = 0$ . Hence  $p_g(N) = 0$ . Because there is only one 3-divisible set supported on  $N$ , by Lemma 1.2.1,  $\text{Tor}_3(X_N)^G = 0$ . From Corollary 1.2.3,  $\text{Tor}_3(X_N) = 0$ .

Assume that there are 4 cusps on  $\Sigma$ . Consider the code  $K'$ . Because  $k' \geq 2$  (Lemma 3.2.1),  $K'$  contains at least two linearly independent elements, i.e., there exist at least two 3-divisible sets  $N$  and  $N'$  contained in the 4 singular points. Obviously they have different supports. So the 3-divisibility of  $N'$  implies that the 3 cusps on  $X_N$  form a 3-divisible set  $N_1$ . We can prove similarly that  $p_g(N_1) = q(N_1) = 0$ . Therefore  $Y = X(N_1)$  is a fake projective plane.  $\square$

The surfaces  $Y_4$  and  $Y_5$  in Theorem I are cyclic étale coverings of the fake projective plane  $Y = X(N_1)$ , so  $q(Y_4)$  and  $q(Y_5)$  are zero. Then it is easy to prove (c) and (d) of (4) in Theorem I.

## 6. 3-divisible sets of cusps on a plane curve

### 6.1. 3-divisible sets of cusps on a curve of even degree

Let  $B$  be a curve of degree  $2d$  in the projective plane with  $n$  cusps  $N = \{p_1, \dots, p_n\}$  as its singularities, and let  $\pi_0 : \Sigma \rightarrow \mathbb{P}^2$  be the double covering ramified over  $B$ . It is well known that  $\Sigma$  and its minimal resolution  $Y$  are simply connected. Hence  $\text{Tor}_3(Y) = 0$ .

Let  $N$  be a 3-divisible set of cusps on  $B$ . We still use  $N$  to denote the 3-divisible set  $\pi_0^{-1}(N)$  on  $\Sigma$ . Then we have the following formulas.

$$\begin{aligned} K_\Sigma^2 &= 2(d-3)^2, \quad \chi(\mathcal{O}_\Sigma) = \frac{(d-1)(d-2)}{2} + 1, \quad q(\Sigma) = 0, \\ K_{X_N}^2 &= 6(d-3)^2, \quad \chi(\mathcal{O}_{X_N}) = \frac{3(d-1)(d-2)}{2} + 3 - \frac{2}{3}n. \end{aligned}$$

Hence

$$(10) \quad p_g(N) - q(N) = (d-1)(d-2) + 2 - \frac{2}{3}n.$$

Because  $K_{X_N} = \pi_N^* \pi_0^*(\mathcal{O}_{\mathbb{P}^2}(d-3))$ ,  $X_N$  is of general type when  $\deg(B) \geq 8$ . By (6), we get

$$(11) \quad n \leq \frac{d(5d-3)}{4}.$$

For the dual curve  $B^*$  of  $B$ , put

$$\delta = \deg B^*, \quad \gamma = \#\{\text{cusps of } B^*\}, \quad \nu = \#\{\text{nodes of } B^*\}.$$

From Plücker's formula, we have

$$\begin{aligned} \delta &= 2d(2d-1) - 3n, \\ \gamma &= 12d(d-1) - 8n, \\ \nu &= \frac{1}{2}(3n - 4d(d-1))(3n - 4d^2 + 9). \end{aligned}$$

If  $d \geq 4$ , then from (11) and  $\nu \geq 0$ , we get

$$(12) \quad n \leq \frac{4}{3}d(d-1).$$

This bound is better than (11) only when  $d = 4$  or  $5$ .

In [Ta2] we gave a construction of 3-divisible set of cusps on a surface. The following is a typical construction of 3-divisible set of cusps on a curve of degree  $6m$ .

**Proposition 6.1.1.** *Let  $f_{2m} = 0$  and  $g_{3m} = 0$  be respectively two general plane curves of degrees  $2m$  and  $3m$ . Let  $f = f_{2m}^3 + g_{3m}^2$ . Then the curve  $f = 0$  has a 3-divisible set of  $6m^2$  cusps.*

We leave the proof to the next section.

**Theorem 6.1.2.** *Let  $B$  be an irreducible plane curve of degree  $2d$  with  $n$  cusps. If the  $n$  cusps form a 3-divisible set  $N$ , then*

$$(13) \quad n \geq \frac{2}{3}d^2.$$

*Proof.* We claim first that the fundamental group  $\pi_1 = \pi_1(\mathbb{P}^2 \setminus B)$  is not Abelian. Then by [No] we get  $6n \geq B^2 = 4d^2$ .

Indeed, if  $\pi_1$  is Abelian, then it is a cyclic group of order  $2d$  ([Za1, Theorem 3]). On the other hand, we denote by  $\tau_0$  the involution of  $\Sigma$  or  $Y$  determined by  $\pi_0$ , and by  $\sigma$  the automorphism of order 3 of  $X_N$  determined by  $\pi_N$ . By the canonical resolution of the singularities of double coverings, we can see easily that  $\tau_0^*(B_i) = C_i$  (cf. Definition A for the notations). Now from the construction of Galois triple coverings, we can check that  $\tau_0$  can be lifted to  $X_N$  as an involution  $\tau$ , and  $\tau\sigma\tau^{-1} = \sigma^2$ , because locally

$$\tau(x, y, z, w) = (x, y, w, z), \quad \sigma(x, y, z, w) = (x, y, \xi z, \xi^2 w),$$

where  $\xi$  is the primitive third root of unity (cf. Sect. 1.1). Hence  $\pi_0 \circ \pi_N$  determines a unramified Galois covering of  $\mathbb{P}^2 \setminus B$  with the non-Abelian group  $G = \langle \sigma, \tau \rangle$  of order 6. Then we have a surjective homomorphism from  $\pi_1$  to  $G$ , which is impossible. Thus  $\pi_1$  can not be Abelian.  $\square$

*Remark 6.1.3.* From Proposition 6.1.1, we can see that the inequality (13) is sharp.

**Corollary 6.1.4.** *The 3-divisible set  $N$  of cusps constructed in Proposition 6.1.1 is minimal. Thus  $N$  is regular, i.e.,*

$$p_g(N) = (m-1)(5m-4), \quad q(N) = 0.$$

**Example 6.1.5.** (Hirzebruch [Hi], Ivinskis [Iv]) *Let  $C$  be a generic plane curve  $C$  with 9 cusps defined by  $F(x, y, z) = 0$ . Set  $C_m = \{F(x^m, y^m, z^m) = 0\}$ . Then  $C_m$  is an irreducible curve of degree  $6m$  with  $n = 9m^2$  cusps.*

**Proposition 6.1.6.** *The  $9m^2$  cusps on  $C_m$  are 3-divisible.*

*Proof.* Note that the finite covering  $\varphi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  defined by  $[x, y, z] \mapsto [x^m, y^m, z^m]$  is ramified over the axes which meet  $C_m$  normally.  $C_m$  is the pullback of  $C$  under  $\varphi$ . Then it is easy to check the desired divisibility from definition.  $\square$

### 6.2. 3-divisible sets of cusps on a quartic curve

**Theorem 6.2.1.** *Let  $B$  be an irreducible curve of degree 4 with a 3-divisible set  $N$  of  $n$  cusps. Then*

$$n = 3, \quad p_g(N) = q(N) = 0.$$

*Furthermore, 3 cusps on  $B$  are 3-divisible.*

*Proof.* The invariants of  $\Sigma$  are:  $K_\Sigma^2 = 2$ ,  $\chi(\mathcal{O}_\Sigma) = 1$ ,  $\Sigma$  is a del Pezzo surface. Because  $K_{X_N} = \pi^*K_\Sigma$ ,  $X_N$  is also a del Pezzo surface. Hence  $\chi(\mathcal{O}_{X_N}) = 1$  and  $n = 3$ .

If  $B$  admits 3 cusps, then we get a  $[3, k', d]$  code  $K'$  over  $\mathbb{F}_3$  with  $k' \geq 1$  (Lemma 3.2.1) as  $b_2(Y) = 8$ ,  $d$  must be 3. Hence there is an element  $x \in K$  with length 3. Thus the 3 cusps are 3-divisible.  $\square$

*Remark 6.2.2.* It is known [BG] that any quartic curve with 3 cusps is projectively isomorphic to the curve defined by

$$x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x + y + z) = 0.$$

### 6.3. 3-divisible sets of cusps on a curve of degree 6

**Theorem 6.3.1.** *Let  $B$  be an irreducible curve of degree 6 with a 3-divisible set  $N$  of  $n$  cusps. Then*

- 1)  $n = 6$ ,  $p_g(N) = 0$ ,  $q(N) = 0$ ,
- 2)  $n = 9$ ,  $p_g(N) = 0$ ,  $q(N) = 2$ .

*9 cusps form a 3-divisible set of type 2), and 8 cusps contain a 3-divisible set of type 1).*

*Proof.* This follows from Theorem 4.2.1 because  $\Sigma$  is a K3 surface.  $\square$

### 6.4. 3-divisible sets of cusps on a curve of degree 8

**Theorem 6.4.1.** *Let  $B$  be an irreducible curve of degree 8 with a 3-divisible set  $N$  of  $n$  cusps. Then*

$$n = 12, \quad p_g(N) = 0, \quad q(N) = 0.$$

$\pi_0 \circ \pi_N$  is the canonical map of  $X_N$ .

*Proof.* From (11) and (13), we have  $11 \leq n \leq 17$ . Hence  $n = 12$  or  $15$ . By Corollary 3.2.2,  $q(N) = 0$ . From (10),  $p_g(N) = 8 - \frac{2}{3}n \geq 0$ , hence  $n = 12$  and  $p_g(N) = 0$ .  $\square$

Zariski [Za2] showed that any irreducible plane curve of degree 8 has at most 15 cusps.

**Corollary 6.4.2.** *If an irreducible curve of degree 8 possesses 15 cusps, then they must contain a 3-divisible set of 12 cusps.*

*Proof.* Let  $B$  possess  $n = 15$  cusps. Since  $b_2(Y) = 44$ , from Lemma 3.2.1, there is a code  $K'$  of type  $[15, k', 12]$  with  $k' \geq 1$ . Hence, there is a 3-divisible set contained in the 15 cusps.  $\square$

### 6.5. 3-divisible sets of cusps on a curve of degree 10

**Theorem 6.5.1.** *Let  $B$  be an irreducible curve of degree 10 with a 3-divisible set  $N$  of  $n$  cusps. Then*

- 1)  $n = 18$ ,  $p_g(N) = 2$ ,  $q(N) = 0$ .
- 2)  $n = 21$ ,  $p_g(N) = 0$ ,  $q(N) = 0$ .

*$B$  possesses at most 26 cusps. If  $B$  has 25 cusps, then it contains a 3-divisible set of cusps.*

*Proof.* From (12) and (13), we get  $17 \leq n \leq 26$ . Hence  $n = 18, 21, 24$ .

If  $n = 24$ , then  $q(N) = p_g(N) + 2 \geq 2$ . Thus there is a  $[24, k, d]$  code with  $k \geq 3$  and  $d \geq 18$ . By Lemma 2.1.2 (i),  $24 = n \geq 3^3 - 1$ , a contradiction. Hence this case can not occur.

If  $n = 18$  or  $21$ , then  $q(N) = 0$  by Corollary 3.2.2. From (10) we can compute the geometric genus.

Assume that  $B$  has 25 cusps. Let  $\Sigma$  be the double covering of  $\mathbb{P}^2$  ramified over  $B$ , and let  $Y$  be the minimal resolution of  $\Sigma$ . From the formulas in Sect. 6.1, we have  $b_2(Y) = 74$ . By Lemma 3.2.1, we have a code  $K'$  of type  $[25, k', 18]$  with  $k' \geq 1$ . Hence there is a non zero element in  $K'$ . This proves the result.  $\square$

## 7. A characterization of the branch locus of a generic triple covering

Let  $B$  be an irreducible curve with only nodes and cusps as its singularity, let  $X$  be a smooth surface, and let  $\pi : X \rightarrow \mathbb{P}^2$  be a finite covering ramified over  $B$ .

**Definition 7.1.1.**  $\pi$  is called a *generic covering* if it satisfies the following conditions:

- 1)  $\pi$  is unramified over  $\mathbb{P}^2 \setminus B$ ,
- 2)  $\pi^*(B) = 2R + C$ , where  $R$  is irreducible and nonsingular, and  $C$  is reduced,
- 3)  $\pi|_R : R \rightarrow B$  coincides with the normalization.

Two generic covering  $\pi : X \rightarrow \mathbb{P}^2$  and  $\pi' : X' \rightarrow \mathbb{P}^2$  branched along  $B$  are called equivalent, if there is an isomorphism  $\psi : X \rightarrow X'$  such that  $\pi = \pi' \circ \psi$ .

If  $\pi$  is of degree 3, then  $B$  has no nodes. We first consider the problem on the existence of generic triple coverings branched along  $B$ . In fact, the existence condition is that the cusps are 3-divisible.

**Theorem 7.1.2.** *Let  $B$  be an irreducible curve of even degree with  $n$  cusps. Then  $B$  is the branch locus of a generic triple covering if and only if the  $n$  cusps on  $\Sigma$  are 3-divisible.*

*Proof.* Let  $B$  be the branch locus of a generic triple covering  $\pi : X \rightarrow \mathbb{P}^2$ . In this case,  $R$  and  $C$  are both nonsingular [Mi]. We consider the normalization  $\tilde{X}$  of the pull-back surface  $\Sigma \times_{\mathbb{P}^2} X$ .

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\pi}_0} & X \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ \Sigma & \xrightarrow{\pi_0} & \mathbb{P}^2 \end{array}$$

It is easy to check that  $\tilde{\pi}_0$  is a double covering ramified over  $C$ . Because  $C$  is nonsingular,  $\tilde{X}$  is smooth. We can see also that  $\tilde{\pi}$  is ramified exactly over the cusps of  $\Sigma$ . Because  $\tilde{\pi}$  has no non-totally ramified locus and  $\mathbb{P}^2$  is simply connected, by the criterion for Galoisness given in [To, Ta3, TaZ],  $\tilde{\pi}$  must be a cyclic triple covering. Then by Lemma 3.1.1, the  $n$  cusps form a 3-divisible set  $N$  and  $\tilde{\pi} = \pi_N$ . Hence one direction of the theorem is proved.

Now assume that the  $n$  cusps  $p_1, \dots, p_n$  form a 3-divisible set  $N$ . Then we know that the triple covering  $\tilde{\pi} : \tilde{X} \rightarrow \Sigma$  exists and  $\tilde{X}$  is smooth, in fact,  $\tilde{X} = X_N$  and  $\tilde{\pi} = \pi_N$ . From the proof of Theorem 6.1.2, we can see that  $\tilde{X}$  admits two automorphisms  $\sigma$  and  $\tau$  of order 3 and 2 respectively.  $\sigma$  is induced by  $\tilde{\pi}$  and  $\tau$  is the lift of the involution  $\tau_0$  on  $\Sigma$

induced by  $\pi_0$ . By the canonical resolution of the singularities of double and cyclic triple coverings (cf. [BPV, AK, Ta1, Ta2 or Ta3]), we know that the inverse image  $\tilde{B}$  of  $B$  in  $\tilde{X}$  is a curve with  $n$  singular points (lying over the cusps), the singularity is three nonsingular local curves meeting at one point.

Then we consider the fixed locus  $\tilde{C}$  of  $\tau$ . Because  $\tau$  is the lift of  $\tau_0$ ,  $\tilde{\pi}(\tilde{C})$  is contained in  $\pi_0^{-1}(B)$ , and  $\tau$  has an action on the three points  $\tilde{\pi}^{-1}(p)$  for any smooth point  $p$  of  $\Sigma$  on  $\pi_0^{-1}(B)$ . Hence  $\tau$  has at least one fixed point  $q$  in  $\tilde{\pi}^{-1}(p)$ . Now from  $\tau\sigma\tau^{-1} = \sigma^2$  we can see that  $\tau(\sigma(q)) = \sigma^2(q) \neq \sigma(q)$ , thus we know that  $q$  is the only fixed point of  $\tau$  in  $\tilde{\pi}^{-1}(p)$ . Hence  $\tilde{B}$  splits into two curves  $\tilde{R} + \tilde{C}$ . In particular,  $\tilde{C}$  is nonsingular. Let  $X$  be the quotient of  $\tilde{X}$  under the group  $\langle 1, \tau \rangle$ . Then  $X$  is nonsingular and there is an induced triple covering  $\pi : X \rightarrow \mathbb{P}^2$ . Now it is easy to see that  $\pi$  is a generic triple covering branched along  $B$ .  $\square$

**Corollary 7.1.3.** *Two generic triple coverings  $\pi$  and  $\pi'$  ramified over  $B$  are equivalent if and only if  $\tilde{\pi}$  and  $\tilde{\pi}'$  are equivalent.*

*Proof.* The proof is obvious if we know that the two constructions in the proof above are invertible.  $\square$

**Theorem 7.1.4.** *Let  $N$  be a 3-divisible set of  $n$  cusps on a curve  $B$  of degree  $2d$ . Then*

- 1)  *$N$  is unique, i.e.,  $\pi_N$  is unique. Hence there is only one generic triple covering ramified over  $B$ .*
- 2) *If  $n < d^2$ , then there is no other generic covering ramified over  $B$ .*
- 3) *If  $n \geq d^2$ , then all other generic coverings ramified over  $B$  have degrees 4 or 5.*

*Proof.* To prove 1), we note that if there is another 3-divisible set  $N'$  consisting of the  $n$  cusps in  $N$ , then from Lemma 3.2.1, we have a code  $K \subset \mathbb{F}_3^n$  such that there are two independent elements  $\alpha$  and  $\beta$  in  $K$  with weight  $n$ . We can see that

$$w(\alpha + \beta) + w(\alpha - \beta) = n.$$

Hence there is a non zero element  $z = \alpha + \beta$  or  $\alpha - \beta$  in  $K$  with weight  $\leq n/2$ . On the other hand, by Theorem 6.1.2, this weight  $w(z) \geq 2d^2/3$ , thus  $n \geq 4d^2/3$ , which contradicts (11). This proves 1).

For 2), we use Nori's result (cf. [No] and [Fu]): if  $\pi : X \rightarrow \mathbb{P}^2$  is a generic covering ramified over  $B$ , then  $\deg \pi \leq 4d^2/(2d^2 - n) < 4$ . Hence  $\pi$  is a triple covering. 3) can be proved similarly by using (11).  $\square$

*Remark 7.1.5.* A well known example is the sextic with 9 cusps, there are 4 generic coverings ramified over this sextic, one is of degree 3, the others are of degree 4. Example 6.1.5 is also such an example with  $n = d^2$ .

*The proof of Proposition 6.1.1.* It is well known that there is a triple covering  $\pi : X \rightarrow \mathbb{P}^2$  ramified over the curve  $f = 0$ . In fact, the triple covering is constructed by  $z^3 + 3f_{2m}z + 2g_{3m} = 0$  (cf. [Ta3]). Because  $f_{2m}$  and  $g_{3m}$  are generic, we can assume that they define two smooth curves intersecting transversally. Then it is easy to see that  $X$  is smooth and  $\pi$  is a generic triple covering (cf. [Mi] or [Ta3]). By Theorem 7.1.2, the cusps on the branch locus defined by  $f = 0$  are 3-divisible.  $\square$

*Acknowledgments.* The author would like to thank W. Barth for useful discussions. He thanks also Prof. M. Teicher for introducing him to the very interesting problems on cuspidal plane curves.

## References

- [AK] Ashikaga, T., Konno, K.: Examples of degenerations of Castelnuovo surfaces. J. Math. Soc. Japan **42** (1991), no. 2, 229–246.

- [Ba1] Barth, W.: K3-surfaces with nine cusps. *Geom. Dedicata* **72** (1998), 171–178.
- [Ba2] Barth, W.: On the classification of K3 surfaces with nine cusps. In: *Complex Analysis and Algebraic Geometry*, (eds: T. Peternell, F.-O. Schreyer) (2000), Walter de Gruyter.
- [Ba3] Barth, W.: A quintic surface with 15 three-divisible cusps. Preprint.
- [BaR] Barth, W., Rams, S.: Projective surfaces with a three-divisible set of cusps. *math.AG/0112046*.
- [Be1] Beauville, A.: Sur le nombre maximum de points doubles d’une surface dans  $\mathbb{P}^3$  ( $\mu(5) = 31$ ). *Journées de géométrie algébrique d’Angers (1979)*, Sijthoff & Noordhoff (1980), 207–215.
- [Be2] Beauville, A.: L’application canonique pour les surfaces de type général. *Invent. Math.* **55** (1979), 121–140.
- [BPV] Barth, W., Peters A., Van de Ven, A.: *Compact Complex Surfaces*. Berlin, Heidelberg, New York: Springer, 1984.
- [BG] Bruce, J. W., Giblin, P. J.: A stratification of the space of plane quartic curves. *Proc. London Math. Soc.* **42** (1981), 270–298.
- [Ch1] Chisini, O.: Sulla identità birazionale delle funzioni algebriche di due variabili dotate di una medesima curva di diramazione. *Rend. Istit. Lombardo* **77** (1944), 339–356.
- [Ch2] Chisini, O.: Courbes de diramation des plans multiples et tresses algébriques. *Liege (Proc. 2nd Colloq. Algebraic Geom., Liege, 1952)* (1952), Masson, Paris, 11–27.
- [De] Debarre, O.: Inegalites numeriques pour les surfaces de type general. Appendice: “L’inegalite  $p_g \geq 2q - 4$  pour les surfaces de type general” par A. Beauville. *Bull. Soc. Math. Fr.* **110** (1982), no. 3, 319–346.
- [En] Enriques, F.: Sulla costruzione delle funzioni algebriche di due variabili possedenti una data curve di diramazione. *Annali di matematica pura ed applicata*, **1** (1923), no. Ser. 4, 185–198.
- [Fu] Fulton, W.: On the topology of algebraic varieties. *Proc. Symp. Pure Math.*, vol. 46, Part 1, Providence, RI: Amer. Math. Soc., 1987, pp. 15–46.
- [Hi] Hirzebruch, F.: Singularities of algebraic surfaces and characteristic numbers. *The Lefschetz Centennial Conf. Part I (Mexico City 1984)*, *Contemp. Math.*, vol. 58, 1985, pp. 141–155.
- [Iv] Ivinskis, K.: Normale Flächen und die Miyaoka-Kobayashi Ungleichung. Diplomarbeit, Boon, 1985.
- [Ku] Kulikov, V.: On Chisini’s conjecture. *Izv. Math.* **63** (1999), no. 6, 1139–1170.
- [Li] Libgober, A.: Fundamental groups of the complements to plane singular curves. *Proc. Symp. Pure Math.*, vol. 46, Part 2, Providence, RI: Amer. Math. Soc., 1987, pp. 29–45.
- [Lin] Van Lint, J. H.: *Introduction to Coding Theory*. Springer, Berlin, 1992.
- [Mi] Miranda, R.: Triple covers in algebraic geometry. *Amer. J. Math.* **107** (1985), 1123–1158.
- [Miy] Miyaoka, Y.: The maximal number of quotient singularities on surfaces with given numerical invariants. *Math. Ann.* **268** (1984), 159–171.
- [Ne] Nemirovskij, S.Yu.: Kulikov’s theorem on the Chisini conjecture. *Izv. Math.* **65** (2001), no. 1, 71–74.
- [Ni] Nikulin, V. V.: On Kummer surfaces. *Math. USSR Izv.* **9** (1975), no. 2, 261–275.
- [No] Nori, M.: Zariski’s conjecture and related problems. *Ann. Sci. École Norm. Sup.* **16** (1983), no. 4, 305–344.
- [Se] Segre, B.: Sulla caratterizzazione delle curve di diramazione dei piani multipli generale. *R. Accad. d’Ital. Cl. di Sci. Fis. Mat. et Nat., Memorie* **1** (1930), 71–97.
- [Ta1] Tan, S.-L.: Galois triple covers of surfaces. *Science in China* **34** (1991), no. Ser. A, 935–942.
- [Ta2] Tan, S.-L.: Surfaces whose canonical maps are of odd degrees. *Math. Ann.* **292** (1992), 13–29.
- [Ta3] Tan, S.-L.: Triple covers on smooth algebraic varieties. In: *Geometry and Nonlinear Partial Differential Equations*, AMS/IP Studies in Advanced Mathematics **29** (2002), 143–164.
- [TaT] Tan, S.-L., Teicher, M.: Fundamental groups of the branch curves of generic triple coverings. (In preparation).
- [TaZ] Tan, S.-L., Zhang, D.-Q.: The determination of integral closure and geometric applications. *Advances in Mathematics* (to appear).
- [To] Tokunaga, H.: Triple coverings of algebraic surfaces according to the Cardano formula. *J. Math. Kyoto Univ.* **31** (1991), no. 2, 359–375.
- [Za1] Zariski, O.: On the problem of the existence of algebraic functions. *Amer. J. Math.* **51** (1929), 305–328.
- [Za2] Zariski, O.: On the non-existence of curves of order 8 with 16 cusps. *Amer. J. Math.* **53** (1931), 309–318.

DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200062, P. R. OF CHINA

*E-mail address:* sltan@math.ecnu.edu.cn