

Extremal sparsity property of the Jordan canonical form

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1, Main result

One of the central results in linear algebra is the Jordan canonical form theorem which states that every square complex matrix A is similar to a Jordan matrix

$$J(A) = \text{diag}(J_{n_1}(\lambda_1), \dots, J_{n_k}(\lambda_k))$$

where

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

is a Jordan block of order n_i . This Jordan canonical form $J(A)$ is unique up to permutations of the diagonal Jordan blocks.

Two matrices are similar if and only if they have a common Jordan canonical form.

We use $J(A)$ to mean any of the Jordan canonical forms of A .

$\mathcal{S}(A)$: the set of all complex matrices that are similar to A .

It seems that $J(A)$ has the simplest form among the matrices in $\mathcal{S}(A)$. But in what sense? Structure? Sparsity? In 2005 one of the authors asked whether for any matrix A , $J(A)$ has the largest number of zero entries among all the matrices in $\mathcal{S}(A)$.

The answer is no, as shown by the following example due to Chi-Kwong Li:

$$A = \begin{bmatrix} 0 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

A has 11 zero entries while $J(A)$ has 10 zero entries.

The correct direction is to consider off-diagonal entries.

We show that for any complex matrix A , $J(A)$ has the largest number of *off-diagonal* zero entries among all the matrices in $\mathcal{S}(A)$, and we characterize the matrices in $\mathcal{S}(A)$ that attain this largest number.

A square complex matrix is called a *monomial matrix* if it has exactly one nonzero entry in each row and each column. Let Γ_n be the set of monomial matrices of order n . Then $M \in \Gamma_n$ if and only if there exist a permutation matrix P and a nonsingular diagonal matrix D such that $M = PD$ if and only if there exist a permutation matrix Q and a nonsingular diagonal matrix E such that $M = EQ$. Obviously, Γ_n is a multiplicative group.

$\phi(A)$: the number of off-diagonal nonzero entries of a matrix A .

Theorem *Let A be a square complex matrix and $B \in \mathcal{S}(A)$. Then*

$$\phi(B) \geq \phi(J(A)).$$

Equality holds if and only if there exists a monomial matrix M such that

$$M^{-1}BM = J(A).$$

The Theorem shows that up to permutation similarity, $J(A)$ is the unique zero-nonzero pattern among the matrices in $\mathcal{S}(A)$ that attains the largest number of off-diagonal zero entries.

This gives a combinatorial characterization of the Jordan canonical form.

2, Key lemmas in the proof

The following purely combinatorial lemma is the basis of our analysis.

Lemma 1 *Let n, k be positive integers with $1 \leq k \leq n$. If a matrix A of order n satisfies $\phi(A) \leq n - k$, then there exists a permutation matrix P such that*

$$P^T A P = \text{diag}(A_1, A_2, \dots, A_k)$$

where A_j is square and non-void for $j = 1, \dots, k$.

Proof. With $A = (a_{ij})$ we associate a graph G with vertex set $V = \{1, 2, \dots, n\}$ where there is an edge between vertices i and j if and only if $i \neq j$, and $a_{ij} \neq 0$ or $a_{ji} \neq 0$.

Another ingredient is

Lemma 2 *Let*

$$A = \begin{bmatrix} a & x^T \\ 0 & B \end{bmatrix}$$

be a complex matrix of order n where B has order $n - 1$. If $J(A)$ has only one Jordan block, then $J(B)$ has only one Jordan block.

Proof. To the contrary, suppose $J(B)$ has at least two Jordan blocks. Then we can get a contradiction.

3, The real case

For real numbers a, b denote

$$C(a, b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

and denote by $C_k(a, b)$ the matrix of order $2k$:

$$C_k(a, b) = \begin{bmatrix} C(a, b) & I & 0 & \cdots & 0 \\ 0 & C(a, b) & I & \cdots & 0 \\ 0 & 0 & C(a, b) & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C(a, b) \end{bmatrix}$$

where I is the identity matrix.

It is known that every real square matrix A is similar via a real matrix to a block diagonal real matrix of the form

$$\text{diag}(C_{n_1}(a_1, b_1), \dots, C_{n_t}(a_t, b_t), J_{m_1}(\mu_1), \dots, J_{m_s}(\mu_s))$$

where $a_j + ib_j$ are the nonreal eigenvalues of A , $j = 1, \dots, t$ and μ_q are the real eigenvalues of A , $q = 1, \dots, s$.

This is called the *real Jordan canonical form* of A .

A referee asked whether a conclusion similar to that in Theorem holds for the real Jordan canonical form, that is, if A is a real square matrix, does the real Jordan canonical form of A have the largest number of off-diagonal zero entries among all *real* matrices in $\mathcal{S}(A)$?

That this is not so is illustrated by the following example:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then A is a matrix in real Jordan canonical form with eigenvalues $1 \pm i$, each of multiplicity 2, and

$$T^{-1}AT = \begin{bmatrix} 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

We have $\phi(A) = 6 > 5 = \phi(T^{-1}AT)$.

On the other hand, we can prove that *the real Jordan canonical form of a diagonalizable real matrix contains the largest number of off-diagonal zero entries among all the real matrices similar to A .*

In contrast to the Jordan canonical form, there are cases of equality other than the type given in Theorem. For example, let

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then the characteristic polynomial of A is $\lambda^4 + 1$, and it has four distinct, non-real eigenvalues. Thus A is diagonalizable over the complex number field, and the real Jordan canonical form B of A has four off-diagonal nonzero entries, the same number as A . But clearly $M^{-1}AM \neq B$ for any monomial matrix M .

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Three problems solved

1. Let $z(n)$ be the largest integer z such that every $n \times n$ complex matrix is unitarily similar to a matrix with at least z zero entries. Schur's unitary triangularization theorem implies that

$$z(n) \geq (n-1)n/2.$$

At the 5th China Matrix Theory and Applications Conference, Shanghai, August 2002 and the 12th ILAS Conference, Regina, Canada, June 2005, I asked whether

$$z(n) = (n-1)n/2?$$

D.Z. Dokovic and C.R. Johnson have answered this question affirmatively by using a dimension argument from differential manifolds in their paper *Unitarily achievable zero patterns and traces of words in A and A^** , **Linear Algebra Appl.**,421(2007).

2. Denote by $s_1(G) \geq s_2(G) \geq \dots$ the singular values of a matrix G in decreasing order. In 2000 I posed the following

Conjecture. Let A, B be positive semidefinite matrices of the same order and $0 \leq t \leq 1$. Then

$$s_j(A^t B^{1-t} + A^{1-t} B^t) \leq s_j(A + B), \quad j = 1, 2, \dots$$

K.M.R. Audenaert has proved this conjecture in *A singular value inequality for Heinz means*,

Linear Algebra Appl., 422(2007).

3. For every positive integer n we have

(i) $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor$;

(ii) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} \rfloor = \lfloor \sqrt{9n+8} \rfloor$;

(iii) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} \rfloor = \lfloor \sqrt{16n+20} \rfloor$;

(iv) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} \rfloor = \lfloor \sqrt{25n+49} \rfloor$.

In the paper *Formulae for sums of consecutive square roots*, **Math. Intelligencer**, **27(2005)**, no.4, I proved (iv) and made the following

Conjecture. For any $k \geq 6$, there exists no constant c depending only on k such that

$$\left\lfloor \sum_{i=0}^{k-1} \sqrt{n+i} \right\rfloor = \left\lfloor \sqrt{k^2n+c} \right\rfloor$$

holds for all positive integers n .

P.W. Saltzman and P.Z. Yuan have proved this conjecture in their paper *On sums of consecutive integral roots*, **Amer. Math. Monthly**, March 2008.

A new book:

X. Zhan, Matrix Theory (in Chinese),

Higher Education Press, Beijing, June 2008

Thank you!