

Extremal numbers of positive entries of imprimitive nonnegative matrices

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A square nonnegative matrix A is said to be primitive if A^p is a positive matrix for some positive integer p ; otherwise A is called imprimitive. A primitive matrix is necessarily irreducible. The *imprimitivity index* k of an irreducible nonnegative matrix A is the number of eigenvalues of A whose moduli are equal to the spectral radius of A . A is primitive if $k = 1$, and imprimitive if $k > 1$.

Let $\sigma(A)$ denote the number of positive entries of a nonnegative matrix A . For a real number x , denote by $\lfloor x \rfloor$ the largest integer not exceeding x .

Suppose A is an $n \times n$ irreducible nonnegative matrix with imprimitivity index k . In 1993 M. Lewin proved that if $k \leq 4$ then $\sigma(A) \leq \lfloor n^2/k \rfloor$ and this bound is sharp, and if $k \geq 5$ then $\sigma(A) \leq \lfloor n^2/4 \rfloor$. The case $n = 16$ and $k = 15$ shows that the bound $\lfloor n^2/4 \rfloor$ is not sharp. In fact, it is easy to see that if A is 16×16 and of imprimitivity index 15 then $\sigma(A) \leq 17$, while $\lfloor 16^2/4 \rfloor = 64$.

We will determine the sharp upper and lower bounds for $\sigma(A)$, and consider related applications. One interesting thing here is that the rule for the maximum $\sigma(A)$ changes when k turns from 4 to 5.

Theorem 1. *Let $\Gamma(n, k)$ be the set of $n \times n$ irreducible nonnegative matrices with imprimitivity index k . Then*

$$\max\{\sigma(A) | A \in \Gamma(n, k)\} = \begin{cases} \lfloor n^2/k \rfloor & \text{if } 1 \leq k \leq 4, \\ 2n - k + \lfloor (n - k)^2/4 \rfloor & \text{if } k \geq 5. \end{cases}$$

Denote by $\text{ind}(A)$ the imprimitivity index of A . The next result estimates this index by the number of positive entries.

Corollary 2. *Let A be an $n \times n$ irreducible nonnegative matrix. If $n^2 < 4\sigma(A)$ then*

$$\text{ind}(A) \leq \lfloor n^2/\sigma(A) \rfloor.$$

If $n^2 \geq 4\sigma(A)$ then

$$\text{ind}(A) \leq \lfloor n + 2 - 2\sqrt{\sigma(A) - n + 1} \rfloor.$$

We remark that Corollary 2 includes Lewin's result that if $\sigma(A) > n^2/2$ then $\text{ind}(A) = 1$, i.e., A is primitive.

Theorem 3. *Let $\Gamma(n, k)$ be the set of $n \times n$ irreducible nonnegative matrices with imprimitivity index k . Then*

$$\min\{\sigma(A) | A \in \Gamma(n, k)\} = \begin{cases} n + 1 & \text{if } k < n, \\ n & \text{if } k = n. \end{cases}$$

The proofs of the above results involve study of a cyclic quadratic form and construction of special graphs.

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Two problems solved

1. Let $z(n)$ be the largest integer z such that every $n \times n$ complex matrix is unitarily similar to a matrix with at least z zero entries. Schur's unitary triangularization theorem implies that

$$z(n) \geq (n - 1)n/2.$$

At the 5th China Matrix Theory and Applications Conference, Shanghai, August 2002 and the 12th ILAS Conference, Regina, Canada, June 2005, I asked whether

$$z(n) = (n - 1)n/2?$$

D.Z. Dokovic and C.R. Johnson have answered this question affirmatively by using a dimension argument from differential manifolds in their paper Unitarily achievable zero patterns and traces of words in A and A^* , which is to appear in Linear Algebra Appl.

2. For every positive integer n we have

(i) $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor$;

(ii) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} \rfloor = \lfloor \sqrt{9n+8} \rfloor$;

(iii) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} \rfloor = \lfloor \sqrt{16n+20} \rfloor$;

(iv) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} \rfloor = \lfloor \sqrt{25n+49} \rfloor$.

In the paper *Formulae for sums of consecutive square roots*, *Math. Intelligencer*, 27(2005), no.4, I proved (iv) and made the following

Conjecture. For any $k \geq 6$, there exists no constant c depending only on k such that

$$\left\lfloor \sum_{i=0}^{k-1} \sqrt{n+i} \right\rfloor = \left\lfloor \sqrt{k^2n+c} \right\rfloor$$

holds for all positive integers n .

I also mentioned this conjecture at the 6th China Matrix Theory and Applications Conference, Harbin, July 2004.

P.W. Saltzman and P.Z. Yuan have proved this conjecture in their paper *On sums of consecutive integral roots*, which is to appear in *Amer. Math. Monthly*.

Two open problems

Let $S_n[a, b]$ denote the set of $n \times n$ real symmetric matrices whose entries are in the interval $[a, b]$. For an $n \times n$ real symmetric matrix A , we denote the eigenvalues of A in decreasing order by $\lambda_1(A) \geq \dots \geq \lambda_n(A)$. The *spread* of such an A is defined to be $s(A) = \lambda_1(A) - \lambda_n(A)$.

Problem 1. For a given integer j with $2 \leq j \leq n - 1$, determine

$$\begin{aligned} & \max\{\lambda_j(A) : A \in S_n[a, b]\}, \\ & \min\{\lambda_j(A) : A \in S_n[a, b]\} \end{aligned}$$

and determine those matrices which attain the maximum and those matrices which attain the minimum. The cases $j = 1, n$ are solved in [1].

Problem 2. For generic $a < b$ determine

$$\max\{s(A) : A \in S_n[a, b]\}$$

and determine those matrices which attain the maximum. The case $a = -b$ is solved in [1].

[1], X. Zhan, Extremal eigenvalues of real symmetric matrices with entries in an interval. SIAM J. Matrix Anal. Appl. 27(2005), no.3.