Singular, Nonsingular, and Bounded Rank Completions of ACI-Matrices

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September 15, 2010

Abstract

An affine column independent matrix is a matrix whose entries are polynomials of degree at most 1 in a number of indeterminates where no indeterminate appears with a nonzero coefficient in two different columns. A completion is a matrix obtained by giving values to each of the indeterminates. Affine column independent matrices are more general than partial matrices where each entry is either a constant or a distinct indeterminate. We determine when the rank of all completions of an affine column independent matrix is bounded by a given number, generalizing known results for partial matrices. We also characterize the square partial matrices over a field all of whose completions are nonsingular. The maximum number of free entries in such

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 The author's research was supported by the NSFC grant 10971070.

matrices of a given order is determined as well as the partial matrices with this maximum number of free entries.

AMS classifications: 05C50,15A15, 15A99

Keywords: partial matrix, affine column independent matrix, completion, determinant, singular, nonsingular, rank.

1 Introduction

Let $\mathbf{F}$ be a field. A *partial matrix* over $\mathbf{F}$ is a matrix in which some entries are specified as elements of $\mathbf{F}$, and the other entries are unspecified and can be freely chosen within $\mathbf{F}$. A *completion* of a partial matrix is a specific choice of values from $\mathbf{F}$ for its unspecified entries. We also use completion to refer to the matrix obtained by choosing values for its unspecified entries. In a partial matrix, we call the unspecified entries *indeterminates*, since they are free to range over $\mathbf{F}$.

There is a substantial literature on matrix completions of which for our purposes [1, 4, 5, 6] are particularly relevant. Matrix completions have found applications in collaborative filtering and image processing [3] and in system analysis [7].

Let $M_{m,n}(\mathbf{F})$ be the set of $m \times n$ matrices whose entries belong to $\mathbf{F}$, and let $P_{m,n}(\mathbf{F})$ be the set of $m \times n$ partial matrices over $\mathbf{F}$. If $m = n$, then we use the abbreviations $M_n(\mathbf{F})$ and $P_n(\mathbf{F})$, respectively. We call elements in $\mathbf{F}$ *constants* and call matrices in $M_{m,n}(\mathbf{F})$ *matrices of constants* or, more simply, *constant matrices*, in contrast to indeterminates and partial matrices respectively.

First consider the following general problem.

Problem 1 *Characterize the square partial matrices over* $\mathbf{F}$ *each of whose completions has the same determinant.*

For example, over any field $\mathbf{F}$ and for every choice of the indeterminates, here denoted by "?", it is easy to see (successively subtract row i from row $i-1$ for $i = 2, 3, \dots, n$) that

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ ? & 0 & 1 & 1 & \cdots & 1 \\ ? & ? & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ ? & ? & \cdots & ? & 0 & 1 \end{bmatrix} = 1.$$

Let A be a partial matrix in $P_n(\mathbf{F})$ with k unspecified entries. We label the unspecified entries of A with distinct indeterminates $x_1, x_2, \dots, x_k$ over $\mathbf{F}$, and then the determinant of A is a polynomial in $x_1, x_2, \dots, x_k$ with coefficients from $\mathbf{F}$:

$$\det A = p_A(x_1, x_2, \dots, x_k).$$

For example, the above determinant when $n = 4$ becomes

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ x_1 & 0 & 1 & 1 \\ x_2 & x_3 & 0 & 1 \end{bmatrix} = 1.$$

In general, the degree of each indeterminate in $p_A(x_1, x_2, \dots, x_k)$ is at most 1. In what follows we consider a more general class of matrices with the same property.

Let $\mathbf{F}[x_1, x_2, \dots, x_k]$ be the ring of polynomials in the indeterminates $x_1, x_2, \dots, x_k$ with coefficients from the field $\mathbf{F}$. A matrix A with entries from $\mathbf{F}[x_1, x_2, \dots, x_k]$ is called an *affine column independent* matrix, abbreviated ACI-matrix, provided the following two conditions hold:

(i) each entry of A is a polynomial in $\mathbf{F}[x_1, \dots, x_k]$ of degree at most one, that is, has the form

$$c_0 + c_1 x_1 + \cdots + c_k x_k$$

for some constants $c_0, c_1, \dots, c_k$, and

(ii) an indeterminate does not appear with a nonzero coefficient in two different columns of A.

Thus, in an ACI-matrix, the indeterminates in the unspecified entries in a column are independent of the indeterminates in the unspecified entries in every other column. In a partial matrix, each unspecified entry is independent of all the other unspecified entries. An *affine row independent matrix* can be defined in a very similar way to affine column independent matrices. By combining affine combinations of indeterminates to new indeterminates, it is easy to see that partial matrices and matrices that are both affine row and column independent are essentially equivalent.

Let $\mathcal{A}_{m,n}(\mathbf{F}[x_1, x_2, \ldots, x_k])$ be the set of $m \times n$ affine column independent matrices over $\mathbf{F}[x_1, x_2, \ldots, x_k]$. This notation is abbreviated to $\mathcal{A}_n(\mathbf{F}[x_1, x_2, \ldots, x_k])$ when $m = n$. By a *completion* of a matrix in $\mathcal{A}_n(\mathbf{F}[x_1, x_2, \ldots, x_k])$ we mean an assignment of values in $\mathbf{F}$ to the indeterminates $x_1, x_2, \ldots, x_k$. Obviously, a partial matrix over $\mathbf{F}$ is an ACI-matrix, and every submatrix of an ACI-matrix is also an ACI-matrix. If A is a square ACI-matrix, then the determinant $p_A(x_1, x_2, \ldots, x_k)$ of A is a polynomial in $x_1, x_2, \ldots, x_k$ with degree in each x_i equal to 0 or 1.

Problem 1 for the larger set of square ACI matrices asks to characterize those matrices such that $p_A(x_1, x_2, \ldots, x_k)$ is a constant polynomial. In case this constant is required to be nonzero, the following lemma shows that it is enough to know only that $p_A(x_1, x_2, \ldots, x_k)$ is nonzero for all choices of values for $x_1, x_2, \ldots, x_k$ in $\mathbf{F}$.

Lemma 1 *Let $A \in \mathcal{A}_n(\mathbf{F}[x_1, x_2, \ldots, x_k])$ be an ACI-matrix. Then all the completions of A have the same nonzero determinant if and only if all the completions of A are nonsingular.*

Proof. All the completions of A have the same nonzero determinant if and only if $\det A = p_A(x_1, x_2, \ldots, x_k)$ is a nonzero, constant polynomial. If $p_A(x_1, x_2, \ldots, x_k)$ is a nonzero constant, then all completions of A are nonsingular. Now suppose that $p_A(x_1, x_2, \ldots, x_k)$ is not a nonzero constant. Since $p_A(x_1, x_2, \ldots, x_k)$ is a polynomial whose degree in each x_i is at most 1, then for some choice of values with $x_1 = c_1, x_2 = c_2, \ldots, x_k = c_k$, we have $p_A(c_1, c_2, \ldots, c_k) = 0$, and hence this completion of A is singular. □

There are the following three possibilities for a given square ACI-matrix: (i) all completions are nonsingular, (ii) all completions are singular, and (iii) there are both nonsingular and singular completions, that is, neither (i) nor (ii) holds. A partial matrix or ACI-matrix A each of whose completions is singular, that is, $p_A(x_1, x_2, \ldots, x_k)$ is an identically zero polynomial, may have a very general structure. For example, if for some positive integer

t, A has t rows each of whose entries is specified and these t rows are linearly dependent, then all completions of A are singular. Because of Lemma 1, Problem 1 splits into the following two problems.

Problem 2 *Characterize ACI-matrices all of whose completions are singular.*

Problem 3 *Characterize ACI-matrices all of whose completions are nonsingular.*

A problem more general than Problem 2 is:

Problem 4 *Given a nonnegative integer ρ, characterize ACI-matrices all of whose completions have rank less than or equal to ρ.*

Problem 2 has been solved for partial matrices by Hartfiel and Loewy in [5] resulting in a theorem which is a determinantal generalization of the well-known combinatorial Frobenius-König theorem. Their proof is long and complicated. The more general Problem 4, restricted to partial matrices, has been solved in [1, 4, 6, 7] with much simpler proofs. Bapat [1] obtains a more abstract theorem from which the solution follows.

We first solve Problem 4 for ACI-matrices from which we obtain the solution for partial matrices. We then solve Problem 3 in the case that the field $\mathbf{F}$ has at least $n+1$ elements where n is the order of the matrix. We also determine the maximum number of indeterminates (unspecified entries) that a partial matrix of a given order can have if all of its completions are nonsingular; the partial matrices that attain this maximum number are then characterized.

2 Main Results

The following lemma is elementary.

Lemma 2 *Let A be an $m \times n$ ACI-matrix over $\mathbf{F}[x_1, \ldots, x_k]$. If $T \in M_m(\mathbf{F})$ is a constant matrix and $P \in M_n(\mathbf{F})$ is a permutation matrix, then TAP is an ACI-matrix over $\mathbf{F}[x_1, \ldots, x_k]$.*

Proof. Let $A = (a_1, a_2, \ldots, a_n)$ where $a_1, a_2, \ldots, a_n$ are the columns of A. Then $TAP = (Ta_1, Ta_2, \ldots, Ta_n)P$, and the result follows. □

Theorem 3 *Let $\mathbf{F}$ be a field and A be an $m \times n$ ACI-matrix over $\mathbf{F}[x_1, \ldots, x_k]$. Let ρ be a nonnegative integer less than* $\min\{m, n\}$. *Then the following two statements are* *equivalent:*

(i) *All completions of A have rank $\leq \rho$.*

(ii) *There exists a nonsingular constant matrix T such that for some positive integers r and s with $r+s=m+n-\rho$, TA has an $r \times s$ zero submatrix.*

Proof. First assume that (ii) holds. Multiplying TA on the left and right by permutation matrices if necessary, we may assume that

$$TA = \left[\begin{array}{c|c} A_{11} & O_{m+n-\rho-s,s} \\ \hline A_{21} & A_{22} \end{array}\right].$$

We always use $O_{r,s}$ to denote the $r \times s$ zero matrix. If we use the same letter A to mean any completion of A, then the rank of TA is at most the rank of

$$\left[\begin{array}{c} A_{11} \\ \hline A_{21} \end{array}\right]$$

plus the rank of A_{22}. The former has $n-s$ columns and so its rank is at most $n-s$. The latter, that is, A_{22}, has $\rho+s-n$ rows and so its rank does not exceed $\rho+s-n$. Hence the rank of TA, and so the rank of A, is at most

$$(n-s)+(\rho+s-n)=\rho.$$

Therefore (i) holds.

We use induction on m to prove that (i) implies (ii). If $m=1$ or $n=1$, then $\rho=0$, and A is a zero matrix. In either case (ii) clearly holds. Next let $m \geq 2$, $n \geq 2$ and assume that the result holds for ACI-matrices with row number less than m.

First suppose that A is a matrix in $M_{m,n}(\mathbf{F})$. Then the rank of A is at most ρ, and hence there exists a nonsingular matrix T bringing A to row-echelon form so that TA has $m-\rho$ zero rows and hence an $m-\rho$ by n zero submatrix. Letting $r=m-\rho$ and $s=n$, we have an r by s zero submatrix of TA with $r+s=m+n-\rho$.

Now assume that A contains at least one indeterminate with a nonzero coefficient. By left and right multiplication by permutation matrices if necessary, without loss of generality we may assume that x_1 occurs with a nonzero coefficient in the $(1,1)$-position of A; since A is an ACI-matrix, x_1 does not occur with a nonzero coefficient in any entry in columns $2,3,\ldots,n$ of A. By elementary row operations applied to A (multiplication on the left by a nonsingular constant matrix) and using Lemma 2, we can assume that x_1 occurs with a nonzero coefficient only in the first entry of column 1. Let A' be obtained from A by deleting row 1 and column 1. Then A' is an ACI-matrix. Using the fact that x_1 occurs in A only in the (1,1)-position, we conclude that all completions of A' have rank less than or equal to $\rho - 1$. By induction there exists a nonsingular constant matrix T' such that $T'A'$ has a $p \times q$ zero submatrix for some p and q with

$$p + q = (m - 1) + (n - 1) - (\rho - 1) = m + n - \rho - 1.$$

The matrix $T = [1] \oplus T'$ is a nonsingular constant matrix. There exist permutation matrices P, Q such that

$$PTAQ = \left[\begin{array}{c|c} C_{11} & O_{p,q} \\ \hline C_{21} & C_{22} \end{array}\right] \quad (p + q = m + n - \rho - 1),$$

where C_{11} is a $p \times (n - q)$ ACI-matrix and C_{22} is an $(m - p) \times q$ ACI-matrix.

Since $PTAQ$ is an ACI-matrix by Lemma 2, C_{11} and C_{22} contain no indeterminate in common. Hence every completion of A can be obtained by first completing C_{11} and C_{22} and then choosing values for those indeterminates in A that do not occur in C_{11} and C_{22}. Suppose that both C_{11} has a completion of rank $\rho - (m - p) + 1$ or greater and C_{22} has a completion of rank $\rho - (n - q) + 1$ or greater. Then $PTAQ$ and hence A has a completion of rank at least

$$\begin{aligned} (\rho - (m - p) + 1) + (\rho - (n - q) + 1) &= 2\rho - (m + n) + (p + q) + 2 \\ &= 2\rho - (m + n) + (m + n - \rho - 1) + 2 \\ &= \rho + 1, \end{aligned}$$

a contradiction. Thus either all completions of C_{11} have rank at most $\rho - (m - p)$, or all completions of C_{22} have rank at most $\rho - (n - q)$. Suppose the former holds. It is easy to verify that $\rho - (m - p) < \min\{p, n - q\}$. Then using induction on C_{11}, there are a nonsingular constant matrix S and a permutation matrix Q_1 such that

$$(SPT)A(QQ_1) = S(PTAQ)Q_1 = \left[\begin{array}{c|c|c} & O_{a,b} & \\ \cline{1-2} & & O_{p,q} \\ \hline \multicolumn{2}{c|}{C_{21}} & C_{22} \end{array}\right]$$

where a and b are positive integers with

$$a + b = p + (n - q) - (\rho - (m - p)) = m + n - \rho - q.$$

Hence $(SPT)A$ has an $a \times (b+q)$ zero submatrix where

$$a + (b+q) = (a+b) + q = (m+n-\rho-q) + q = m+n-\rho,$$

as desired. A similar argument works if all completions of C_{22} have rank at most $\rho-(n-q)$. Therefore (ii) holds, completing the proof of the theorem. □

Theorem 3 in the case that $m = n$ and $\rho = n-1$ gives the following corollary which, like for partial matrices, can be regarded as a generalization of the Frobenius-König theorem.

Corollary 4 *Let $\mathbf{F}$ be a field and A be a square ACI-matrix of order n over $\mathbf{F}[x_1, \ldots, x_k]$. Then the determinant of A is identically 0 if and only if there exists a nonsingular constant matrix $T \in M_n(\mathbf{F})$ such that TA has an $r \times s$ zero submatrix for some integers r and s with $r + s = n + 1$.*

Since partial matrices over a field are ACI-matrices, Corollary 4 immediately implies the following result.

Corollary 5 *Let $\mathbf{F}$ be a field and let $A \in P_n(\mathbf{F})$ be a partial matrix. Then all completions of A have determinant equal to 0 if and only if there exists a nonsingular constant matrix $T \in M_n(\mathbf{F})$ such that TA has an $r \times s$ zero submatrix for some integers r and s with $r + s = n + 1$.*

In the case of partial matrices, an additional equivalent condition can be included in Theorem 3. We will need the following elementary and well-known fact.

Lemma 6 *Let A be an $r \times s$ matrix and B be an $s \times t$ matrix over a field $\mathbf{F}$. If $AB = 0$, then* $\operatorname{rank} A + \operatorname{rank} B \leq s$.

Proof. $AB = 0$ implies that every column of B is in the null space of A. Since the dimension of the null space of A is $s - \operatorname{rank} A$, $\operatorname{rank} B \leq s - \operatorname{rank} A$. □

Theorem 7 *Let $\mathbf{F}$ be a field and A be an $m \times n$ partial matrix over $\mathbf{F}$. Let ρ be a nonnegative integer less than $\min\{m, n\}$. Then the following three statements are equivalent:*

(i) *All completions of A have rank $\leq \rho$.*

(ii) *There exists a nonsingular constant matrix T such that for some positive integers r and s with $r + s = m + n - \rho$, TA has an $r \times s$ zero submatrix.*

(iii) *For some positive integers r and s with $r + s \geq m + n - \rho$, A has an $r \times s$ constant submatrix whose rank is at most $r + s - (m + n - \rho)$.*

Proof. By Theorem 3 we know the equivalence of (i) and (ii). Thus it suffices to show the equivalence of (ii) and (iii). First assume that (iii) holds. Without loss of generality, we may assume that

$$A = \left[\begin{array}{c|c} & B \\ \hline & \end{array}\right],$$

where B is an $r \times s$ constant matrix with $r+s \geq m+n-\rho$ and rank $B \leq r+s-(m+n-\rho)$. There exists a nonsingular constant matrix T such that multiplication of A on the left by T puts B in row-echelon form with at least

$$r - ((r+s) - (m+n-\rho)) = m+n-\rho-s$$

zero rows. Hence TA has an $(m+n-\rho-s) \times s$ zero submatrix with

$$(m+n-\rho-s) + s = m+n-\rho.$$

Thus (ii) holds.

To complete the proof we show that when A is a partial matrix, (ii) implies (iii). Let $T = [t_{ij}]$ be a nonsingular constant matrix such that $G = [g_{ij}] = TA$ has an $r \times s$ zero submatrix with $r + s = m + n - \rho$. Without loss of generality we assume that

$$TA = \left[\begin{array}{c|c} O_{r,s} & \\ \hline & \end{array}\right].$$

Let

$$\gamma = \{1, 2, \ldots, r\}, \beta = \{1, 2, \ldots, s\}, M = \{1, 2, \ldots, m\},$$

and let

$$\alpha = \{j : t_{ij} \neq 0 \text{ for some } i \in \gamma\}, \alpha' = M \setminus \alpha.$$

Then $G[\gamma|\beta] = O$ and $T[\gamma|\alpha'] = O$. (Here e.g. the notation $G[\gamma|\beta]$ designates the submatrix of G determined by the rows with indices in γ and columns with indices in β.) We claim that

(a) $|\alpha| + |\beta| \geq m + n - \rho$, and that

(b) $B = A[\alpha|\beta]$ is a constant matrix with rank $B \leq |\alpha| + |\beta| - (m + n - \rho)$.

These two claims show that (iii) holds.

Since T is nonsingular and $T[\gamma|\alpha'] = O$, rank $T[\gamma|M] =$ rank $T[\gamma|\alpha] = r$. Thus $|\alpha| \geq r$ and so

$$|\alpha| + |\beta| \geq r + s = m + n - \rho,$$

and (a) holds.

Suppose that there exists $i_0 \in \alpha$ and $j_0 \in \beta$ such that a_{i_0,j_0} is an indeterminate z. By the definition of α, there exists $i \in \gamma$ such that $t_{i,i_0} \neq 0$. But then

$$g_{i,j_0} = \sum_{k=1}^{m} t_{ik} a_{k,j_0} = t_{i,i_0} z + \cdots ,$$

since $a_{i_0,j_0} = z$ occurs only once in this summation. This contradicts $G[\gamma|\beta] = O$. Thus B is a constant matrix. Since $G[\gamma|\beta] = O$ and $T[\gamma|\alpha'] = O$, we have by block multiplication that

$$O = G[\gamma|\beta] = T[\gamma|\alpha]A[\alpha|\beta] + T[\gamma|\alpha']A[\alpha'|\beta] = T[\gamma|\alpha]A[\alpha|\beta].$$

We now apply Lemma 6 to get

$$\text{rank } A[\alpha|\beta] \leq |\alpha| - \text{rank } T[\gamma|\alpha] = |\alpha| - r.$$

Thus

$$\begin{aligned}
\text{rank } B &= \text{rank } A[\alpha|\beta] \\
&\leq |\alpha| - r \\
&= |\alpha| - r + 0 \\
&= |\alpha| - r + (|\beta| - s) \\
&= |\alpha| + |\beta| - (r + s) \\
&= |\alpha| + |\beta| - (m + n - \rho).
\end{aligned}$$

Hence (iii) holds and the proof is complete. □

We note that if A is an ACI-matrix, (iii) of Theorem 7 implies (i) but (i) does not imply (iii). For example, let A be the $n \times n$ ACI-matrix

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

Then all completions of A have rank $\rho \leq 1$, but A does not contain an $r \times s$ constant submatrix with $r + s \geq 2n - 1$.

Corollary 8 (Hartfiel and Loewy [5]) *Let* $\mathbf{F}$ *be a field and* $A \in P_n(\mathbf{F})$ *be a partial matrix. Then* $\det A = 0$ *if and only if for some integers* r *and* s *with* $r + s \geq n + 1$, A *contains an* $r \times s$ *constant submatrix* $B \in M_{r,s}(\mathbf{F})$ *such that* $\operatorname{rank} B \leq r + s - n - 1$.

Recall that the *term rank* $\rho(A)$ of a $(0, 1)$-matrix A is the maximum number of 1s with no two from the same row or column.

Corollary 9 (König-Egerváry) *Let* A *be an* $m \times n$ $(0, 1)$*-matrix. Then the term rank of* A *equals the minimum number of rows and columns that contain all the* 1*s of* A*; equivalently, the term rank of* A *equals*

$$(m + n) - \max\{r + s : r \geq 0, s \geq 0, A \textit{ has an } r \times s \textit{ zero submatrix}\}.$$

Proof. We show that this corollary is a special case of Theorem 7. Here if $r = 0$ or $s = 0$, the zero submatrix is vacuous. First if the term rank $\rho(A) = \min\{m, n\}$, the result holds obviously. Next we assume $\rho(A) < \min\{m, n\}$. Let A' be the matrix obtained from A by replacing each 1 by a different indeterminate. Then A' is an $m \times n$ partial matrix. Moreover, it follows easily that the maximum rank of a completion of A' equals $\rho(A)$. By Theorem 7, all completions of A' have rank $\leq \rho(A)$ if and only if for some integers r and s, A' has an $r \times s$ constant submatrix with $r + s \geq m + n - \rho(A)$ whose rank is at most $r + s - (m + n - \rho(A))$. The only constant submatrices of A' are zero matrices and they have rank equal to 0. Thus all completions of A' have rank $\leq \rho(A)$ if and only if A', and hence A, has an $r \times s$ zero submatrix for some integers r and s with $r + s \geq m + n - \rho(A)$. The corollary now follows. □

Next we study ACI-matrices all of whose completions are nonsingular.

Theorem 10 *Let* $\mathbf{F}$ *be a field with at least* $n + 1$ *elements. Let* A *be an* $n \times n$ *ACI-matrix over* $\mathbf{F}[x_1, \ldots, x_k]$*. Then* $\det A$ *is a nonzero constant if and only if there exist a nonsingular constant matrix* $T \in M_n(\mathbf{F})$ *and a permutation matrix* $Q \in M_n(\mathbf{F})$ *such that* TAQ *is an upper triangular matrix with nonzero constant diagonal entries.*

Proof. The condition is obviously sufficient. To prove the necessity we use induction on the order n. The conclusion for $n = 1$ is trivially true. Now let $n \geq 2$, and assume that the conclusion holds for all ACI-matrices of order $n - 1$. Let A be an ACI-matrix of order n over $\mathbf{F}[x_1, \ldots, x_k]$ with det A equal to a nonzero constant.

We assert that A has at least one column with only constant entries. To the contrary, suppose each column of A contains at least one indeterminate implying that the number of indeterminates in A is at least n. Without loss of generality we assume that x_j appears in the j-th column of $A = [a_{ij}]$ for $j = 1, \ldots, n$. Let

$$a_{ij} = a_{ij}^{(0)} + b_{ij}x_j + \sum_{u \neq j} a_{ij}^{(u)} x_u \quad (1 \leq i \leq n),$$

where for each j, $b_{1j}, b_{2j}, \ldots, b_{nj}$ are not all zero. We show that there exist constants $t_2, t_3, \ldots, t_n$ in $\mathbf{F}$ such that

$$b_{1j} + t_2 b_{2j} + \cdots + t_n b_{nj} \neq 0, \text{ for } j = 1, \ldots, n. \tag{1}$$

In fact we may successively choose $t_2, \ldots, t_n$ such that if $b_{i,j_0} \neq 0$ for some i and j_0, then

$$b_{1,j_0} + t_2 b_{2,j_0} + \cdots + t_i b_{i,j_0} \neq 0. \tag{2}$$

(1) will follow from (2) since the square matrix $B = [b_{ij}]$ of order n has no zero column. If the second row of B is a zero row, we choose $t_2 = 0$. Otherwise let $b_{2,j_1}, \ldots, b_{2,j_s}$ be the nonzero entries in the second row of B. For every $p = 1, \ldots, s$, the equation $b_{1,j_p} + y b_{2,j_p} = 0$ has only one solution, i.e., $y = -b_{2,j_p}^{-1} b_{1,j_p}$. Since $s \leq n$ and $|\mathbf{F}| \geq n+1$, there exists $t_2 \in \mathbf{F}$ such that $b_{1,j_p} + t_2 b_{2,j_p} \neq 0$ holds for all $p = 1, \ldots, s$. Next if the third row of B is a zero row, choose $t_3 = 0$. Otherwise, as above there exists $t_3 \in \mathbf{F}$ such that $b_{1,j} + t_2 b_{2,j} + t_3 b_{3,j} \neq 0$ holds for all those j for which $b_{3,j} \neq 0$. Continuing in this way we can find $t_2, t_3, \ldots, t_n$ satisfying (2). Now in A adding t_i times the i-th row to the first row for $i = 2, \ldots, n$ we get a matrix A_1 with the same determinant as A. Note that A_1 is an ACI-matrix. Since in the position $(1, j)$ of A_1, $j = 1, \ldots, n$, the coefficient of x_j is nonzero by (1), there is a choice of values for the indeterminates such that the first row of A_1 becomes a zero row. This contradicts the condition that det A is a nonzero constant.

Suppose the q-th column of A contains only constant entries. This column has at least one nonzero entry, say, in the r-th row. We interchange columns 1 and q, and then interchange rows 1 and r to get a matrix $A_0 = P_1 A Q_1 = [\tilde{a}_{ij}]$ with $\tilde{a}_{11} \neq 0$ where P_1, Q_1 are permutation matrices. Adding $-\tilde{a}_{i,1}/\tilde{a}_{11}$ times the first row to the i-th row in A_0 for $i = 2, \ldots, n$ successively we get a matrix $A_1 = T_1 A_0$, where $T_1 \in M_n(\mathbf{F})$ is the nonsingular matrix corresponding to these elementary row operations. Partition A_1 as

$$A_1 = T_1 A_0 = \begin{bmatrix} \tilde{a}_{11} & u^T \\ 0 & H \end{bmatrix}$$

where H is of order $n-1$. Then $\det A_1 = \tilde{a}_{11}\det H$ and

$$\det H = \det A_1/\tilde{a}_{11} = \det(T_1P_1Q_1)\det \mathrm{A}/\tilde{a}_{11}$$

is a nonzero constant.

By Lemma 2, A_1 and hence H is an ACI-matrix. Using the induction hypothesis on H, we know that there exist a nonsingular constant matrix $T_2 \in M_{n-1}(\mathbf{F})$ and a permutation matrix $Q_2 \in M_{n-1}(\mathbf{F})$ such that T_2HQ_2 is an upper triangular matrix with nonzero constant diagonal entries.

Set $T = (1 \oplus T_2)T_1P_1$ and $Q = Q_1(1 \oplus Q_2)$. Then $T \in M_n(\mathbf{F})$ is a nonsingular constant matrix, $Q \in M_n(\mathbf{F})$ is a permutation matrix and

$$\begin{aligned} TAQ &= (1 \oplus T_2)T_1P_1AQ_1(1 \oplus Q_2) \\ &= (1 \oplus T_2)A_1(1 \oplus Q_2) \\ &= \begin{bmatrix} \tilde{a}_{11} & u^TQ_2 \\ 0 & T_2HQ_2 \end{bmatrix} \end{aligned}$$

is an upper triangular matrix with nonzero constant diagonal entries. □

By Lemma 1 and Theorem 10 we have the following corollary.

Corollary 11 *Let* $\mathbf{F}$ *be a field with at least* $n+1$ *elements. Let* A *be an ACI-matrix over* $\mathbf{F}[x_1, \ldots, x_k]$. *Then all the completions of* A *are nonsingular if and only if there exist a nonsingular constant matrix* $T \in M_n(\mathbf{F})$ *and a permutation matrix* $Q \in M_n(\mathbf{F})$ *such that* TAQ *is an upper triangular matrix with nonzero constant diagonal entries.*

We remark that the proof of Theorem 10 provides a polynomial algorithm to decide whether a given partial matrix A is always nonsingular: A is always nonsingular if and only if it can be transformed to an upper triangular form with nonzero constant diagonal entries by a *modified Gaussian elimination,* in which the three elementary operations: (1) permutation of columns; (2) permutation of rows; (3) addition of a scalar multiple of one row to another row are permitted. For example:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ x & 0 & 1 & 1 \\ y & z & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & x \\ 1 & z & 0 & y \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & x-1 \\ 0 & z-1 & -1 & y-1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & z-1 & -1 & y-1 \\ 0 & -1 & 0 & x-1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & z-1 & y-1 \\ 0 & 0 & -1 & x-1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

We conclude that all the completions of A are nonsingular. Here the operations we performed are: interchanging the first and the fourth columns; adding -1 times the first row to the second, third and fourth row; interchanging the second and fourth rows; interchanging the second and third columns.

Next we study the possible numbers of indeterminates in a partial matrix all of whose completions are nonsingular. Obviously it suffices to determine the maximum number.

Theorem 12 *Let $\mathbf{F}$ be a field with at least $n+1$ elements. Let $A \in P_n(\mathbf{F})$ be a partial matrix all of whose completions are nonsingular. Then the number of indeterminates of A is less than or equal to $n(n-1)/2$. This maximum number is attained if and only if there exist permutation matrices P, Q such that PAQ is upper triangular with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates.*

Proof. By Corollary 11 there exist a nonsingular constant matrix $T = [t_{ij}] \in M_n(\mathbf{F})$ and a permutation matrix $Q \in M_n(\mathbf{F})$ such that $B = TAQ$ is an upper triangular matrix with nonzero constant diagonal entries. Let $\tilde{A} = AQ = [a_{ij}]$. Then $B = [b_{ij}] = T\tilde{A}$, where $b_{ij} = 0$ for $i > j$ and $b_{ii} \in \mathbf{F}$ is nonzero for $i = 1, \ldots, n$.

We assert that the j-th column of $\tilde{A}$ contains at most $j-1$ indeterminates, $j = 1, \ldots, n$. To the contrary suppose the j-th column of $\tilde{A}$ has exactly p indeterminates, say, $a_{i_1,j}, a_{i_2,j}, \ldots, a_{i_p,j}$ with $p \geq j$. Since

$$b_{ij} = \sum_{k=1}^{n} t_{ik} a_{kj} = \sum_{1 \leq s \leq p} t_{i,i_s} a_{i_s,j} + d_{ij}, \; d_{ij} \in \mathbf{F}$$

and b_{ij} are constants for $i \geq j$, we have $t_{i,i_s} = 0$ for $j \leq i \leq n$ and $1 \leq s \leq p$. So T has an $(n-j+1) \times p$ zero submatrix and $(n-j+1) + p \geq n+1$. By the easy part of the Frobenius-König theorem [2], $\det T = 0$, which contradicts the fact that T is nonsingular. Therefore, the number of indeterminates of A, which is equal to that of $\tilde{A}$, is less than or equal to

$$0 + 1 + \cdots + (n-1) = \frac{n(n-1)}{2}.$$

If A has the properties stated in the theorem, then A has $n(n-1)/2$ indeterminates and all completions are nonsingular. Now suppose that the number of indeterminates of A is equal to $n(n-1)/2$ and that all completions are nonsingular. From the above argument we see that the j-th column of $\tilde{A}$ has exactly $j-1$ indeterminates. Note that $\tilde{A}$ is a partial matrix. To complete our proof of Theorem 12, it suffices to prove the following statement:

(S) *Let $G \in P_n(\mathbf{F})$ be a partial matrix and $T \in M_n(\mathbf{F})$ be a nonsingular constant matrix such that TG is an upper triangular matrix with nonzero constant diagonal entries. If the j-th column of G has exactly $j-1$ indeterminates for $j = 1, 2, \ldots, n$, then there exists a permutation matrix P such that PG is upper triangular with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates.*

We use induction on the order n to prove (S). It holds trivially for the case $n = 1$. Next let $n \geq 2$ and assume (S) holds for matrices of order $n-1$. There exists a permutation matrix P_1 such that if we denote $P_1 G = [g_{ij}]$, then g_{nn} is a constant and all the other entries $g_{1n}, g_{2n}, \ldots, g_{n-1,n}$ in the last column are indeterminates. Let $TP_1^T = [t_{ij}]$ and denote

$$U = TG = (TP_1^T)(P_1 G) = [u_{ij}].$$

Since

$$u_{nn} = \sum_{k=1}^{n-1} t_{nk} g_{kn} + t_{nn} g_{nn} \in \mathbf{F},$$

we have $t_{nk} = 0$ for $1 \leq k \leq n-1$, and the above equality reduces to $u_{nn} = t_{nn} g_{nn}$. Then $u_{nn} \neq 0$ implies $t_{nn} \neq 0$ and $g_{nn} \neq 0$. From

$$0 = u_{nj} = \sum_{k=1}^{n} t_{nk} g_{kj} = t_{nn} g_{nj}, \text{ for } j = 1, 2, \ldots, n-1,$$

we get $g_{nj} = 0$ for $j = 1, 2, \ldots, n-1$. Partition

$$TP_1^T = \begin{bmatrix} T_1 & v \\ 0 & t_{nn} \end{bmatrix}, \quad P_1 G = \begin{bmatrix} G_1 & w \\ 0 & g_{nn} \end{bmatrix}$$

where $T_1 \in M_{n-1}(\mathbf{F})$ and $G_1 \in P_{n-1}(\mathbf{F})$. Since T is nonsingular and $t_{nn} \neq 0$, T_1 is nonsingular. Clearly the j-th column of G_1 has exactly $j-1$ indeterminates for $j = 1, 2, \ldots, n-1$. From

$$U = (TP_1^T)(P_1 G) = \begin{bmatrix} T_1 G_1 & * \\ 0 & t_{nn} g_{nn} \end{bmatrix}$$

we deduce that $T_1 G_1$ is an upper triangular matrix with nonzero constant diagonal entries. By the induction hypothesis there exists a permutation matrix P_2 of order $n-1$ such that

P_2G_1 is upper triangular with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates. Set $P = (P_2 \oplus 1)P_1$. Then P is a permutation matrix and

$$PG = \begin{bmatrix} P_2G_1 & P_2w \\ 0 & g_{nn} \end{bmatrix}$$

is upper triangular with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates. This completes the proof. $\square$

Finally we pose two problems.

Problem 5 *Let A be an $m \times n$ ACI-matrix and let r be an integer with $r \leq \min\{m, n\}$. When do all completions of A have rank r? When do all completions of A have rank $\geq r$?*

The maximum rank of a partial matrix is investigated in [4].

Problem 6 *We call a partial matrix A maximal-nonsingular provided all the completions of A are nonsingular and replacing any specified entry with a new indeterminate results in a partial matrix B at least one of whose completions is singular. Characterize maximal-nonsingular partial matrices.*

Upper triangular partial matrices with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates are maximal-nonsingular. The matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ x & 0 & 1 & 1 \\ y & z & 0 & 1 \end{bmatrix}$$

is also maximal-nonsingular.

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