ACI-matrices all of whose completions have the same rank

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Abstract

We characterize the ACI-matrices all of whose completions have the same rank, determine
the largest number of indeterminates in such partial matrices of a given size, and determine
the partial matrices that attain this largest number.

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1 Introduction

A partial matrix over a set Ω is a matrix in which some entries all from Ω are specified and the
other entries are free to be chosen from Ω. A completion of a partial matrix is a specific choice
of values from Ω for its unspecified entries. A completion may also mean a completed matrix
of a partial matrix. We call the unspecified entries indeterminates since they are free to range
over Ω.

Let $M_{m,n}(Ω)$ be the set of $m \times n$ matrices whose entries are from a given set Ω, and let
$P_{m,n}(Ω)$ be the set of $m \times n$ partial matrices over Ω. If $m = n$, then we use the abbreviations
$M_n(Ω)$ and $P_n(Ω)$ respectively. We call elements in Ω constants and call matrices in $M_{m,n}(Ω)$
constant matrices, in contrast to indeterminates and partial matrices respectively.

To study partial matrices, it is more convenient to consider a larger class of matrices for
technical reasons. Let $F[x_1, \ldots, x_k]$ be the ring of polynomials in the indeterminates $x_1, \ldots, x_k$

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with coefficients from a field \( F \). We call a matrix \( A \) over \( F[x_1, \ldots, x_k] \) an **affine column independent** (abbreviated as ACI) matrix if each entry of \( A \) is a polynomial of degree at most one and no indeterminate appears in two distinct columns of \( A \). Obviously, every submatrix of an ACI-matrix is also an ACI-matrix. Given a partial matrix, we may label its unspecified entries with distinct indeterminates. Thus partial matrices are ACI-matrices. By a **completion** of a matrix over \( F[x_1, \ldots, x_k] \) we mean an assignment of values in \( F \) to the indeterminates \( x_1, \ldots, x_k \). A completion may also mean a completed polynomial matrix.

The ACI-matrices all of whose completions are nonsingular and the ACI-matrices all of whose completions are singular are characterized in [1]. The following problem was also posed in [1, Problem 5]:

**Problem 1** Let \( F \) be a field. Characterize the ACI-matrices over \( F[x_1, \ldots, x_k] \) all of whose completions have the same rank.

We will solve this problem under a minor condition on the field \( F \) and determine the maximum number of indeterminates in such partial matrices as well as the matrices attaining this maximum number.

## 2 Main Results

First we give some lemmas which will be used to prove our main results. \( F \) is a field throughout.

**Lemma 1** ([?]) Let \( A \) be an \( m \times n \) ACI-matrix over \( F[x_1, \ldots, x_k] \). If \( T \in M_m(F) \) is a constant matrix and \( P \in M_n(F) \) is a permutation matrix, then \( TAP \) is an ACI-matrix over \( F[x_1, \ldots, x_k] \).

A **proper ACI-matrix** is an ACI-matrix containing at least one indeterminate, i.e., it is not a constant matrix.

**Lemma 2** Let \( A \) be an \( m \times n \) proper ACI-matrix over \( F[x_1, \ldots, x_k] \). Then there exists a nonsingular constant matrix \( T \in M_m(F) \) and a permutation matrix \( Q \in M_n(F) \) such that

\[
TAQ = \begin{bmatrix}
  b_1 & * & * & \cdots & * & * \\
  c_1^{(1)} & b_2 & * & \cdots & * & * \\
  c_1^{(2)} & c_2^{(1)} & b_3 & \cdots & * & * \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  c_1^{(s-1)} & c_2^{(s-2)} & c_3^{(s-3)} & \cdots & b_s & * \\
  c_1^{(s)} & c_2^{(s-1)} & c_3^{(s-2)} & \cdots & c_s^{(1)} & B
\end{bmatrix}
\]

where for \( j = 1, \ldots, s \), \( b_j \) is a column vector each of whose components is a polynomial of degree 1 in which there is an indeterminate that appears nowhere else in \( TAQ \), \( c_j^{(i)} \) are constant column
Proof. We use induction on \( n \). The case for \( n = 1 \) is easy to check. Assume that the result holds for all proper ACI-matrices with \( n - 1 \) columns and let \( A \) be an \( m \times n \) proper ACI-matrix.

Suppose \( A \) has an entry in the position \(( r, t)\) that contains an indeterminate, say, \( x_1 \). We interchange rows 1 and \( r \), and then interchange columns 1 and \( t \) to get a matrix \( A_1 = P_0AQ_0 = (\tilde{a}_{ij}) \), where

\[
\tilde{a}_{ij} = \tilde{a}^{(0)}_{ij} + \sum_{u=1}^{k} \tilde{a}^{(u)}_{ij}x_u,
\]

\( \tilde{a}^{(1)}_{11} \neq 0 \), \( P_0 \in M_m(\mathbb{F}) \) and \( Q_0 \in M_n(\mathbb{F}) \) are permutation matrices. Adding \(-\tilde{a}^{(1)}_{11}/\tilde{a}^{(1)}_{11}\) times the first row to the \( i \)-th row in \( A_1 \) for \( i = 2, \ldots , m \) successively we get a matrix \( A_2 = T_1A_1 \), where \( T_1 = M_m(\mathbb{F}) \) is the nonsingular matrix corresponding to these elementary row operations. Now \( x_1 \) appears only in the \((1,1)\) position of \( A_2 \).

If there is another indeterminate in the first column of \( A_2 \) but not in the \((1,1)\) position, say, \( x_2 \) in the \((i_1,1)\) position, then interchange row \( i_1 \) and row 2 we get a new matrix \( A_3 = PA_2 \) where \( P \) is a permutation matrix. Suppose the coefficient of \( x_2 \) in position \((i,1)\) of \( A_3 \) is \( u_{i1} \), \( i = 1, 2, \ldots , m \). Adding \(-u_{i1}/u_{11}\) times the second row to the \( i \)-th row for \( i = 1, 3, 4, \ldots , m \) we get a new matrix \( A_4 = T_2A_3 \) where \( T_2 = M_m(\mathbb{F}) \) is a nonsingular constant matrix. Now \( x_1 \) and \( x_2 \) appear only in the \((1,1)\) position and the \((2,1)\) position of \( A_4 \) respectively. If there is an indeterminate in the last \( m - 2 \) components of the first column of \( A_4 \), continue in this way until we get

\[
A_5 = T_3A_4 = \begin{bmatrix}
\tilde{b}_1 & B_1 \\
c_1 & B_2
\end{bmatrix}
\]

where \( c_1 \) is a constant column vector and \( \tilde{b}_1 \) is a column vector each of whose components is a polynomial of degree 1 in which there is an indeterminate that appears nowhere else in \( A_5 \).

If \( c_1 \) is void in \( A_5 \) or \( B_2 \) is a constant matrix, we have already finished the proof since \( A_5 \) has the form in (1). Otherwise let the length of \( c_1 \) be \( l \geq 1 \) and assume that \( B_2 \) contains at least one indeterminate. Note that \( B_2 \) is an ACI-matrix by Lemma 1. Use the induction we know that there exists a nonsingular constant matrix \( T_4 \in M_l(\mathbb{F}) \) and a permutation matrix \( Q_1 \in M_{n-1}(\mathbb{F}) \) such that \( T_4B_2Q_1 \) has the form in (1), i.e.,

\[
T_4B_2Q_1 = \begin{bmatrix}
b_2 & * & \cdots & * & * \\
c_2^{(s-2)} & c_3^{(s-3)} & \cdots & b_s & * \\
c_2^{(s-1)} & c_3^{(s-2)} & \cdots & c_3^{(1)} & B
\end{bmatrix}
\]
where \( b_2, \ldots, b_s \) are column vectors each of whose components is a polynomial of degree 1 in which there is an indeterminate that appears nowhere else in \( T_4B_2Q_1 \), \( c^{(i)}_j \) are constant column vectors for \( 2 \leq j \leq s, 1 \leq i \leq s - j + 1 \), and \( B \) is a constant matrix.

Denote by \( I_t \) the identity matrix of order \( t \). Let

\[
A_6 = (I_{m-\ell} \oplus T_4)A_5(1 \oplus Q_1) = \begin{bmatrix} \tilde{b}_1 & B_1Q_1 \\ T_4c_1 & T_4B_2Q_1 \end{bmatrix}.
\]

If some entries of \( B_1Q_1 \) contain the same indeterminates as those in \( b_2, \ldots, b_s \) that appear only once in \( T_4B_2Q_1 \) mentioned above, by using elementary row operations on \( A_6 \) we can make these indeterminates vanish in \( B_1Q_1 \), i.e., there exists a nonsingular matrix \( T_5 \in M_m(\mathbb{F}) \) such that

\[
T_5A_6 \text{ has form (1)}.
\]

\[\begin{aligned}
&\text{We use } |S| \text{ to denote the cardinality of a set } S.
\end{aligned}\]

**Lemma 3** Let \( m \geq n \) be positive integers, let \( \mathbb{F} \) be a field with \(|\mathbb{F}| \geq m \) and let \( A \) be an \( m \times n \) ACI-matrix over \( \mathbb{F}[x_1, \ldots, x_k] \). If all the completions of \( A \) have rank \( n \), then there exists a nonsingular constant matrix \( T \in M_m(\mathbb{F}) \) and a permutation matrix \( Q \in M_n(\mathbb{F}) \) such that

\[
TAQ = \begin{bmatrix} \ast \\ U \end{bmatrix}
\]

where \( U \) is an \( n \times n \) upper triangular ACI-matrix with nonzero constant diagonal entries.

**Proof.** We prove the lemma in the equivalent form:

There exists a nonsingular constant matrix \( T \in M_m(\mathbb{F}) \) and a permutation matrix \( Q \) such that

\[
TAQ = \begin{bmatrix} L \\ \ast \end{bmatrix}
\]

where \( L \) is an \( n \times n \) lower triangular ACI-matrix with nonzero constant diagonal entries.

We use induction on \( n \) to prove this equivalent version. For \( n = 1 \) the result follows from Lemma 2. Assume the result holds for all ACI-matrices over \( \mathbb{F}[x_1, \ldots, x_k] \) with \( n - 1 \) columns and let \( A \) be an \( m \times n \) ACI-matrix all of whose completions have rank \( n \).

Case 1. \( A \) has a constant column, say, the \( j \)-th column which has a nonzero entry, say, the \( t \)-th entry, since \( A \) cannot have zero columns. Without loss of generality, \( A_0 = P_1AQ_1 = (\tilde{a}_{ij}) \)
with \( \tilde{a}_{11} \neq 0 \) where \( P_1, Q_1 \) are permutation matrices of orders \( m \) and \( n \) respectively, and the entries in the first column of \( A_0 \) are constants. Adding \(-\tilde{a}_{11}/\tilde{a}_{11}\) times the first row to the \( i \)-th row of \( A_0 \) for \( i = 2, \ldots, m \) successively we get a matrix \( A_1 = T_1 A_0 \), where \( T_1 \in M_m(\mathbf{F}) \) is the nonsingular matrix corresponding to these elementary row operations. Partition \( A_1 \) as

\[ A_1 = T_1 A_0 = \begin{bmatrix} \tilde{a}_{11} & u^T \\ 0 & H \end{bmatrix}. \tag{3} \]

By Lemma 1, \( A_1 \) and hence \( H \) is an ACI-matrix. Now all completions of \( A_1 \) have rank \( n \). Thus all completions of \( H \) have rank \( n - 1 \). By the induction hypothesis on \( H \), there exists a nonsingular constant matrix \( T_2 \in M_{m-1}(\mathbf{F}) \) and a permutation matrix \( Q_2 \in M_{n-1}(\mathbf{F}) \) such that \( T_2 HQ_2 \) is of form (2).

Denote

\[ Q_0 = \begin{bmatrix} 0 & I_{n-1} \\ 1 & 0 \end{bmatrix}, \]

the basic circulant permutation matrix. Set \( T = (Q_0 \oplus I_{m-n}) (1 \oplus T_2) T_1 P_1 \) and \( Q = Q_1 (1 \oplus Q_2) Q_0^T \). Then \( T \in M_m(\mathbf{F}) \) is a nonsingular constant matrix, \( Q \) is a permutation matrix and \( TAQ \) is of form (2).

**Case 2.** \( A \) has no constant column. By Lemma 2 there exists a nonsingular constant matrix \( T_3 \in M_m(\mathbf{F}) \) and a permutation matrix \( Q_3 \) such that \( T_3 AQ_3 \) has form (1). Set

\[ A_2 \equiv T_3 AQ_3 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \]

where \( B_2 = (c_1^{(s)}, c_2^{(s-1)}, \ldots, c_s^{(1)}, B) \). We claim that \( B_2 \) is nonvoid and \( B_2 \neq 0 \). Otherwise, in \( A_2 \) adding the \( j \)-th column to the first column for \( 2 \leq j \leq s \) and choosing suitable values for the indeterminates successively we can make the sum of the first \( s \) columns be a zero vector by the property of \( b_1, \ldots, b_s \) stated in Lemma 2. This contradicts the fact that all completions of \( A \) and hence all completions of \( A_2 \) have rank \( n \).

Let \( B_2 \) be \( p \times n \) and let the rank of \( B_2 \) be \( r \geq 1 \). Then there exists a nonsingular constant matrix \( T_4 \in M_p(\mathbf{F}) \) and a permutation matrix \( Q_4 \in M_n(\mathbf{F}) \) such that

\[ T_4 B_2 Q_4 = \begin{bmatrix} D & E \\ 0 & 0 \end{bmatrix} \]

where \( D = \text{diag}(d_1, \ldots, d_r) \) is a diagonal matrix with each \( d_i \) nonzero. Let \( T_5 = I_{m-p} \oplus T_4 \) and

\[ A_3 \equiv T_5 A_2 Q_4 = T_5 T_3 AQ_3 Q_4 = \begin{bmatrix} B_1 Q_4 \\ T_4 B_2 Q_4 \end{bmatrix}. \]
If \( r = n \), i.e., \( E \) is void, then by considering row permutations of \( A_3 \) we easily see that the conclusion holds. Next suppose \( r < n \). Now we prove that \( E \) has at least one zero row. To the contrary suppose that \( E \) has no zero row.

Let \( t = m - p \), \( g = n - r \). Partition \( B_1Q_4 = (C_1, C_2) \) where \( C_1 \) is \( t \times r \) and \( C_2 \) is \( t \times g \). Let

\[
R_1 = \begin{bmatrix}
I_r & -D^{-1}E \\
0 & I_{n-r}
\end{bmatrix}, \quad A_4 = A_3R_1 = \begin{bmatrix}
C_1 & Z \\
D & 0 \\
0 & 0
\end{bmatrix}
\]

where \( Z = -C_1D^{-1}E + C_2 \). By Lemma 2, each row of \( B_1Q_4 = (C_1, C_2) \) contains an indeterminate that appears only in one position of \( A_2 \). Without loss of generality we may suppose that \( x_1, \ldots, x_t \) are these indeterminates and \( x_i \) appears in the \( i \)-th row of \( (C_1, C_2) \). Since \( x_i \) appears only in one position of \( (C_1, C_2) \) and \( D^{-1}E \) has no zero row, \( x_i \) appears in the \( i \)-th row of \( Z = -C_1D^{-1}E + C_2 \). Note that \( Z \) is \( t \times g \).

Let

\[
Z = \begin{bmatrix}
w_{11}x_1 + \cdots + w_{12}x_1 + \cdots + w_{1g}x_1 + \cdots \\
w_{21}x_2 + \cdots + w_{22}x_2 + \cdots + w_{2g}x_2 + \cdots \\
\vdots & \vdots & \ddots & \vdots \\
w_{t1}x_t + \cdots + w_{2t}x_t + \cdots + w_{tg}x_t + \cdots
\end{bmatrix}
\]

where \( w_{ij} \in F \) and for each \( 1 \leq i \leq t \), \( w_{ij}, j = 1, \ldots, g \) are not all zero. We show that there exist \( k_j \in F, j = 1, \ldots, g - 1 \) such that

\[
w_{ig} + k_{g-1}w_{i,g-1} + \cdots + k_1w_{i1} \neq 0 \quad \text{for} \quad i = 1, \ldots, t. \tag{4}
\]

In fact we may successively choose \( k_j \) such that if \( w_{i0,j} \neq 0 \) for some \( i_0, j \), then

\[
w_{i0,g} + k_{g-1}w_{i0,g-1} + \cdots + k_jw_{i0,j} \neq 0. \tag{5}
\]

(4) will follow from (5) since \( w_{ij}, j = 1, \ldots, g \) are not all zero for each \( 1 \leq i \leq t \). If \( w_{i,g-1} = 0 \) for all \( 1 \leq i \leq t \), we choose \( k_{g-1} = 0 \). Otherwise let \( w_{i1,g-1}, \ldots, w_{i0,g-1} \) be the nonzero elements among \( w_{1,g-1}, \ldots, w_{l,g-1} \). For every \( q = 1, \ldots, v \), the equation \( w_{iq,g} + yw_{iq,g-1} = 0 \) has only one solution, i.e., \( y = -w_{iq,g-1}^{-1}w_{iq,g} \). Since \( v \leq t < m \) and \( |F| \geq m \), there exists \( k_{g-1} \in F \) such that \( w_{iq,g} + k_{g-1}w_{iq,g-1} \neq 0 \) holds for all \( q = 1, \ldots, v \). Next if \( w_{i,g-2} = 0 \) for all \( 1 \leq i \leq t \), we choose \( k_{g-2} = 0 \). Otherwise, as above there exists \( k_{g-2} \in F \) such that \( w_{ig} + k_{g-1}w_{i,g-1} + k_{g-2}w_{i,g-2} \neq 0 \) holds for all those \( i \) for which \( w_{i,g-2} \neq 0 \). Continuing in this way we can find all the \( k_{g-1}, k_{g-2}, \ldots, k_1 \) satisfying (5).

In \( Z \) adding \( k_j \) times column \( j \) to column \( g \) for \( 1 \leq j \leq g - 1 \) we get a new matrix \( G = ZR_2 \), where \( R_2 \in M_g(F) \) is the nonsingular matrix corresponding to these elementary column operations. Since \( x_i \) appears with a nonzero coefficient in the \( i \)-th component of the last
column of $G$ for $i = 1, \ldots, t$, we can choose suitable values for $x_1, \ldots, x_t$ successively to make the last column of $G$ be a zero column, which means that there is at least one completion of $G$ which has rank $\leq g - 1 = n - r - 1$. But on the other hand, all completions of $A_4 = T_5T_3AQ_3Q_4R_1$ have rank $n$. Therefore all completions of $Z$ and hence all completions of $G = ZR_2$ have rank $n - r$, which is a contradiction.

So $E$ has a zero row, say, the $f$-th row being zero. In $A_3$ interchanging the $(t + f)$-th row and the first row, and then interchanging the $f$-th column and the first column, we get a new ACI-matrix

$$A_5 = P_2A_3Q_5 = P_2T_5T_3AQ_3Q_4Q_5 = \begin{bmatrix} d_f & 0 \\ * & H_2 \end{bmatrix},$$

where $P_2 \in M_m(F)$ and $Q_5 \in M_n(F)$ are permutation matrices corresponding to these elementary row and column operations respectively.

Since all completions of $A_5$ have rank $n$, all completions of $H_2$ have rank $n - 1$. By Lemma 1, $H_2$ is an ACI-matrix. So using the induction hypothesis on $H_2$, there exists a nonsingular constant matrix $T_6 \in M_{m-1}(F)$ and a permutation matrix $Q_6 \in M_{n-1}(F)$ such that $T_6H_2Q_6$ is of form (2).

Set $T = (1 \oplus T_6)P_2T_5T_3$ and $Q = Q_3Q_4Q_5(1 \oplus Q_6)$. Then $T \in M_m(F)$ is nonsingular, $Q \in M_n(F)$ is a permutation matrix and $TAQ$ is of form (2). This completes the proof.

For a matrix $G$, we denote by $G(i, j)$ the entry of $G$ in the position $(i, j)$.

**Lemma 4** Let $F$ be a field with $|F| \geq n + 1$ and $A$ be an $m \times n$ ACI-matrix over $F[x_1, \ldots, x_k]$. If all completions of $A$ have the same rank $r < n$, then $A$ has at least one column with only constant entries.

**Proof.** To the contrary suppose each column of $A$ contains at least one indeterminate, which implies that the number of indeterminates in $A$ is at least $n$. Without loss of generality we assume that $x_j$ appears in the $j$-th column of $A$ for $j = 1, \ldots, n$. Let $A = (a_{ij})$ with

$$a_{ij} = a_{ij}^{(0)} + b_{ij}x_j + \sum_{u \neq j} a_{ij}^{(u)}x_u,$$

where for each $j$, $b_{ij}$, $i = 1, \ldots, m$, are not all zero. We show that there exist $t_i \in F$, $i = 2, \ldots, m$ such that

$$b_{1j} + t_2b_{2j} + \cdots + t_nb_{nj} \neq 0 \text{ for } j = 1, \ldots, n. \quad (6)$$

In fact we may successively choose $t_2, \ldots, t_n$ such that if $b_{i,j_0} \neq 0$ for some $i, j_0$, then

$$b_{1,j_0} + t_2b_{2,j_0} + \cdots + t_nb_{nj_0} \neq 0. \quad (7)$$
(6) will follow from (7) since the matrix \( B \equiv (b_{ij})_{m \times n} \) has no zero column. If the second row of \( B \) is a zero row, we choose \( t_2 = 0 \). Otherwise let \( b_{2,j_1}, \ldots, b_{2,j_k} \) be the nonzero entries in the second row of \( B \). For every \( p = 1, \ldots, s \), the equation \( b_{1,j_p} + yb_{2,j_p} = 0 \) has only one solution, i.e., \( y = -b_{2,j_p}^{-1}b_{1,j_p} \). Since \( s \leq n \) and \(|F| \geq n + 1\), there exists \( t_2 \in F \) such that \( b_{1,j_p} + t_2b_{2,j_p} \neq 0 \) holds for all \( p = 1, \ldots, s \). Next if the third row of \( B \) is a zero row, choose \( t_3 = 0 \). Otherwise, as above there exists \( t_3 \in F \) such that \( b_{1,j_3} + t_2b_{2,j_3} + t_3b_{3,j} \neq 0 \) holds for all \( j \) for which \( b_{3,j} \neq 0 \). Continuing in this way we can find all the \( t_2, t_3, \ldots, t_n \) satisfying (7). Now in \( A \) adding \( t_i \) times the \( i \)-th row to the first row for \( i = 2, \ldots, n \) we get a matrix \( A_1 \) all of whose completions have the same rank \( r \). By Lemma 1, \( A_1 \) is an ACI-matrix. By the condition (6), \( x_j \) appears in \( A_1(1, j) \) with a nonzero coefficient.

Obviously there is a choice of values for the indeterminates such that the first row of \( A_1 \) becomes a zero row. For this completion of \( A_1 \), without loss of generality suppose the first \( r \) columns are linearly independent. Now in \( A_1 \) change the value of \( x_{r+1} \) such that \( A_1(1, r + 1) \neq 0 \) and keep the values of other indeterminates unchanged. Now for the second choice of values for the indeterminates, the rank of the completed matrix is \( r + 1 \), which is a contradiction. \( \Box \)

Now we are ready to characterize the ACI-matrices all of whose completions have the same rank. In a block matrix, the condition that some block \( B \) is \( s \times t \) with \( s = 0 \) means that the rows in which \( B \) lies are void, i.e., they do not appear, and the condition that some block \( B \) is \( s \times t \) with \( t = 0 \) means that the columns in which \( B \) lies are void.

**Theorem 5** Let \( m, n \) be positive integers, \( F \) be a field with \(|F| \geq \max\{m, n + 1\} \) and \( A \) be an \( m \times n \) ACI-matrix over \( F[x_1, \ldots, x_k] \). Then all completions of \( A \) have the same rank \( r \) if and only if there exists a nonsingular constant matrix \( T \in M_m(F) \) and a permutation matrix \( Q \in M_n(F) \) such that

\[
TAQ = \begin{bmatrix}
U_1 & * & * \\
0 & 0 & * \\
0 & 0 & U_2
\end{bmatrix}
\]

where \( U_1 \) and \( U_2 \) are square upper triangular ACI-matrices with nonzero constant diagonal entries and the sum of their orders equals \( r \).

**Proof.** The sufficiency of the condition is obvious. To prove the necessity we use induction on \( n \). For \( n = 1 \) the conclusion is easy to verify by using Lemma 2. Now let \( n \geq 2 \) and assume that the result holds for all ACI-matrices with \( n - 1 \) columns. Let \( A \) be an \( m \times n \) ACI-matrix all of whose completions have rank \( r \).

If \( r = n \), which implies \( m \geq n \), then the result follows from Lemma 3 (with the first block row and the first two block columns in (8) void). If \( A \) has a zero column, say, column \( j \), interchanging column 1 and \( j \) we get a matrix \( A_0 = AQ_0 = (0, B) \), where \( Q_0 \in M_n(F) \) is a permutation matrix
and $B$ is an $m \times (n - 1)$ ACI-matrix all of whose completions have rank $r$. Using the induction hypothesis on $B$, there exists a nonsingular matrix $T_0 \in M_m(F)$ and a permutation matrix $Q_1 \in M_{n-1}(F)$ such that

$$T_0BQ_1 = \begin{bmatrix} U_1 & * & * \\ 0 & 0 & * \\ 0 & 0 & U_2 \end{bmatrix}$$

where $U_1$ and $U_2$ are $r_1 \times r_1$ and $r_2 \times r_2$ upper triangular ACI-matrices with nonzero constant diagonal entries respectively, $r_1 + r_2 = r$.

We have

$$A_1 \equiv T_0A_0(1 \oplus Q_1) = (0, T_0BQ_1) = \begin{bmatrix} 0 & U_1 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & U_2 \end{bmatrix}.$$

In $A_1$ interchange column 1 and the second block column, and denote the corresponding permutation matrix by $Q_2$. Set $T = T_0$ and $Q = Q_0(1 \oplus Q_1)Q_2$. Then

$$TAQ = T_0AQ_0(1 \oplus Q_1)Q_2 = A_1Q_2$$

has form (8).

Next we consider the case that $r < n$ and $A$ has no zero column. By Lemma 4, $A$ has a column with only constant entries, which are not all zero, say, the $j$-th column with the $i$-th entry nonzero. Interchanging rows 1 and $i$, and then interchanging columns 1 and $j$ we get a new matrix $A_2 = P_1AQ_3 = (\tilde{a}_{ij})$ with $\tilde{a}_{11} \neq 0$ and $\tilde{a}_{i1} \in F$ for $1 \leq i \leq m$, where $P_1, Q_3$ are permutation matrices. In $A_2$ adding $-\tilde{a}_{i1}/\tilde{a}_{11}$ times the first row to the $i$-th row for $i = 2, \ldots, m$ successively we get a matrix $A_3 = T_1A_2$, where $T_1 \in M_m(F)$ is the nonsingular matrix corresponding to these elementary row operations. Partition $A_3$ as

$$A_3 = T_1A_2 = \begin{bmatrix} \tilde{a}_{11} & u^T \\ 0 & H \end{bmatrix}.$$

By Lemma 1, $A_3$ is an ACI-matrix all of whose completions have rank $r$. Therefore $H$ is an $(m - 1) \times (n - 1)$ ACI-matrix all of whose completions have rank $r - 1$. Using the induction hypothesis on $H$, we know that there exists a nonsingular constant matrix $T_2 \in M_{m-1}(F)$ and a permutation matrix $Q_4 \in M_{n-1}(F)$ such that

$$T_2HQ_4 = \begin{bmatrix} V_1 & * & * \\ 0 & 0 & * \\ 0 & 0 & U_2 \end{bmatrix}$$

where $V_1$ and $U_2$ are $\tilde{r}_1 \times \tilde{r}_1$ and $r_2 \times r_2$ upper triangular ACI-matrices with nonzero constant diagonal entries respectively, $\tilde{r}_1 + r_2 = r - 1$. 

9
Set \( T = (1 \oplus T_2)T_1P_1 \) and \( Q = Q_3(1 \oplus Q_4) \). Then \( T \in M_m(\mathbb{F}) \) is a nonsingular constant matrix, \( Q \in M_n(\mathbb{F}) \) is a permutation matrix and
\[
TAQ = \begin{bmatrix}
\tilde{a}_{11} & * & * & * \\
0 & V_1 & * & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & U_2
\end{bmatrix} = \begin{bmatrix}
U_1 & * & * \\
0 & 0 & * \\
0 & 0 & U_2
\end{bmatrix}.
\]

Let \( r_1 = \tilde{r}_1 + 1 \). Then \( U_1 \) and \( U_2 \) are \( r_1 \times r_1 \) and \( r_2 \times r_2 \) upper triangular ACI-matrices with nonzero constant diagonal entries and \( r_1 + r_2 = \tilde{r}_1 + r_2 + 1 = r \).

We remark that in the form (8) some block rows or/and block columns may be void. For example (8) includes the following forms as special cases:
\[
U_1, \begin{bmatrix} U_1 & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * \\ 0 & U_2 \end{bmatrix}.
\]

Now we study the possible numbers of indeterminates in the partial matrices of a given size all of whose completions have the same rank. Obviously it suffices to determine the largest number.

**Theorem 6** Let \( m \geq n \) be positive integers, \( \mathbb{F} \) be a field with \( |\mathbb{F}| \geq \max\{m, n+1\} \) and \( A \) be an \( m \times n \) partial matrix over \( \mathbb{F} \) all of whose completions have the same rank \( r \). Then the number of indeterminates of \( A \) is less than or equal to \( mr - r(r+1)/2 \). This maximum number is attained at \( A \) if and only if there exist permutation matrices \( P \in M_m(\mathbb{F}), Q \in M_n(\mathbb{F}) \) such that
\[
PAQ = \begin{bmatrix}
V_1 & C_1 & C_2 \\
0 & 0 & C_3 \\
0 & 0 & V_2
\end{bmatrix}
\]
where \( C_1, C_2, C_3 \) are partial matrices all of whose entries are indeterminates, \( V_1 \) and \( V_2 \) are \( s \times s \) and \( (r-s) \times (r-s) \) upper triangular matrices with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates, and \( s = 0 \) when \( m > n \).

**Proof.** By Theorem 5 there exists a nonsingular constant matrix \( T = (t_{ij}) \in M_m(\mathbb{F}) \) and a permutation matrix \( Q \in M_n(\mathbb{F}) \) such that
\[
TAQ = \begin{bmatrix}
U_1 & * & * \\
0 & 0 & * \\
0 & 0 & U_2
\end{bmatrix}
\]
where \( U_1 \) and \( U_2 \) are \( r_1 \times r_1 \) and \( r_2 \times r_2 \) upper triangular ACI-matrices with nonzero constant diagonal entries respectively, \( r_1 + r_2 = r \). Let \( \tilde{A} = AQ = (\tilde{a}_{ij}) \) and \( B = (b_{ij}) = T \tilde{A} = TAQ \).
We assert that the $j$-th column of $\tilde{A}$ contains at most $j-1$ indeterminates for $j=1,\ldots,r_1$, at most $r_1$ indeterminates for $j=r_1+1,\ldots,n-r_2$ and at most $m-n+j-1$ indeterminates for $j=n-r_2+1,\ldots,n$.

Suppose the $j$-th column of $\tilde{A}$ has exactly $p$ indeterminates, say, $a_{i_1,j},a_{i_2,j},\ldots,a_{i_p,j}$. From (9) we have

$$b_{ij} = \sum_{k=1}^{m} t_{ik}a_{kj} = \sum_{h=1}^{p} t_{i,ih}a_{ih,j} + d_{ij}, \quad d_{ij} \in F.$$ 

Since for $1 \leq j \leq r_1$ and $i \geq j$, $b_{ij}$ are constants, we have $t_{i,ih} = 0$ for $j \leq i \leq m$ and $1 \leq h \leq p$. So $T$ has an $(m-j+1) \times p$ zero submatrix. If $p \geq j$, then $(m-j+1) + p \geq m+1$ and by the Frobenius-König theorem [?], $\det T = 0$, which contradicts the fact that $T$ is nonsingular. This shows that if $1 \leq j \leq r_1$, then $p \leq j-1$.

Since for $r_1+1 \leq j \leq n-r_2$ and $i \geq r_1+1$, $b_{ij}$ are constants, we have $t_{i,ih} = 0$ for $r_1+1 \leq i \leq m$ and $1 \leq h \leq p$. Thus $T$ has an $(m-r_1) \times p$ zero submatrix. If $p \geq r_1+1$ then $m-r_1+p \geq m+1$ and hence $T$ is singular, contradiction. This shows that if $r_1+1 \leq j \leq n-r_2$, then $p \leq r_1$.

Since for $n-r_2+1 \leq j \leq n$ and $i \geq m-(n-j)$, $b_{ij}$ are constants, we have $t_{i,ih} = 0$ for $m-(n-j) \leq i \leq m$ and $1 \leq h \leq p$. So $T$ has an $(n-j+1) \times p$ zero submatrix. If $p \geq m-n+j$, then $n-j+1+p \geq m+1$ and $T$ is singular, contradiction. This shows that if $j \geq n-r_2+1$, then $p \leq m-n+j-1$.

Denote by $f(A)$ the number of indeterminates of $A$, which is equal to that of $\tilde{A}$. Using $r_1+r_2 = r$ we have

$$f(A) \leq \sum_{j=1}^{r_1} (j-1) + (n-r_2-r_1)r_1 + \sum_{j=n-r_2+1}^{n} (m-n+j-1)$$

$$= (m-n)r_2 + nr - \frac{1}{2}r(r+1)$$

$$\leq (m-n)r + nr - \frac{1}{2}r(r+1)$$

$$= mr - \frac{1}{2}r(r+1).$$

The “if” part of the second conclusion is obvious. Now suppose that the number of indeterminates of $A$ is equal to $mr - r(r+1)/2$. From the above argument we see that

i) if $m = n$, then the $j$-th column of $\tilde{A}$ has exactly $j-1$ indeterminates for $j=1,\ldots,r_1$, $n-r_2+1,\ldots,n$ and $r_1$ indeterminates for $j=r_1+1,\ldots,n-r_2$;

ii) if $m > n$, then $r_1 = 0, r_2 = r$, the $j$-th column of $\tilde{A}$ has no indeterminate for $j=1,\ldots,n-r$ and has exactly $m-n+j-1$ indeterminates for $j=n-r+1,\ldots,n$. 

11
Note that $\tilde{A}$ is a partial matrix. To complete our proof of Theorem 6, it suffices to prove the following two statements:

(S1) Let $G \in P_n(F)$ be a partial matrix and $T \in M_n(F)$ be a nonsingular constant matrix such that

$$TG = \begin{bmatrix} U_1 & * & * \\ 0 & 0 & * \\ 0 & 0 & U_2 \end{bmatrix}$$

(10)

where $U_1$ and $U_2$ are $r_1 \times r_1$ and $r_2 \times r_2$ upper triangular matrices with nonzero constant diagonal entries. If the $j$-th column of $G$ has exactly $j-1$ indeterminates for $j = 1, \ldots, r_1, n-r_2+1, \ldots, n$ and $r_1$ indeterminates for $j = r_1+1, \ldots, n-r_2$, then there exists a permutation matrix $P$ such that

$$PG = \begin{bmatrix} V_1 & C_1 & C_2 \\ 0 & 0 & C_3 \\ 0 & 0 & V_2 \end{bmatrix}$$

(11)

where $C_1, C_2, C_3$ are partial matrices all of whose entries are indeterminates, $V_1$ and $V_2$ are $r_1 \times r_1$ and $r_2 \times r_2$ upper triangular matrices with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates.

(S2) Let $m > n$, let $G \in P_{m,n}(F)$ be a partial matrix and $T \in M_m(F)$ be a nonsingular constant matrix such that

$$TG = \begin{bmatrix} 0 & * \\ 0 & U_2 \end{bmatrix}$$

(12)

where $U_2$ is an $r \times r$ upper triangular matrix with nonzero constant diagonal entries. If the $j$-th column of $G$ has no indeterminates for $1 \leq j \leq n-r$ and has exactly $m-n+j-1$ indeterminates for $n-r+1 \leq j \leq n$, then there exists a permutation matrix $P$ such that

$$PG = \begin{bmatrix} 0 & C \\ 0 & U \end{bmatrix}$$

(13)

where $C$ is a partial matrix all of whose entries are indeterminates, $U$ is an $r \times r$ upper triangular matrix with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates.

Proof of (S1). We use induction on $n$ to prove (S1). It holds trivially for the case $n = 1$. Next let $n \geq 2$ and assume (S1) holds for all matrices of order $\leq n-1$. Let $G$ be an $n \times n$ partial matrix which satisfies the condition of (S1). The case $r_1 = n$ is just the statement (S) in [1, proof of Theorem 12]. So next we suppose $r_1 < n$.

There exists a permutation matrix $P_1 \in M_n(F)$ such that if we denote $P_1G = (g_{ij})$, then the following hold:
i) if $r_2 = 0$, then in the last column, $g_{1n}, g_{2n}, \ldots, g_{r_1n}$ are distinct indeterminates and all the other entries $g_{r_1+1,n}, g_{r_1+2,n}, \ldots, g_{nn}$ are constants; 

ii) if $r_2 \geq 1$, then in the last column, $g_{nn}$ is a constant and all the other entries $g_{1n}, g_{2n}, \ldots, g_{n-1,n}$ are indeterminates.

Let $TP_1^T = (t_{ij})$ and denote $V = TG = (TP_1^T)(P_1G) = (v_{ij})$.

We consider the above two cases separately.

**Case 1.** $r_2 = 0$. Since 

$$v_{in} = \sum_{k=1}^{r_1} t_{ik}g_{kn} + \sum_{k=r_1+1}^{n} t_{ik}g_{kn} = 0, \text{ for } i = r_1 + 1, \ldots, n,$$

we have $t_{ij} = 0$ for $r_1 + 1 \leq i \leq n$ and $1 \leq j \leq r_1$. Partition 

$$TP_1^T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \quad P_1G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

where $T_{11} \in M_{r_1}(F)$ and $G_{11} \in P_{r_1}(F)$. Since $T$ is nonsingular, $T_{11}$ and $T_{22}$ are nonsingular. Thus 

$$V = (TP_1^T)(P_1G) = \begin{bmatrix} U_1 & * \\ 0 & 0 \end{bmatrix}$$

implies $G_{21} = 0$ and $G_{22} = 0$ since $T_{22}G_{21} = 0$ and $T_{22}G_{22} = 0$.

Clearly the $j$-th column of $G_{11}$ has exactly $j - 1$ indeterminates for $j = 1, \ldots, r_1$. From 

$$V = (TP_1^T)(P_1G) = \begin{bmatrix} T_{11}G_{11} & * \\ 0 & 0 \end{bmatrix}$$

we deduce that $T_{11}G_{11} = U_1$ is upper triangular with nonzero constant diagonal entries. By the induction hypothesis there exists a permutation matrix $P_2$ of order $r_1$ such that $P_2G_{11}$ is upper triangular with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates. By assumption the $j$-th column of $G$, and hence $P_1G$, has $r_1$ indeterminates for $j = r_1 + 1, \ldots, n$. Since $G_{22} = 0$, we deduce that all the entries of $G_{12}$ are indeterminates. Set $P = (P_2 \oplus I_{n-r_1})P_1$. Then $P$ is a permutation matrix and 

$$PG = \begin{pmatrix} P_2G_{11} & P_2G_{12} \\ 0 & 0 \end{pmatrix}$$

has form (11).

**Case 2.** $r_2 \geq 1$. Since 

$$v_{mn} = \sum_{k=1}^{n-1} t_{nk}g_{kn} + t_{nn}g_{nn} \in F,$$
we have $t_{nk} = 0$ for $1 \leq k \leq n - 1$, and the above equality reduces to $v_{nn} = t_{nn} g_{nn}$. Then $v_{nn} \neq 0$ implies $t_{nn} \neq 0$ and $g_{nn} \neq 0$. Since $U_2$ is upper triangular, the first $n - 1$ entries in the last row of $V$ are zero. From

$$0 = v_{nj} = \sum_{k=1}^{n} t_{nk} g_{kj} = t_{nn} g_{nj}, \text{ for } j = 1, 2, \ldots, n - 1,$$

we get $g_{nj} = 0$ for $j = 1, 2, \ldots, n - 1$. Partition

$$TP_1^T = \begin{bmatrix} T_1 & v \\ 0 & t_{nn} \end{bmatrix}, \quad P_1 G = \begin{bmatrix} G_1 & w \\ 0 & g_{nn} \end{bmatrix}$$

where $T_1 \in M_{n-1}(F)$ and $G_1 \in P_{n-1}(F)$. Since $T$ is nonsingular, $T_1$ is nonsingular. Clearly the $j$-th column of $G_1$ has exactly $j - 1$ indeterminates for $j = 1, \ldots, r_1, (n-1)-(r_2-1)+1, \ldots, n-1$ and $r_1$ indeterminates for $j = r_1 + 1, \ldots, (n-1)-(r_2-1)$. By the condition ii) all the components of $w$ are indeterminates. From

$$V = (TP_1^T)(P_1 G) = \begin{bmatrix} T_1 G_1 & * \\ 0 & t_{nn} g_{nn} \end{bmatrix}$$

we deduce that $T_1 G_1$ has form (10). By the induction hypothesis there exists a permutation matrix $P_2$ of order $n - 1$ such that $P_2 G_1$ has form (11). Set $P = (P_2 \oplus 1)P_1$. Then $P$ is a permutation matrix and

$$PG = \begin{pmatrix} P_2 G_1 & P_2 w \\ 0 & g_{nn} \end{pmatrix}$$

has form (11). Thus we complete the proof of (S1).

Proof of (S2). If $r = 0$ then $G = 0$, and the result holds trivially. It suffices to prove the case $r \geq 1$. We can use induction on $n$ to prove (S2) by an argument similar to that in the above proof of Case 2 of (S1). We omit the details. The starting step $n = 1$ needs some care. When $m > n = r = 1$, the condition (12) says that $TG$ is a column vector with its last component being a nonzero constant, and the second condition in (S2) says that $G$ has exactly one constant component. The conclusion (13) states that the only constant component of $G$ is nonzero. To the contrary, suppose it is zero. Then each component of $TG$ is either 0 or a polynomial of degree 1, which contradicts the condition that the last component of $TG$ is a nonzero constant.

$\square$

For those $m \times n$ partial matrices with $m < n$, we may apply Theorem 6 by considering their transposes.

Note that the possible generalization of Theorem 6 to ACI-matrices does not make sense. Just consider the matrix

$$A = \begin{bmatrix} 1 & x_1 + x_2 + \cdots + x_k \\ 0 & 1 \end{bmatrix}.$$
References
