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Nonsymmetric normal entry patterns with the maximum number of distinct indeterminates



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ABSTRACT

We prove that a nonsymmetric normal entry pattern of order n ($n \geq 3$) has at most $n(n-3)/2 + 3$ distinct indeterminates and up to permutation similarity this number is attained by a unique pattern which is explicitly described. An open problem is posed.

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1. Introduction

Symmetric matrices, Toeplitz matrices, Hankel matrices and circulant matrices all require repetitions of some entries in prescribed positions. These special matrices suggest that we define a new concept for investigation of the general situation. Given a set S , we

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denote by $M_n(S)$ the set of $n \times n$ matrices whose entries are from S . If $S = \{x_1, \dots, x_k\}$ is a finite set, we write $M_n\{x_1, \dots, x_k\}$ for $M_n(S)$.

Definition 1. Let x_1, x_2, \dots, x_k be distinct indeterminates. We call a matrix in $M_n\{x_1, x_2, \dots, x_k\}$ an *entry pattern*.

Thus an entry pattern is a matrix whose entries are indeterminates some of which may be equal. For example, among

$$A = \begin{bmatrix} x & y \\ y & z \end{bmatrix}, \quad B = \begin{bmatrix} 2x & x+y \\ -z & w \end{bmatrix}, \quad C = \begin{bmatrix} 3 & x \\ y & z \end{bmatrix}$$

A is an entry pattern while B and C are not. Rectangular entry patterns are defined similarly.

The spirit of entry patterns is that sometimes we can deduce properties of certain special matrices by just looking at the patterns of their entries without knowing the actual entries. This is possible. For example, every real symmetric matrix has all real eigenvalues and every complex circulant matrix is normal [4, p. 5]. Entry patterns will serve the study of matrices over fields. To avoid unnecessary technical complications we consider only real matrices.

The *pattern class* of an entry pattern A , denoted $A(\mathbb{R})$, is the set of the real matrices obtained by specifying the values of the indeterminates of A . Thus

$$\begin{bmatrix} 2 & 3 & 5 \\ 5 & 2 & 3 \\ 3 & 5 & 2 \end{bmatrix} \in A(\mathbb{R}) \quad \text{with} \quad A = \begin{bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{bmatrix}.$$

Conversely,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 4 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 5 & 5 \\ 6 & 7 & 8 \\ 6 & 8 & 7 \end{bmatrix} \quad \text{have the same entry pattern} \quad \begin{bmatrix} x & x & x \\ y & z & w \\ y & w & z \end{bmatrix}.$$

We denote by A^T the transpose of a matrix A . Recall that a real matrix A is said to be *normal* if $AA^T = A^T A$. Including symmetric matrices and orthogonal matrices as subclasses, normal matrices have nice properties and they are an important topic in matrix analysis. See [1, Chapters VI and VII] and [2, Chapter 8].

Definition 2. A square entry pattern A is said to be *normal* if every matrix in $A(\mathbb{R})$ is normal.

Symmetric entry patterns are obviously normal. There are many nonsymmetric normal entry patterns. We will determine the maximum number of distinct indeterminates

in a nonsymmetric normal entry pattern of a given order and the patterns that attain this number. At the end we pose an open problem.

2. Main results

The main result is the following theorem.

Theorem 1. *Let $n \geq 3$ be an integer, and let A be a nonsymmetric normal entry pattern of order n with k distinct entries. Then $k \leq n(n - 3)/2 + 3$, where equality holds if and only if A is permutation similar to a pattern of the form*

$$\left[\begin{array}{cccc|ccc}
 x_{11} & x_{12} & \cdots & x_{1,n-3} & y_1 & y_1 & y_1 \\
 x_{12} & x_{22} & \cdots & x_{2,n-3} & y_2 & y_2 & y_2 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 x_{1,n-3} & x_{2,n-3} & \cdots & x_{n-3,n-3} & y_{n-3} & y_{n-3} & y_{n-3} \\
 \hline
 y_1 & y_2 & \cdots & y_{n-3} & z & u & v \\
 y_1 & y_2 & \cdots & y_{n-3} & v & z & u \\
 y_1 & y_2 & \cdots & y_{n-3} & u & v & z
 \end{array} \right] \tag{1}$$

where $u, v, z, y_i, x_{ij}, 1 \leq i \leq j \leq n - 3$, are distinct indeterminates.

Note that the matrix in (1) is of the form $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$ where X is symmetric, the entries in each row of Y are equal and Z is a circulant matrix of order 3.

It follows from Theorem 1 that the number of distinct indeterminates in a nonsymmetric normal entry pattern of order n ($n \geq 3$) can be any number in the interval $[2, n(n - 3)/2 + 3]$. A nonsymmetric normal entry pattern with the minimum number of distinct entries is obtained from (1) by letting $u = t_1$ and by letting $v = z = x_{ij} = y_k = t_2$ for all i, j, k .

To prove Theorem 1 we need some lemmas. Denote by J_n the $n \times n$ matrix with all entries equal to 1. An entry pattern A in $M_n\{x_1, \dots, x_k\}$ can be expressed uniquely as $A = \sum_{i=1}^k x_i A_i$ where A_1, \dots, A_k are 0-1 matrices with $\sum_{i=1}^k A_i = J_n$. We call A_i the coefficient matrix of x_i in A . Here and in the sequel we view an entry pattern as a matrix whose entries are polynomials over the field of real numbers \mathbb{R} so that addition and multiplication of entry patterns are defined in the usual way.

Lemma 2. *Let A_i be the coefficient matrix of x_i in an entry pattern $A \in M_n\{x_1, \dots, x_k\}$, $i = 1, \dots, k$. Then A is a normal entry pattern if and only if each A_i is normal and*

$$A_i A_j^T + A_j A_i^T = A_i^T A_j + A_j^T A_i \text{ for all } 1 \leq i < j \leq k. \tag{2}$$

Proof. Since $A = \sum_{i=1}^k x_i A_i$, we have

$$AA^T = \left(\sum_{i=1}^k x_i A_i\right)\left(\sum_{j=1}^k x_j A_j\right)^T = \sum_{i=1}^k x_i^2 A_i A_i^T + \sum_{1 \leq i < j \leq k} x_i x_j (A_i A_j^T + A_j A_i^T)$$

and

$$A^T A = \left(\sum_{i=1}^k x_i A_i\right)^T \left(\sum_{j=1}^k x_j A_j\right) = \sum_{i=1}^k x_i^2 A_i^T A_i + \sum_{1 \leq i < j \leq k} x_i x_j (A_i^T A_j + A_j^T A_i).$$

Now suppose A is normal. Then $AA^T = A^T A$ for any real values x_1, \dots, x_k . In this equality fixing any i with $1 \leq i \leq k$ and setting $x_i = 1$ and all other $x_j = 0$ we obtain $A_i A_i^T = A_i^T A_i$; i.e., A_i is normal. Then fixing any pair i, j with $1 \leq i < j \leq k$ and setting $x_i = x_j = 1$ and $x_t = 0$ for all $t \in \{1, \dots, k\} \setminus \{i, j\}$ we obtain (2).

The converse implication is obvious. \square

Lemma 2 and its proof show that an entry pattern A in $M_n\{x_1, \dots, x_k\}$ is normal if and only if A is normal for all $x_1, \dots, x_k \in \{0, 1\}$.

Let $B = (b_{ij})$ be a normal 0–1 matrix of order n . For $1 \leq i \leq n$, denote by r_i and c_i the i -th row sum and the i -th column sum of B respectively, and denote by r'_i (c'_i) the sum of off-diagonal entries in the i -th row (column) of B . Then $r'_i = r_i - b_{ii}$, $c'_i = c_i - b_{ii}$ for $1 \leq i \leq n$. Equating the i -th diagonal entries of both sides of $BB^T = B^T B$ we have

$$r_i = c_i \quad \text{and} \quad r'_i = c'_i, \quad i = 1, \dots, n. \tag{3}$$

Corollary 3. *Let A_1 be the coefficient matrix of x_1 in $A \in M_n\{x_1, x_2\}$. Then A is a normal entry pattern if and only if A_1 is normal.*

Proof. If A is a normal entry pattern, then by **Lemma 2** A_1 is normal. Conversely suppose A_1 is normal. The equalities $r_i = c_i$, $i = 1, \dots, n$ in (3) imply $J_n A_1^T + A_1 J_n = J_n A_1 + A_1^T J_n$, from which it follows that A_2 is normal and $A_1 A_2^T + A_2 A_1^T = A_1^T A_2 + A_2^T A_1$ where we have used the fact that $A_2 = J_n - A_1$. Applying **Lemma 2** again in another direction we conclude that A is normal. \square

We remark that there is no known characterization of normal 0–1 matrices; see [3].

Throughout we denote by $f(B)$ the number of ones in a 0–1 matrix B , by O_k the zero matrix of order k and by I_k the identity matrix of order k . Sometimes we omit the subscript k if the order is clear from the context. For square matrices A, B the notation $A \oplus B$ means the block diagonal matrix $\text{diag}(A, B)$. The notation \equiv means that we denote something.

Lemma 4. Let B be an $n \times n$ normal 0–1 matrix with $n \geq 2$. Then

- (i) $f(B) = 1$ if and only if B is permutation similar to $1 \oplus O_{n-1}$.
- (ii) $f(B) = 2$ if and only if B is permutation similar to $I_2 \oplus O_{n-2}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus O_{n-2}$.
- (iii) $f(B) = 3$ if and only if B is permutation similar to one of the following four matrices:

$$\begin{aligned}
 (a) \quad & I_3 \oplus O_{n-3}, & (b) \quad & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \oplus O_{n-2}, \\
 (c) \quad & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \oplus O_{n-3}, & (d) \quad & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \oplus O_{n-3}.
 \end{aligned} \tag{4}$$

Proof. The sufficiency is clear. Next we prove the necessity.

If $f(B) = 1$, by (3) the only nonzero entry must be a diagonal entry. Hence B is permutation similar to $1 \oplus O_{n-1}$.

If $f(B) = 2$ and one diagonal entry is nonzero, then by (3) the other nonzero entry of B is also a diagonal entry. Thus B is permutation similar to $I_2 \oplus O_{n-2}$.

If $f(B) = 2$ and the diagonal entries are all zero, then by (3) B is permutation similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus O_{n-2}$.

If $f(B) = 3$, we distinguish three cases according to the number of nonzero diagonal entries. By (3) we know that B cannot have exactly two nonzero diagonal entries. If B has three nonzero diagonal entries, then all the off-diagonal entries of B are zero and B is permutation similar to the matrix in (a). If B has only one nonzero diagonal entry, then B must be symmetric. Hence it is permutation similar to (b) or (c). If $B \equiv (b_{ij})$ has no nonzero diagonal entry, by (3) we know that each row (column) of B has at most one nonzero entry. Using a permutation similarity transformation if necessary we may assume that the first row of B has an entry equal to one, say, $b_{12} = 1$. Then there is an entry $b_{j1} = 1$. If $j = 2$, then $f(B) = 3$ forces the third nonzero entry in B to be a diagonal entry, a contradiction. Hence $j \notin \{1, 2\}$ and we may assume $j = 3$, since we can permute rows j and 3 and then permute the columns j and 3 if necessary. Now we have proved that B is permutation similar to

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & ? & ? & \cdots & ? \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & ? & ? & \cdots & ? \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & ? & ? & \cdots & ? \end{bmatrix} \equiv (c_{ij}).$$

Again, by (3) we know that the third nonzero entry in C is in the second row and in the third column, which means $c_{23} = 1$. Hence B is permutation similar to (d). \square

Lemma 5. Let $B = \begin{bmatrix} B_1 & B_2 \\ B_2^T & B_3 \end{bmatrix}$ be a square real matrix or an entry pattern. If B_1 is symmetric, then B is normal if and only if B_3 is normal and $B_2B_3 = B_2B_3^T$.

Proof. Since

$$BB^T = \begin{bmatrix} B_1^2 + B_2B_2^T & B_1B_2 + B_2B_3^T \\ B_2^TB_1 + B_3B_2^T & B_2^TB_2 + B_3B_3^T \end{bmatrix}$$

and

$$B^TB = \begin{bmatrix} B_1^2 + B_2B_2^T & B_1B_2 + B_2B_3 \\ B_2^TB_1 + B_3^TB_2^T & B_2^TB_2 + B_3^TB_3 \end{bmatrix},$$

the conclusion is clear. \square

Lemma 6. Let A be a normal entry pattern in $M_n\{x_1, \dots, x_k\}$ and let A_i be the coefficient matrix of x_i in A .

(i) If A_i has exactly one nonzero entry, then A is permutation similar to

$$\begin{bmatrix} x_i & a \\ a^T & B \end{bmatrix}.$$

(ii) If A_i has exactly two nonzero entries, then A is permutation similar to

$$\begin{bmatrix} x_i & x_j & b \\ x_j & x_i & c \\ b^T & c^T & B \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x_j & x_i & b \\ x_i & x_k & c \\ b^T & c^T & B \end{bmatrix} \tag{5}$$

where $j \neq i$ and $k \neq i$.

Proof. (i) By Lemma 2, A_i is a normal 0–1 matrix with only one nonzero entry. By Lemma 4, A_i is permutation similar to $1 \oplus O_{n-1}$. Without loss of generality we assume $A_i = 1 \oplus O_{n-1}$. For $j \in \{1, \dots, k\} \setminus \{i\}$, partition A_j as

$$A_j = \begin{bmatrix} 0 & b_j \\ c_j^T & B_j \end{bmatrix}.$$

Then

$$A_iA_j^T + A_jA_i^T = \begin{bmatrix} 0 & c_j \\ c_j^T & O \end{bmatrix} \quad \text{and} \quad A_i^TA_j + A_j^TA_i = \begin{bmatrix} 0 & b_j \\ b_j^T & O \end{bmatrix}.$$

By (2) we have $b_j = c_j$. Hence $A = \sum_j x_jA_j$ has the required form in (i).

(ii) Applying Lemma 4, we may assume $A_i = I_2 \oplus O_{n-2}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus O_{n-2}$. If $A_i = I_2 \oplus O_{n-2}$, for $j \in \{1, \dots, k\} \setminus \{i\}$, partition A_j as

$$A_j = \begin{bmatrix} 0 & \lambda_j & b_j \\ \theta_j & 0 & c_j \\ r_j^T & s_j^T & B_j \end{bmatrix}$$

where $\lambda_j, \theta_j \in \{0, 1\}$. Using the same arguments as above, we have $r_j = b_j, s_j = c_j$ for all $j \neq i$. Moreover, by (3) we have $\lambda_j = \theta_j$ for all $j \neq i$. Therefore, A is of the first form in (5).

If $A_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus O_{n-2} \equiv P \oplus O_{n-2}$, then for $j \in \{1, \dots, k\} \setminus \{i\}$, we partition A_j as

$$A_j = \begin{bmatrix} B_{j1} & B_{j2} \\ B_{j3} & B_{j4} \end{bmatrix}$$

with B_{j1} a 2×2 matrix. Since $\sum_t A_t = J_n$, B_{j1} is diagonal. We have

$$A_i A_j^T + A_j A_i^T = \begin{bmatrix} PB_{j1} + B_{j1}P & PB_{j3}^T \\ B_{j3}P & O \end{bmatrix}$$

and

$$A_i^T A_j + A_j^T A_i = \begin{bmatrix} PB_{j1} + B_{j1}P & PB_{j2} \\ B_{j2}^T P & O \end{bmatrix}.$$

By (2) we have $B_{j3} = B_{j2}^T$. Hence $A = \sum_t x_t A_t$ has the second form in (5). \square

Lemma 7. *A 2×2 entry pattern is normal if and only if it is symmetric.*

Proof. It is clear that any symmetric entry pattern is normal. If a 2×2 entry pattern A is normal, Lemma 2 implies that each coefficient matrix A_i is a 2×2 normal 0–1 matrix. Then (3) implies that each A_i is symmetric and hence A is symmetric. \square

Lemma 8. *Theorem 1 is true for $n = 3$.*

Proof. By a direct computation one can verify that any entry pattern of the form

$$\begin{bmatrix} z & u & v \\ v & z & u \\ u & v & z \end{bmatrix} \tag{6}$$

is normal.

Conversely, let A be a nonsymmetric normal entry pattern of order 3. Suppose A has at least 4 distinct entries. Since A has only 9 entries, there is an entry x_i , say, x_1 , which

appears exactly once or twice in A , i.e., $f(A_1) \leq 2$. If $f(A_1) = 1$, then by Lemma 6(i), A is permutation similar to

$$\begin{bmatrix} x_1 & a \\ a^T & B \end{bmatrix}.$$

Moreover, by Lemma 5, B is a 2×2 normal entry pattern, which is symmetric by Lemma 7. Hence A is symmetric, a contradiction. If $f(A_1) = 2$, by Lemma 6(ii), A is symmetric, a contradiction. Hence A has at most 3 distinct entries. Note that with $n = 3$, $n(n - 3)/2 + 3 = 3$.

Suppose A has exactly 3 distinct entries x_1, x_2, x_3 . If one of x_1, x_2, x_3 appears exactly once or twice in A , then using the same arguments as above we deduce that A is symmetric. Hence we have $f(A_1) = f(A_2) = f(A_3) = 3$. By Lemma 2, A_1, A_2, A_3 are all normal 0–1 matrices. Applying Lemma 4(iii) we conclude that each of A_1, A_2 and A_3 is permutation similar to one of the following four matrices:

$$I_3, \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Note that the first three matrices are symmetric. Since A is nonsymmetric, one of A_1, A_2 and A_3 is permutation similar to P . Since $A_1 + A_2 + A_3 = J_3$, by considering the diagonal entries we deduce that one of the other two coefficient matrices is I_3 . Thus, there is a permutation matrix Q such that

$$\{A_1, A_2, A_3\} = \{I_3, Q^T P Q, Q^T P^T Q\}.$$

It follows that A is permutation similar to a pattern of the form (6). \square

Proof of Theorem 1. Denote by $\phi(G)$ the number of distinct entries in an entry pattern G . First we use induction on the order n to prove that if A is a nonsymmetric normal entry pattern of order n , then $\phi(A) \leq n(n - 3)/2 + 3$. Lemma 8 shows that this is true for $n = 3$. Now let $n \geq 4$ and assume that the conclusion is true for all entry patterns of order $n - 1$. Let A be a nonsymmetric normal entry pattern of order n .

To the contrary, assume that $\phi(A) \geq n(n - 3)/2 + 4$. Then A contains an entry, say, x_1 , which appears exactly once or twice in A . Otherwise, each entry appears at least 3 times and we have

$$3[n(n - 3)/2 + 4] > n^2,$$

which contradicts the fact that A has only n^2 entries. Let A_i be the coefficient matrix of x_i in A .

If $f(A_1) = 1$, then by Lemma 6(i) A is permutation similar to

$$A^{(1)} = \begin{bmatrix} x_1 & a \\ a^T & B \end{bmatrix}. \tag{7}$$

Since A is nonsymmetric, so is B . By Lemma 5, B is a normal entry pattern of order $n - 1$. Applying the induction hypothesis to B , we have $\phi(B) \leq (n - 1)(n - 4)/2 + 3$. Denote by $\theta(a)$ the number of those distinct entries in $A^{(1)}$ that appear only in a and a^T . Then

$$\begin{aligned} \theta(a) &= \phi(A) - \phi(B) - 1 \\ &\geq [n(n - 3)/2 + 4] - [(n - 1)(n - 4)/2 + 3] - 1 \\ &= n - 2. \end{aligned}$$

Since a has $n - 1$ components, it follows that there are at least $n - 3$ distinct entries in a that appear exactly twice in $A^{(1)}$. Without loss of generality, we assume $f(A_2) = \dots = f(A_{n-2}) = 2$. Lemma 6(ii) indicates that for each $2 \leq i \leq n - 2$, the two rows of $A^{(1)}$ in which the two x'_i s lie and the corresponding two columns are symmetric. Thus, $A^{(1)}$ is permutation similar to

$$A^{(2)} = \left[\begin{array}{c|ccc|c} x_1 & x_2 & \cdots & x_{n-2} & b \\ \hline x_2 & & & & \\ \vdots & & C & & E \\ \hline x_{n-2} & & & & \\ \hline b^T & & E^T & & F \end{array} \right] \tag{8}$$

where C is symmetric. By Lemma 5, F is a 2×2 normal entry pattern which must be symmetric by Lemma 7. Hence $A^{(2)}$ is symmetric and consequently A is symmetric, a contradiction.

If $f(A_1) = 2$, then by Lemma 6(ii), A is permutation similar to a matrix of one of the two forms in (5) with $i = 1$. Repartition

$$\begin{bmatrix} x_1 & x_j & b \\ x_j & x_1 & c \\ b^T & c^T & B \end{bmatrix} = \begin{bmatrix} x_1 & a \\ a^T & S \end{bmatrix}, \quad \begin{bmatrix} x_j & x_1 & b \\ x_1 & x_k & c \\ b^T & c^T & B \end{bmatrix} = \begin{bmatrix} x_j & a \\ a^T & S \end{bmatrix}. \tag{9}$$

In both cases we have $\theta(a) \geq n - 2$. Using the same arguments as above we deduce that A is symmetric, a contradiction.

Therefore, $\phi(A) \leq n(n - 3)/2 + 3$.

Next we use induction on the order n to prove that if A is a nonsymmetric normal entry pattern of order n with $\phi(A) = n(n - 3)/2 + 3$, then A is permutation similar to a pattern of the form (1). Lemma 8 shows that this is true for $n = 3$. Now let $n \geq 4$ and assume that the conclusion is true for all entry patterns of order $n - 1$. Let A be a nonsymmetric normal entry pattern of order n with $\phi(A) = n(n - 3)/2 + 3$.

From $4[n(n - 3)/2 + 3] > n^2$ we know that there is at least one entry that appears less than 4 times in A . Suppose x_1 is an entry that appears the least times in A . Then $f(A_1) \leq 3$. We distinguish three cases.

Case 1. $f(A_1) = 1$. By Lemma 6(i) A is permutation similar to a pattern of the form (7). Applying Lemma 5 we deduce that the matrix B in (7) is a normal entry pattern of order $n - 1$. By what we have proved above,

$$\phi(B) \leq (n - 1)(n - 4)/2 + 3.$$

If $\phi(B) < (n - 1)(n - 4)/2 + 3$, then there are at least

$$\phi(A) - \phi(B) - 1 \geq n - 2$$

distinct entries in a that do not appear in B . Using the same arguments as above, we conclude that A is symmetric, a contradiction. Hence we have $\phi(B) = (n - 1)(n - 4)/2 + 3$. By the induction hypothesis, B is permutation similar to a matrix of the form (1) and hence A is permutation similar to

$$H = \begin{bmatrix} x_1 & c_2 & \cdots & c_{n-3} & c_{n-2} & c_{n-1} & c_n \\ c_2 & x_{22} & \cdots & x_{2,n-3} & y_2 & y_2 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ c_{n-3} & x_{2,n-3} & \cdots & x_{n-3,n-3} & y_{n-3} & y_{n-3} & y_{n-3} \\ c_{n-2} & y_2 & \cdots & y_{n-3} & z & u & v \\ c_{n-1} & y_2 & \cdots & y_{n-3} & v & z & u \\ c_n & y_2 & \cdots & y_{n-3} & u & v & z \end{bmatrix}.$$

Suppose there are exactly d distinct entries among c_2, \dots, c_n that do not appear in B . Then

$$1 + d + \phi(B) = n(n - 3)/2 + 3,$$

yielding $d = n - 3$. Now it suffices to prove that $c_{n-2} = c_{n-1} = c_n$, which forces c_2, \dots, c_{n-3} to be distinct entries, and hence H has the required form.

Let $y_i = x_{ij} = 0$ for all $2 \leq i \leq j \leq n - 3$. Then H is permutation similar to

$$K = \begin{bmatrix} & & & c_2 & & & & \\ & & & \vdots & & & & \\ & & & c_{n-3} & & & & \\ c_2 & \cdots & c_{n-3} & x_1 & c_{n-2} & c_{n-1} & c_n & \\ & & & c_{n-2} & z & u & v & \\ & & & c_{n-1} & v & z & u & \\ & & & c_n & u & v & z & \end{bmatrix} \equiv \begin{bmatrix} O & p^T & O \\ p & x_1 & q \\ O & q^T & F \end{bmatrix}$$

where $p = (c_2, \dots, c_{n-3})$, $q = (c_{n-2}, c_{n-1}, c_n)$. Now $KK^T = K^TK$ implies $qF = qF^T$. Setting $z = v = 0$ and $u = 1$ in F we get

$$0 = q(F - F^T) = (c_n - c_{n-1}, c_{n-2} - c_n, *),$$

which gives $c_{n-2} = c_{n-1} = c_n$.

Case 2. $f(A_1) = 2$. We will show that this case cannot happen. By Lemma 6(ii), A is permutation similar to one of the two forms in (5). In both cases, we can repartition A as (9). Applying Lemma 5 we deduce that the matrix S in (9) is a nonsymmetric normal entry pattern of order $n - 1$. By what we have proved above,

$$\phi(S) \leq (n - 1)(n - 4)/2 + 3.$$

If $\phi(S) < (n - 1)(n - 4)/2 + 3$, then

$$\theta(a) \geq \phi(A) - \phi(S) - 1 \geq n - 2.$$

Using the same arguments as above, we deduce that A is permutation similar to a matrix of form (8), which is symmetric, a contradiction. Hence we have $\phi(S) = (n - 1)(n - 4)/2 + 3$. By the induction hypothesis, S is permutation similar to a matrix of the form (1) and hence A is permutation similar to

$$\begin{bmatrix} w & c_2 & \cdots & c_{n-3} & c_{n-2} & c_{n-1} & c_n \\ c_2 & x_{22} & \cdots & x_{2,n-3} & y_2 & y_2 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ c_{n-3} & x_{2,n-3} & \cdots & x_{n-3,n-3} & y_{n-3} & y_{n-3} & y_{n-3} \\ c_{n-2} & y_2 & \cdots & y_{n-3} & z & u & v \\ c_{n-1} & y_2 & \cdots & y_{n-3} & v & z & u \\ c_n & y_2 & \cdots & y_{n-3} & u & v & z \end{bmatrix}$$

where $w = x_1$ or $w = x_j$. As in Case 1, we can prove that $c_{n-2} = c_{n-1} = c_n$. Since this matrix has $n(n - 3)/2 + 3$ distinct entries by assumption, it follows that each of the diagonal entries $w, x_{22}, \dots, x_{n-3,n-3}$ appears exactly once, which contradicts the assumptions that $f(A_1) = 2$ and x_1 appears the least times in A .

Case 3. $f(A_1) = 3$. We will show that this case cannot happen. If $n \geq 7$, then $3[n(n - 3)/2 + 3] > n^2$, a contradiction. Hence we have $n \leq 6$. Moreover, if $n = 6$, each of the $n(n - 3)/2 + 3$ distinct entries appears exactly 3 times. If $n = 4$ or $n = 5$, then one of the $n(n - 3)/2 + 3$ distinct entries appears exactly 4 times and the others appear exactly 3 times. By Lemma 4(iii), we may assume A_1 has one of the forms in (4).

Subcase 3.1. A_1 has form (b) in (4). For any $2 \leq j \leq n(n - 3)/2 + 3$, partition A_j as

$$A_j = \begin{bmatrix} B_{j1} & B_{j2} \\ B_{j3} & B_{j4} \end{bmatrix} \tag{10}$$

with B_{j1} a 2×2 matrix. Applying (2) we deduce that $B_{j3} = B_{j2}^T$ and hence A has the form

$$A = \begin{bmatrix} C_1 & C_2 \\ C_2^T & C_3 \end{bmatrix} \tag{11}$$

with $C_1 = \begin{bmatrix} x_1 & x_1 \\ x_1 & \beta \end{bmatrix}$. By Lemma 5 and the assumption on A , C_3 is a nonsymmetric normal entry pattern.

If $n = 4$, the 2×2 normal pattern C_3 must be symmetric, a contradiction.

If $n = 5$, C_3 contains at least 3 distinct entries. By Lemma 8, C_3 is permutation similar to

$$\begin{bmatrix} z & u & v \\ v & z & u \\ u & v & z \end{bmatrix}. \tag{12}$$

Note that since C_2 and C_2^T are in symmetric positions in A , the only way for an entry in C_2 to appear exactly 3 times in A is appearing once in a diagonal position. Since all diagonal entries in C_3 are equal, then at most two entries in C_2 appear exactly 3 times in A . Thus at least two entries in C_2 appear an even number of times, a contradiction.

If $n = 6$, it is impossible for all the 8 entries in C_2 to appear exactly 3 times in A , since A has only 6 diagonal positions.

Subcase 3.2. A_1 has form (a), (c) or (d) in (4). Again, for any $2 \leq j \leq n(n-3)/2+3$, partition A_j as form (10) with B_{j1} a 3×3 matrix. Applying (2) we deduce that A has form (11) with C_1 a 3×3 entry pattern.

If $n = 4$ or $n = 5$, then by (3), C_3 is symmetric, and hence C_1 is a nonsymmetric normal entry pattern by Lemma 5. C_1 has 3 distinct entries. Applying Lemma 8, we deduce that C_1 is permutation similar to a pattern of form (12). If $n = 4$, then C_2 has at least one entry, say, x_2 , appears exactly 3 times in A . It follows that $C_3 = x_2$ and A_2 is permutation similar to $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \oplus O_2$. Applying Subcase 3.1 we get a contradiction. If $n = 5$, then C_2 has 6 entries at least 4 of which appear exactly 3 times in A , which is impossible since C_1 has three identical diagonal entries and C_3 has only 2 diagonal entries.

If $n = 6$, then all the 9 entries in C_2 appear exactly 3 times in A , which is impossible since A has only 6 diagonal positions.

So far we have proved that a nonsymmetric normal entry pattern of order n with $n(n-3)/2+3$ distinct entries is permutation similar to a pattern of the form (1). It remains to verify that an entry pattern A of the form (1) is normal. Partition A as $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$ where X is symmetric, the entries in each row of Y are equal and Z is a circulant matrix of order 3. A direct computation shows that $AA^T = A^T A$. This completes the proof. \square

Many questions on sign patterns can be asked for entry patterns. We give only one example. We say that an entry pattern A requires all real eigenvalues if the eigenvalues of any matrix in $A(\mathbb{R})$ are all real numbers.

Of course, every symmetric entry pattern requires all real eigenvalues. But there are such nonsymmetric entry patterns. For example, the entry patterns

$$\begin{bmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ x_2 & x_2 & x_2 & \cdots & x_2 \\ x_3 & x_3 & x_3 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_n & x_n & x_n & \cdots & x_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x & x & x & \cdots & x \\ x & y & y & \cdots & y \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x & y & y & \cdots & y \\ x & x & x & \cdots & x \end{bmatrix}$$

are nonsymmetric and require all real eigenvalues. We pose the following

Open problem. Characterize those entry patterns that require all real eigenvalues.

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