Partial matrices all of whose completions have the same spectrum

Zejun Huang

School of Sciences, Zhejiang A & F University, Hangzhou, Zhejiang 311300, China

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong

huangzejun@yahoo.cn

Xingzhi Zhan*

Department of Mathematics, East China Normal University, Shanghai 200241, China

zhan@math.ecnu.edu.cn

Abstract

We characterize the square partial matrices over a field all of whose completions have the same spectrum, and determine the maximum number of indeterminates in such partial matrices of a given order as well as the matrices that attain this maximum number.

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1 Introduction

A *partial matrix* over a set Ω is a matrix in which some entries all from Ω are specified and the other entries are free to be chosen from Ω. A *completion* of a partial matrix over Ω is a specific choice of values from Ω for its unspecified entries. A completion may also mean

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a completed matrix of a partial matrix. We call the unspecified entries *indeterminates* since they are free to range over Ω.

Denote by $M_n(\Omega)$ the set of $n \times n$ matrices whose entries are from a given set Ω. We call elements in Ω *constants* and call matrices in $M_n(\Omega)$ *constant matrices*, in contrast to indeterminates and partial matrices respectively.

Let A be a matrix of order n over a field $\mathbf{F}$. The characteristic polynomial $f(x)$ of A has n roots $\lambda_1, \ldots, \lambda_n$ (multiplicities counted) in the algebraic closure of $\mathbf{F}$, more precisely, in the splitting field of $f(x)$. The elements $\lambda_1, \ldots, \lambda_n$ are called the *eigenvalues* of A and the set $\{\lambda_1, \ldots, \lambda_n\}$ is called the *spectrum* of A. Note that if A has repeated eigenvalues then its spectrum is a multi-set. Clearly, two matrices over a field have the same spectrum if and only if they have the same characteristic polynomial.

In this paper we characterize the square partial matrices over a field all of whose completions have the same spectrum, and determine the maximum number of indeterminates in such partial matrices of a given order as well as the matrices that attain this maximum number. In [2] and [3], the partial matrices over a field all of whose completions have the same determinant, have a bounded rank or have the same rank are studied.

2 Main results

Denote by $\mathbf{F}[x_1, \ldots, x_k]$ the ring of polynomials in the indeterminates $x_1, \ldots, x_k$ over the field $\mathbf{F}$. We will need the following combinatorial Nullstellensatz of Alon [1] whose proof can also be found in [4, p.330].

Lemma 1 *Let $\mathbf{F}$ be a field and let $f \in \mathbf{F}[x_1, \ldots, x_k]$ be a polynomial of degree d which contains a nonzero coefficient at $x_1^{d_1} \cdots x_k^{d_k}$ with $d_1 + \cdots + d_k = d$. If $S_1, \ldots, S_k$ are subsets of $\mathbf{F}$ such that $|S_i| > d_i$ for all $i = 1, \ldots, k$, then there exists $a_1 \in S_1, \ldots, a_k \in S_k$ such that $f(a_1, \ldots, a_k) \neq 0$.*

Let A be a partial matrix over a set Ω. If Δ is a subset of Ω, then a completion of A over Δ means that we choose values from Δ for the indeterminates of A.

Lemma 2 *Let $\mathbf{F} \subseteq \mathbf{E}$ be a field extension, and A be a square partial matrix over $\mathbf{E}$. Then all the completions of A over $\mathbf{F}$ have the same determinant if and only if all the completions of A over $\mathbf{E}$ have the same determinant.*

Proof. The sufficiency is trivial. We prove the necessity. Suppose that all the completions of A over $\mathbf{F}$ have the same determinant. Let $x_1, x_2, \ldots, x_k$ be the indeterminates in A. Then $\det A$ is a polynomial $f(x_1, \ldots, x_k)$ over $\mathbf{E}$ in which the degree of each x_i is at most 1 for $i = 1, 2, \ldots, k$. The condition that all the completions of A over $\mathbf{F}$ have the same determinant implies that $f(x_1, \ldots, x_k)$ assumes a constant value c for $x_i \in \mathbf{F}$. Let $g(x_1, \ldots, x_k) = f(x_1, \ldots, x_k) - c$. Then $g(x_1, \ldots, x_k) = 0$ for all $x_i \in \mathbf{F}$. Since $|\mathbf{F}| \geq 2$, applying Lemma 1 we deduce that g is the zero polynomial. Hence f is a constant polynomial, that is, all the completions of A over $\mathbf{E}$ have the same determinant. □

Lemma 3 *Let A be a square partial matrix over a field $\mathbf{F}$. If all the completions of A have the same spectrum, then all the diagonal entries of A are constants from $\mathbf{F}$.*

Proof. Since all the completions of A have the same spectrum, all the completions of A have the same trace. It follows that all the diagonal entries of A must be constants. □

The following lemma is a special case of [2, Theorem 10]

Lemma 4 *Let $\mathbf{F}$ be a field with at least $n+1$ elements. Let A be an $n \times n$ partial matrix over $\mathbf{F}$. Then all the completions of A have the same nonzero determinant if and only if there exists a nonsingular constant matrix $T \in M_n(\mathbf{F})$ and a permutation matrix $Q \in M_n(\mathbf{F})$ such that TAQ is an upper triangular matrix with nonzero constant diagonal entries.*

Let $\mathbf{F}$ be a field and x be a given indeterminate transcendental over $\mathbf{F}$. We denote by $\mathbf{F}(x)$ the field of rational functions in x over $\mathbf{F}$, i.e., the field

$$\mathbf{F}(x) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in \mathbf{F}[x], g(x) \neq 0. \right\}$$

Note that $\mathbf{F} \subseteq \mathbf{F}(x)$ is a field extension. Denote by I the identity matrix whose order will be clear from the context.

Theorem 5 *Let $\mathbf{F}$ be a field and x be an indeterminate transcendental over $\mathbf{F}$. Let A be an $n \times n$ partial matrix over $\mathbf{F}$. Then all the completions of A have the same spectrum if and only if there exists a nonsingular matrix $T \in M_n(\mathbf{F}(x))$ and a permutation matrix $Q \in M_n(\mathbf{F})$ such that $T(xI - A)Q$ is an upper triangular matrix with nonzero diagonal entries from $\mathbf{F}(x)$.*

Proof. By Lemma 3, $xI - A$ is a partial matrix over $\mathbf{F}(x)$. Thus all the completions of A over $\mathbf{F}$ have the same spectrum if and only if all the completions of $xI - A$ over $\mathbf{F}$ have the same determinant. By Lemma 2, this is equivalent to that all the completions of $xI - A$ over $\mathbf{F}(x)$ have the same determinant. Note that $|\mathbf{F}(x)| = \infty > n+1$ and that as a polynomial, $\det(xI - A)$ is always nonzero. Applying Lemma 4 to $xI - A$ and the field $\mathbf{F}(x)$ we complete the proof. $\square$

As in [2], Theorem 5 provides an algorithm to decide whether all the completions of a given partial matrix A have the same spectrum. This is the case if and only if $xI - A$ can be transformed to an upper triangular form with nonzero constant diagonal entries by the following three elementary operations: (1) permutation of columns; (2) permutation of rows; (3) addition of a scalar multiple of one row to another row.

For example: let

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ y & -1 & -1 & 1 \end{bmatrix}.$$

Then

$$xI - A = \begin{bmatrix} x-1 & -1 & 1 & 0 \\ 1 & x-1 & 1 & 1 \\ 1 & 1 & x-1 & 1 \\ -y & 1 & 1 & x-1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -1 & 1 & x-1 \\ 1 & x-1 & 1 & 1 \\ 1 & 1 & x-1 & 1 \\ x-1 & 1 & 1 & -y \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & x-1 & 1 \\ 1 & x-1 & 1 & 1 \\ 0 & -1 & 1 & x-1 \\ x-1 & 1 & 1 & -y \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & x-1 & 1 \\ 0 & x-2 & 2-x & 0 \\ 0 & -1 & 1 & x-1 \\ 0 & 2-x & 1-(x-1)^2 & -y-x+1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & x-1 & 1 \\ 0 & x-2 & 2-x & 0 \\ 0 & 0 & 0 & x-1 \\ 0 & 0 & -x^2+x+2 & -y-x+1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & x-1 & 1 \\ 0 & x-2 & 2-x & 0 \\ 0 & 0 & -x^2+x+2 & -y-x+1 \\ 0 & 0 & 0 & x-1 \end{bmatrix}$$

We conclude that all completions of A have the same spectrum. Here the operations we performed are: interchanging column 1 and column 4; interchanging row 1 and row 3; adding -1 times row 1 to row 2 and adding $1-x$ times row 1 to row 4; adding $1/(x-2)$ times row 2 to row 3 and adding row 2 to row 4; interchanging row 3 and row 4.

Next we study the maximum number of indeterminates in the partial matrices all of whose completions have the same spectrum. We will use the following lemma [2, Theorem 12]. The noun "diagonal" will always mean the main diagonal.

Lemma 6 *Let* $\mathbf{F}$ *be a field with at least* $n+1$ *elements. Let* A *be a partial matrix of order* n *over* $\mathbf{F}$ *all of whose completions are nonsingular. Then the number of indeterminates of* A *is less than or equal to* $n(n-1)/2$*. This maximum number is attained if and only if there exist permutation matrices* P, Q *such that* PAQ *is upper triangular with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates.*

Theorem 7 *Let* $\mathbf{F}$ *be a field and let* A *be an* $n \times n$ *partial matrix over* $\mathbf{F}$ *all of whose completions have the same spectrum. Then the number of indeterminates of* A *is less than or equal to* $n(n-1)/2$*. This maximum number is attained if and only if* A *is permutation similar to an upper triangular matrix with constant diagonal entries and with all the entries above the diagonal being indeterminates.*

Proof. Denote by $f(G)$ the number of indeterminates in a partial matrix G. Let x be an indeterminate transcendental over $\mathbf{F}$. From the proof of Theorem 5 we know that all the completions of A over $\mathbf{F}$ have the same spectrum if and only if all the completions of $xI - A$ over $\mathbf{F}(x)$ have the same nonzero determinant. Applying Lemma 6 to $xI - A$ and $\mathbf{F}(x)$, we have

$$f(A) = f(xI - A) \leq n(n-1)/2$$

and equality holds if and only if there exist permutation matrices P, Q such that $P(xI - A)Q$ is upper triangular with nonzero constant diagonal entries from $\mathbf{F}(x)$ and with all the entries above the diagonal being indeterminates.

Now suppose $f(A) = n(n-1)/2$. Then there are permutation matrices P, Q satisfying the conditions stated above. Since $P(xI - A)Q = xPQ - PAQ$ and $P(xI - A)Q$ is upper triangular, the permutation matrix PQ must be upper triangular. But an upper triangular permutation matrix is just the identity matrix. Hence $PQ = I$, i.e., $Q = P^T$. Then $P(xI - A)Q = xI - PAP^T$. It follows that PAP^T is an upper triangular matrix with constant diagonal entries from $\mathbf{F}$ and with all the entries above the diagonal being indeterminates. □

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