Partial matrices all of whose completions have the same spectrum

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Abstract

We characterize the square partial matrices over a field all of whose completions have the same spectrum, and determine the maximum number of indeterminates in such partial matrices of a given order as well as the matrices that attain this maximum number.

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1 Introduction

A partial matrix over a set $\Omega$ is a matrix in which some entries all from $\Omega$ are specified and the other entries are free to be chosen from $\Omega$. A completion of a partial matrix over $\Omega$ is a specific choice of values from $\Omega$ for its unspecified entries. A completion may also mean

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a completed matrix of a partial matrix. We call the unspecified entries indeterminates since they are free to range over \( \Omega \).

Denote by \( M_n(\Omega) \) the set of \( n \times n \) matrices whose entries are from a given set \( \Omega \). We call elements in \( \Omega \) constants and call matrices in \( M_n(\Omega) \) constant matrices, in contrast to indeterminates and partial matrices respectively.

Let \( A \) be a matrix of order \( n \) over a field \( F \). The characteristic polynomial \( f(x) \) of \( A \) has \( n \) roots \( \lambda_1, \ldots, \lambda_n \) (multiplicities counted) in the algebraic closure of \( F \), more precisely, in the splitting field of \( f(x) \). The elements \( \lambda_1, \ldots, \lambda_n \) are called the eigenvalues of \( A \) and the set \( \{\lambda_1, \ldots, \lambda_n\} \) is called the spectrum of \( A \). Note that if \( A \) has repeated eigenvalues then its spectrum is a multi-set. Clearly, two matrices over a field have the same spectrum if and only if they have the same characteristic polynomial.

In this paper we characterize the square partial matrices over a field all of whose completions have the same spectrum, and determine the maximum number of indeterminates in such partial matrices of a given order as well as the matrices that attain this maximum number. In [2] and [3], the partial matrices over a field all of whose completions have the same determinant, have a bounded rank or have the same rank are studied.

2 Main results

Denote by \( F[x_1, \ldots, x_k] \) the ring of polynomials in the indeterminates \( x_1, \ldots, x_k \) over the field \( F \). We will need the following combinatorial Nullstellensatz of Alon [1] whose proof can also be found in [4, p.330].

**Lemma 1** Let \( F \) be a field and let \( f \in F[x_1, \ldots, x_k] \) be a polynomial of degree \( d \) which contains a nonzero coefficient at \( x_1^{d_1} \cdots x_k^{d_k} \) with \( d_1 + \cdots + d_k = d \). If \( S_1, \ldots, S_k \) are subsets of \( F \) such that \( |S_i| > d_i \) for all \( i = 1, \ldots, k \), then there exists \( a_1 \in S_1, \ldots, a_k \in S_k \) such that \( f(a_1, \ldots, a_k) \neq 0 \).

Let \( A \) be a partial matrix over a set \( \Omega \). If \( \Delta \) is a subset of \( \Omega \), then a completion of \( A \) over \( \Delta \) means that we choose values from \( \Delta \) for the indeterminates of \( A \).

**Lemma 2** Let \( F \subseteq E \) be a field extension, and \( A \) be a square partial matrix over \( E \). Then all the completions of \( A \) over \( F \) have the same determinant if and only if all the completions of \( A \) over \( E \) have the same determinant.
Proof. The sufficiency is trivial. We prove the necessity. Suppose that all the completions of \( A \) over \( F \) have the same determinant. Let \( x_1, x_2, \ldots, x_k \) be the indeterminates in \( A \). Then \( \det A \) is a polynomial \( f(x_1, \ldots, x_k) \) over \( E \) in which the degree of each \( x_i \) is at most 1 for \( i = 1, 2, \ldots, k \). The condition that all the completions of \( A \) over \( F \) have the same determinant implies that \( f(x_1, \ldots, x_k) \) assumes a constant value \( c \) for \( x_i \in F \). Let \( g(x_1, \ldots, x_k) = f(x_1, \ldots, x_k) - c \). Then \( g(x_1, \ldots, x_k) = 0 \) for all \( x_i \in F \). Since \( |F| \geq 2 \), applying Lemma 1 we deduce that \( g \) is the zero polynomial. Hence \( f \) is a constant polynomial, that is, all the completions of \( A \) over \( E \) have the same determinant.  

Lemma 3 Let \( A \) be a square partial matrix over a field \( F \). If all the completions of \( A \) have the same spectrum, then all the diagonal entries of \( A \) are constants from \( F \).

Proof. Since all the completions of \( A \) have the same spectrum, all the completions of \( A \) have the same trace. It follows that all the diagonal entries of \( A \) must be constants.  

The following lemma is a special case of [2, Theorem 10] 

Lemma 4 Let \( F \) be a field with at least \( n + 1 \) elements. Let \( A \) be an \( n \times n \) partial matrix over \( F \). Then all the completions of \( A \) have the same nonzero determinant if and only if there exists a nonsingular constant matrix \( T \in M_n(F) \) and a permutation matrix \( Q \in M_n(F) \) such that \( TAQ \) is an upper triangular matrix with nonzero constant diagonal entries.

Let \( F \) be a field and \( x \) be a given indeterminate transcendental over \( F \). We denote by \( F(x) \) the field of rational functions in \( x \) over \( F \), i.e., the field 

\[
F(x) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in F[x], g(x) \neq 0 \right\}
\]

Note that \( F \subseteq F(x) \) is a field extension. Denote by \( I \) the identity matrix whose order will be clear from the context.

Theorem 5 Let \( F \) be a field and \( x \) be an indeterminate transcendental over \( F \). Let \( A \) be an \( n \times n \) partial matrix over \( F \). Then all the completions of \( A \) have the same spectrum if and only if there exists a nonsingular matrix \( T \in M_n(F(x)) \) and a permutation matrix \( Q \in M_n(F) \) such that \( T(xI - A)Q \) is an upper triangular matrix with nonzero diagonal entries from \( F(x) \).
Proof. By Lemma 3, $xI - A$ is a partial matrix over $F(x)$. Thus all the completions of $A$ over $F$ have the same spectrum if and only if all the completions of $xI - A$ over $F$ have the same determinant. By Lemma 2, this is equivalent to that all the completions of $xI - A$ over $F(x)$ have the same determinant. Note that $|F(x)| = \infty > n + 1$ and that as a polynomial, $\det(xI - A)$ is always nonzero. Applying Lemma 4 to $xI - A$ and the field $F(x)$ we complete the proof.

As in [2], Theorem 5 provides an algorithm to decide whether all the completions of a given partial matrix $A$ have the same spectrum. This is the case if and only if $xI - A$ can be transformed to an upper triangular form with nonzero constant diagonal entries by the following three elementary operations: (1) permutation of columns; (2) permutation of rows; (3) addition of a scalar multiple of one row to another row.

For example: let

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ y & -1 & -1 & 1 \end{bmatrix}.$$ 

Then

$$xI - A = \begin{bmatrix} x - 1 & -1 & 1 & 0 \\ 1 & x - 1 & 1 & 1 \\ 1 & 1 & x - 1 & 1 \\ -y & 1 & 1 & x - 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 & 1 & x - 1 \\ 1 & x - 1 & 1 & 1 \\ 1 & 1 & x - 1 & 1 \\ x - 1 & 1 & 1 & -y \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & x - 1 & 1 \\ 0 & x - 2 & 2 - x & 0 \\ 0 & -1 & 1 & x - 1 \\ 0 & 2 - x & 1 - (x - 1)^2 & -y - x + 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & x - 1 & 1 \\ 0 & x - 2 & 2 - x & 0 \\ 0 & -1 & 1 & x - 1 \\ 0 & 0 & -x^2 + x + 2 & -y - x + 1 \end{bmatrix}$$

We conclude that all completions of $A$ have the same spectrum. Here the operations we performed are: interchanging column 1 and column 4; interchanging row 1 and row 3; adding $-1$ times row 1 to row 2 and adding $1 - x$ times row 1 to row 4; adding $1/(x - 2)$ times row 2 to row 3 and adding row 2 to row 4; interchanging row 3 and row 4.
Next we study the maximum number of indeterminates in the partial matrices all of whose completions have the same spectrum. We will use the following lemma [2, Theorem 12]. The noun “diagonal” will always mean the main diagonal.

**Lemma 6** Let $F$ be a field with at least $n+1$ elements. Let $A$ be a partial matrix of order $n$ over $F$ all of whose completions are nonsingular. Then the number of indeterminates of $A$ is less than or equal to $n(n−1)/2$. This maximum number is attained if and only if there exist permutation matrices $P, Q$ such that $P AQ$ is upper triangular with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates.

**Theorem 7** Let $F$ be a field and let $A$ be an $n \times n$ partial matrix over $F$ all of whose completions have the same spectrum. Then the number of indeterminates of $A$ is less than or equal to $n(n−1)/2$. This maximum number is attained if and only if $A$ is permutation similar to an upper triangular matrix with constant diagonal entries and with all the entries above the diagonal being indeterminates.

Proof. Denote by $f(G)$ the number of indeterminates in a partial matrix $G$. Let $x$ be an indeterminate transcendental over $F$. From the proof of Theorem 5 we know that all the completions of $A$ over $F$ have the same spectrum if and only if all the completions of $x I - A$ over $F(x)$ have the same nonzero determinant. Applying Lemma 6 to $x I - A$ and $F(x)$, we have

$$f(A) = f(x I - A) \leq n(n−1)/2$$

and equality holds if and only if there exist permutation matrices $P, Q$ such that $P(x I - A)Q$ is upper triangular with nonzero constant diagonal entries from $F(x)$ and with all the entries above the diagonal being indeterminates.

Now suppose $f(A) = n(n−1)/2$. Then there are permutation matrices $P, Q$ satisfying the conditions stated above. Since $P(x I - A)Q = xPQ - PAQ$ and $P(x I - A)Q$ is upper triangular, the permutation matrix $PQ$ must be upper triangular. But an upper triangular permutation matrix is just the identity matrix. Hence $PQ = I$, i.e., $Q = P^T$. Then $P(x I - A)Q = x I - PAP^T$. It follows that $PAP^T$ is an upper triangular matrix with constant diagonal entries from $F$ and with all the entries above the diagonal being indeterminates. \qed
References


