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ORIGINAL PAPER

On Vertex Types of Graphs

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Abstract
The vertices of a graph are classified into seven types by J.T. Hedetniemi, S.M. Hedetniemi, S.T. Hedetniemi and T.M. Lewis and they ask the following questions: (1) What is the smallest order n of a graph having $n-2$ very typical vertices or $n-2$ typical vertices? (2) What is the smallest order of a pantypical graph? We answer these two questions and determine all the possible orders of the graphs in these three classes in this paper.

Keywords Graph · Vertex type · Degree · Smallest order

1 Introduction

We consider finite simple graphs. For a vertex v in a graph, we denote by $d(v)$ and $N(v)$ the degree of v and the neighborhood of v respectively throughout the paper. Motivated by the notions of strong and weak vertices in [3] and [2], Hedetniemi, Hedetniemi, Hedetniemi and Lewis [1] classified the vertices of a graph into the following seven types.

Definition A vertex u in a simple graph is said to be

1. *very strong* if $d(u) \geq 2$ and for every vertex $v \in N(u)$, $d(u) > d(v)$;
2. *strong* if $d(u) \geq 2$ and for every vertex $v \in N(u)$, $d(u) \geq d(v)$, at least one neighbor $x \in N(u)$ has $d(x) < d(u)$ and at least one neighbor $y \in N(u)$ has $d(y) = d(u)$;
3. *regular* if $d(u) \geq 0$ and for every vertex $v \in N(u)$, $d(u) = d(v)$;

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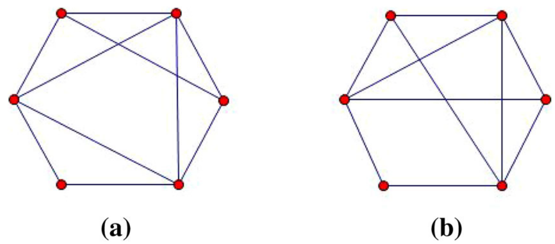


Fig. 1 Non-isomorphic graphs with the same degree sequence

4. *very typical* if $d(u) \geq 2$ and for every vertex $v \in N(u)$, $d(u) \neq d(v)$, at least one neighbor $x \in N(u)$ has $d(x) > d(u)$ and at least one neighbor $y \in N(u)$ has $d(y) < d(u)$;
5. *typical* if $d(u) \geq 3$ and there are three distinct vertices $x, y, z \in N(u)$ satisfying $d(x) < d(u) = d(y) < d(z)$;
6. *weak* if $d(u) \geq 2$ and for every vertex $v \in N(u)$, $d(u) \leq d(v)$, at least one neighbor $x \in N(u)$ has $d(x) > d(u)$ and at least one neighbor $y \in N(u)$ has $d(y) = d(u)$;
7. *very weak* if $d(u) \geq 1$ and for every vertex $v \in N(u)$, $d(u) < d(v)$.

If a graph G has vertices of all seven types, then G is said to be *pantypical.*

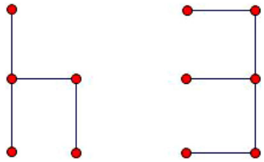
By the definition above, isolated vertices are regular. Now every simple graph G corresponds to a 7-tuple $\Gamma(G) = (n_1, n_2, n_3, n_4, n_5, n_6, n_7)$ where $n_1, n_2, n_3, n_4, n_5, n_6, n_7$ are the numbers of very strong, strong, regular, very typical, typical, weak and very weak vertices of G respectively. We call $\Gamma(G)$ the *vertex type* of G. Clearly, two isomorphic graphs must have the same vertex type. Thus the concept of vertex type provides a new necessary condition for isomorphism when degree sequences cannot distinguish graphs. For example, the two graphs in Fig. 1 have the same degree sequence 4, 4, 4, 3, 3, 2. Since the graph (a) has one very weak vertex while the graph (b) has three very weak vertices, the two graphs have different vertex types and hence they are not isomorphic.

The following two questions are asked in [1].

(1) What is the smallest order n of a graph having $n - 2$ very typical vertices or $n - 2$ typical vertices?
(2) What is the smallest order of a pantypical graph?

The purpose of this paper is to answer these two questions and determine all the possible orders of the graphs in these three classes. Concerning question (1), since by definition a vertex of maximum degree or of minimum degree is neither very typical nor typical, a graph of order n can have at most $n - 2$ very typical vertices and at most $n - 2$ typical vertices. For every order $n \geq 17$ of the form $n = k^2 + 1$ with k an integer, a graph of order n with $n - 2$ very typical vertices is constructed in [1, proof of Theorem 4], and for every order $n \geq 20$ of the form $n = k(k+1)$ with k an integer, a graph of order n with $n - 2$ typical vertices is constructed in [1, proof of Theorem 5]. Concerning question (2), a pantypical graph of order 9 and size 21 is given in [1].

Fig. 2 Graphs with n-4 VT vertices



2 Main Results

The main results are as follows. All the graphs are simple.

Theorem 1 *Let $f(n)$ and $g(n)$ be the maximum number of very typical vertices and the maximum number of typical vertices in a graph of order n respectively. Then*

$$f(n) = \begin{cases} 0 & \text{if} \quad n \leq 4; \\ n-4 & \text{if} \quad 5 \leq n \leq 6; \\ n-3 & \text{if} \quad 7 \leq n \leq 9; \\ n-2 & \text{if} \quad n \geq 10 \end{cases}$$

and

$$g(n) = \begin{cases} 0 & \text{if} \quad n \leq 4; \\ n-3 & \text{if} \quad 5 \leq n \leq 8; \\ n-2 & \text{if} \quad n \geq 9. \end{cases}$$

Corollary 2 *The smallest order n of a graph having $n-2$ very typical vertices is* 10 *and the smallest order n of a graph having $n-2$ typical vertices is* 9.

Theorem 3 *There exists a pantypical graph of order n if and only if $n \geq 9$.*

In the following proofs we abbreviate very strong, strong, regular, very typical, typical, weak and very weak as VS, S, R, VT, T, W and VW respectively. For two vertices u and v, we use the symbol $u \leftrightarrow v$ to mean that u and v are adjacent and use $u \not\leftrightarrow v$ to mean that u and v are non-adjacent. The symbol $\Rightarrow$ means "implies", and ϕ denotes the empty set. For two subsets of vertices P and Q in a graph G, the symbol $[P, Q]$ denotes the set of those edges with one end vertex in P and the other end vertex in Q, and $G[P]$ denotes the subgraph of G induced by P. For a vertex v, $N[v]$ denotes the closed neighborhood of v; i.e., $N[v] = \{v\} \cup N(v)$. Finally $V(G)$ denotes the vertex set of a graph G.

Proof of Theorem 1. We first consider $f(n)$. The case $n \leq 4$ is trivial. Now suppose $n \geq 5$. In Fig. 2 we give graphs of orders $n = 5, 6$ with $n-4$ very typical vertices.

In Fig. 3 we give graphs of orders $n = 7, 8, 9$ with $n-3$ very typical vertices.

In Fig. 4 we give graphs of orders $n = 10, 11$ with $n-2$ very typical vertices.

If $n = 2t \geq 12$, let A be the complete 4-partite graph with partite sets of sizes $1, 2, t-3$ and $t-1$ respectively. Let B be the graph obtained from A by adding one additional vertex that is adjacent to every vertex in the partite set of size $t-1$. Then B is a graph of order n with $n-2$ very typical vertices. If $n = 2t+1 \geq 13$, let C

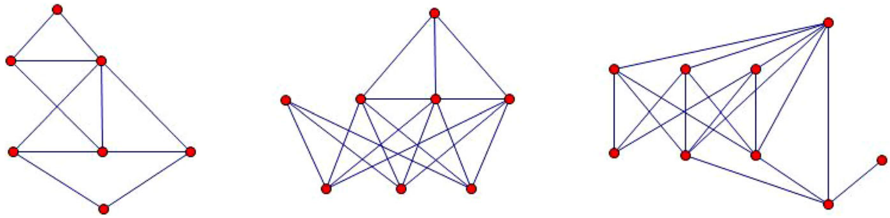


Fig. 3 Graphs with n-3 VT vertices

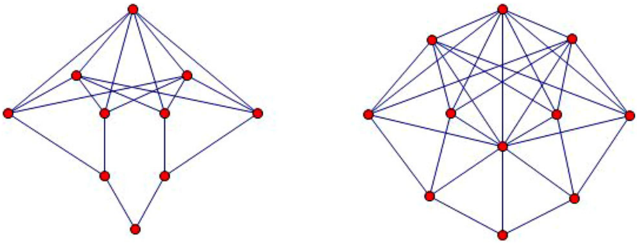


Fig. 4 Graphs with n-2 VT vertices

be the complete 4-partite graph with partite sets of sizes $1, 2, t-3$ and t respectively. Let D be the graph obtained from C by adding one additional vertex that is adjacent to every vertex in the partite set of size t. Then D is a graph of order n with $n-2$ very typical vertices.

Since a vertex of maximum or minimum degree cannot be a very typical vertex, a graph of order n has at most $n-2$ very typical vertices. Thus, the above constructions show that for $5 \leq n \leq 6$, $f(n) \geq n-4$, for $7 \leq n \leq 9$, $f(n) \geq n-3$ and for $n \geq 10$, $f(n) = n-2$. Next we show that a graph of order 9 cannot have $n-2=7$ very typical vertices. The proofs that a graph of orders $n=7,8$ cannot have $n-2$ very typical vertices and that a graph of orders $n=5,6$ cannot have $n-3$ very typical vertices are similar to the case $n=9$, but are easier, so we omit them and assume that the results for these lower orders have been proved.

To the contrary, suppose that there is a graph G of order 9 with 7 VT vertices. Let $V(G) = \{v_1, v_2, \ldots, v_9\}$ with $d(v_1) > d(v_2) \geq \cdots \geq d(v_8) > d(v_9)$. Then $v_2, \ldots, v_8$ are VT. Clearly $d(v_9) = 0$ is impossible. If $d(v_9) = 1$, then $G - v_9$ is a graph of order 8 with 6 VT vertices, a contradiction. From now on, we assume that $d(v_9) \geq 2$. Denote by S_i the set of vertices in $V(G) \setminus \{v_1, v_9\}$ with degree i for $i = 3, 4, 5, 6, 7$. Then each S_i is an independent set. We have the rough estimate that

$$|S_7| \leq 2, \quad |S_6| \leq 3, \quad |S_5| \leq 4, \quad |S_4| \leq 5, \quad |S_3| \leq 2.$$

Note that the vertices in S_3 have v_9 as the only lower degree neighbor. If $|S_3| \geq 3$, then $d(v_9) \geq 3$, implying that $S_3 = \phi$, a contradiction. Thus we have $|S_3| \leq 2$.

Now we show that $S_7 = \phi$. If $|S_7| = 2$, then $d(v_1) = 8$, $d(v_9) \geq 3$ and $S_7 = \{v_2, v_3\}$. But then $G - \{v_1, v_2\}$ is a graph of order 7 with 5 VT vertices, a contradiction. If $|S_7| = 1$, then $d(v_1) = 8$ and $S_7 = \{v_2\}$. Let w be the vertex such that $w \not\leftrightarrow v_2$. If w has a higher degree neighbor other than v_1, then $G - v_1$ is a graph of order 8 with 6 VT vertices, a contradiction. If v_1 is the only higher degree neighbor of w, then in $G - v_1$ we add the edge wv_2 to obtain a graph of order 8 with 6 VT vertices, a contradiction.

If $|S_6| = 3$, then $d(v_9) \geq 4 \Rightarrow S_3 = S_4 = \phi \Rightarrow |S_5| = 4 \Rightarrow d(v_9) \geq 7$, a contradiction. Thus $|S_6| \leq 2$. If $|S_5| = 4$, then $d(v_9) \geq 4 \Rightarrow S_3 = S_4 = \phi \Rightarrow |S_6| = 3$, impossible. If $|S_4| = 5$, then $d(v_9) \geq 5 \Rightarrow S_3 = S_4 = S_5 = \phi$, a contradiction. Thus we have the following sharper estimate that

$$S_7 = \phi, \ \ |S_6| \leq 2, \ \ |S_5| \leq 3, \ \ |S_4| \leq 4, \ \ |S_3| \leq 2.$$

Now we show that $S_6 = \phi$. First suppose $|S_6| = 2$. Then $S_6 = \{v_2, v_3\}$. We distinguish three cases.

Case 1. $v_2 \nleftrightarrow v_9$ and $v_3 \nleftrightarrow v_9$. In this case v_2 and v_3 are adjacent to each vertex in $S_5 \cup S_4 \cup S_3$ and $S_6 \neq \phi \Rightarrow d(v_1) \geq 7$. But then either at least one vertex in $S_5 \cup S_4$ has no lower degree neighbor or $d(v_1) \leq 5$, a contradiction.

Case 2. $v_2 \leftrightarrow v_9$ and $v_3 \leftrightarrow v_9$. We have $S_3 = \phi \Rightarrow |S_4 \cup S_5| = 5$. But $|S_5| \leq 3 \Rightarrow |S_4| \geq 2 \Rightarrow d(v_9) \geq 4 \Rightarrow S_4 = \phi$, a contradiction.

Case 3. One of v_2 and v_3, say $v_2 \leftrightarrow v_9$ and $v_3 \nleftrightarrow v_9$. We have $|S_3| \leq 1$. Note that v_3 is adjacent to each vertex in $S_3 \cup S_4 \cup S_5$. If $|S_3| = 1$, then $|S_4| = 1$. To see this, note that $|S_5| \leq 3 \Rightarrow |S_4| \geq 1$, but $|S_4| > 1$ would imply that at least one vertex in S_4 has no lower degree neighbor. Consequently $|S_5| = 3$, $S_3 = \{v_8\}$ and $S_4 = \{v_7\}$. Now since the three vertices in S_5 can have v_7 as the only lower degree neighbor, we have $d(v_7) \geq 5$, a contradiction. If $|S_3| = 0$, then $|S_5| \leq 3 \Rightarrow |S_4| \geq 2 \Rightarrow d(v_9) \leq 3 \Rightarrow |S_4| \leq 2$. Hence $|S_4| = 2$ and $d(v_9) = 3 \Rightarrow S_4 = \{v_8, v_7\}$. Now the three vertices in S_5 can have either v_8 or v_7 as a lower degree neighbor and v_2 is adjacent to at least one of v_8 and v_7. Hence $N(v_1) \subseteq S_6 \cup S_5 \Rightarrow d(v_1) \leq 5$, a contradiction.

Next suppose $|S_6| = 1$. Then $S_6 = \{v_2\}$. We distinguish two cases.

Case 1 $v_2 \leftrightarrow v_9$. Then we have $|S_3| \leq 1$.

Subcase 1 $|S_3| = 1$. This implies that $S_3 = \{v_8\}$ and $|S_4| \leq 2$. But $|S_5| \leq 3$ and $|S_5| + |S_4| = 5$. Hence $|S_4| = 2$. We deduce that $v_9, v_8, v_1 \notin N(v_1) \Rightarrow d(v_1) \leq 6$, contradicting $d(v_1) \geq 7$.

Subcase 2 $|S_3| = 0$. Now $|S_4| + |S_5| = 6$. $S_3 = \phi$ and $S_4 \neq \phi \Rightarrow d(v_9) \leq 3 \Rightarrow |S_4| \leq 2$. Hence $|S_5| \geq 4$, contradicting $|S_5| \leq 3$.

Case 2 $v_2 \nleftrightarrow v_9$. Since $|S_3| \leq 2$, we distinguish the following three subcases.

Subcase 1 $|S_3| = 0$. We have $|S_4| + |S_5| = 6$ and $|S_5| \leq 3 \Rightarrow |S_4| \geq 3 \Rightarrow d(v_9) \leq 3 \Rightarrow |S_4| = 3 \Rightarrow |S_5| = 3$ and $d(v_9) = 3$. Let $S_5 = \{a_1, a_2, a_3\}$ and $S_4 = \{b_1, b_2, b_3\}$. Note that $N(v_9) = S_4$. Since each $a_i \nleftrightarrow v_9$, S_5 is independent and $d(a_i) = 5$, we deduce that each $a_i \leftrightarrow$ each $b_j \Rightarrow N(b_j) = \{v_9, a_1, a_2, a_3\}$ for every $j = 1, 2, 3 \Rightarrow N(v_2) \subseteq \{v_1\} \cup S_5 \Rightarrow d(v_2) \leq 4$, contradicting $d(v_2) = 6$.

Subcase 2 $|S_3| = 1$. $S_3 \neq \phi \Rightarrow d(v_9) \leq 2$. We have $|S_4| \leq 3$, since otherwise at least one vertex in S_4 would have no lower degree neighbor. $|S_4| + |S_5| = 5$ and $|S_5| \leq 3 \Rightarrow |S_4| \geq 2$. Thus there are two possibilities: $|S_4| = 2$ or 3.

Let $P = \{v_1\} \cup S_6 \cup S_5$ and $Q = S_4 \cup S_3 \cup \{v_9\}$. Note that $d(v_1) \geq 7$ and $d(v_9) = 2$. If $|S_4| = 2$, then $|S_5| = 3$ and

$$\sum_{v \in P} d(v) - \sum_{v \in Q} d(v) \geq 7 + 6 + 3 \times 5 - 2 \times 4 - 3 - 2 = 15. \tag{1}$$

Since every edge in $[P, Q]$ contributes the same degree 1 to both $\sum_{v \in P} d(v)$ and $\sum_{v \in Q} d(v)$, to calculate their difference it suffices to consider the degrees from those edges inside $G[P]$ or $G[Q]$. There are at least three edges inside $G[Q]$. Thus

$$\sum_{v \in P} d(v) - \sum_{v \in Q} d(v) \leq 5 \times 4 - 3 \times 2 = 14,$$

contradicting (1).

If $|S_4| = 3$, then $|S_5| = 2$ and

$$\sum_{v \in P} d(v) - \sum_{v \in Q} d(v) \geq 7 + 6 + 2 \times 5 - 3 \times 4 - 3 - 2 = 6. \tag{2}$$

On the other hand, since there are at least four edges inside $G[Q]$ we have

$$\sum_{v \in P} d(v) - \sum_{v \in Q} d(v) \leq 4 \times 3 - 4 \times 2 = 4,$$

contradicting (2).

Subcase 3. $|S_3| = 2$. $S_3 \neq \phi \Rightarrow d(v_9) = 2$. $|S_4| + |S_5| = 4$ and $|S_5| \leq 3 \Rightarrow |S_4| \geq 1$. Since $d(v_1) \geq 7$ and $v_1 \nleftrightarrow v_9$, each vertex in S_3 is adjacent to v_1. Consequently $|S_4| \leq 2$. Using the same method as in the above subcase 2 to the two possible cases $|S_4| = 1$ and $|S_4| = 2$ we obtain contradictions too.

So far we have proved that $S_7 = S_6 = \phi$. Next according to $|S_3| \leq 2$ we distinguish three cases.

Case 1 $|S_3| = 2$. We have $S_3 = \{v_8, v_7\}$, $d(v_9) = 2$, $|S_4| + |S_5| = 5$ and $|S_5| \leq 3 \Rightarrow |S_4| \geq 2$. $|S_4| \leq 4 \Rightarrow S_5 \neq \phi \Rightarrow d(v_1) \geq 6 \Rightarrow v_1$ is adjacent to at least one of v_8 and $v_7 \Rightarrow 2 \leq |S_4| \leq 3$. If $|S_4| = 2$, then $|S_5| = 3$ and at least one vertex in S_5 has degree ≤ 3, a contradiction. If $|S_4| = 3$, then $|S_5| = 2$ and the two vertices in S_5 have degree ≤ 4, a contradiction.

Case 2 $|S_3| = 1$. We have $S_3 = \{v_8\}$. The vertices in S_4 can only have v_8 or v_9 as their lower degree neighbors. Since $d(v_9) = 2$, $|S_4| \leq 3$. But $|S_4| + |S_5| = 6$ and $|S_5| \leq 3$. Hence $|S_4| = |S_5| = 3$. Then the vertices in S_5 have degree ≤ 4, a contradiction.

Case 3 $|S_3| = 0$. Now $|S_4| + |S_5| = 7$, $|S_4| \leq 4$, $|S_5| \leq 3 \Rightarrow |S_4| = 4$. Since the vertices in S_4 can only have v_9 as their lower degree neighbor, they must be adjacent to v_9. Hence $d(v_9) \geq 4$. But then the vertices in S_4 have no lower degree neighbor, a contradiction. This completes the proof of the result on $f(n)$.

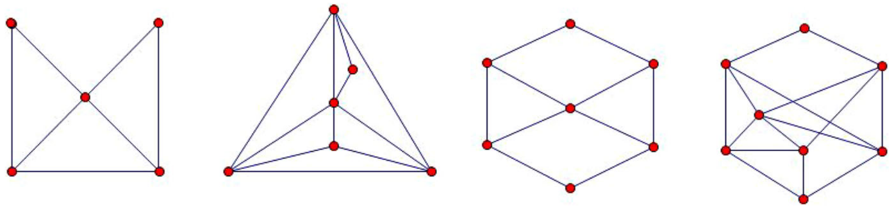


Fig. 5 Graphs with n-3 T vertices

Now we consider $g(n)$. The case $n \leq 4$ is trivial and we assume $n \geq 5$. In Fig. 5 we give graphs of orders $n = 5, 6, 7, 8$ with $n - 3$ T vertices.

For each $n \geq 9$, we construct a graph of order n with $n - 2$ T vertices. Recall that the join of graphs $G_1, G_2, \ldots, G_k$, denoted $G_1 \vee G_2 \vee \cdots \vee G_k$, is the graph obtained from the disjoint union $G_1 + G_2 + \cdots + G_k$ by adding the edges xy with $x \in V(G_i)$ and $y \in V(G_j)$ for all $i \neq j$.

In the following constructions we let K_1 be the graph consisting of one vertex, let C_t be the cycle of order t, let M_{2h} be a graph of order $2h$ and size h whose edges form a perfect matching, and let T_s be a cubic graph of order s.

If $n = 4k + 1 \geq 9$, to the graph $K_1 \vee C_{2k-1} \vee M_{2k}$ add one additional vertex that is adjacent to each vertex in $V(K_1) \cup V(M_{2k})$. Then we obtain a graph of order n with $n - 2$ T vertices.

If $n = 4k + 2 \geq 10$, to the graph $K_1 \vee T_{2k} \vee M_{2k}$ add one additional vertex that is adjacent to each vertex in $V(K_1) \cup V(M_{2k})$. Then we obtain a graph of order n with $n - 2$ T vertices.

If $n = 4k + 3 \geq 11$, to the graph $K_1 \vee T_{2k} \vee C_{2k+1}$ add one additional vertex that is adjacent to each vertex in $V(K_1) \cup V(C_{2k+1})$. Then we obtain a graph of order n with $n - 2$ T vertices.

If $n = 4k \geq 12$, to the graph $K_1 \vee M_{2k-2} \vee M_{2k}$ add one additional vertex that is adjacent to each vertex in $V(M_{2k})$. Then we obtain a graph of order n with $n - 2$ T vertices.

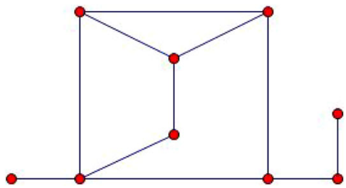
Since a vertex of maximum or minimum degree cannot be a T vertex, a graph of order n can have at most $n - 2$ T vertices. Thus the above constructions show that for $5 \leq n \leq 8$, $g(n) \geq n - 3$ and for $n \geq 9$, $g(n) = n - 2$. It remains to prove that for $5 \leq n \leq 8$, $g(n) \leq n - 3$. The proofs for the cases $5 \leq n \leq 7$ are similar to that for the case $n = 8$ but easier, so we omit them.

Let G be a graph of order 8 with $V(G) = \{v_1, v_2, \ldots, v_8\}$ and $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_8)$. We will show that G has at most 5 T vertices. To the contrary, suppose G has 6 T vertices. Then $v_2, \ldots, v_7$ are T vertices and

$$d(v_1) > d(v_2) = d(v_3) \geq d(v_4) \geq d(v_5) \geq d(v_6) = d(v_7) > d(v_8).$$

Renaming the vertices v_3, v_4, v_5 and v_6 if necessary, we may assume that $v_2 \leftrightarrow v_3$ and $v_6 \leftrightarrow v_7$. Then $v_1v_2v_3$ and $v_6v_7v_8$ are triangles. Denote $a = d(v_2) = d(v_3)$ and $b = d(v_6) = d(v_7)$. Clearly $a > b$. We have $b \geq 3 \Rightarrow a \geq 4 \Rightarrow d(v_1) \geq 5$. Also, $d(v_1) \leq 7 \Rightarrow a \leq 6 \Rightarrow b \leq 5 \Rightarrow d(v_8) \leq 4$. We assert that $d(v_4) > b$. Otherwise

Fig. 6 A pantypical graph of order 9 and size 11



$d(v_4) = d(v_5) = b \Rightarrow v_4, v_5, v_6, v_7 \in N(v_8) \Rightarrow d(v_8) \geq 4 \Rightarrow b \geq 5 \Rightarrow a \geq 6 \Rightarrow d(v_1) = 7 \Rightarrow v_1 \leftrightarrow v_8 \Rightarrow d(v_8) \geq 5$, contradicting $d(v_8) \leq 4$. We further assert that $d(v_5) < a$. Otherwise $d(v_4) = d(v_5) = a$. Let $P = \{v_1, v_2, v_3, v_4, v_5\}$ and $Q = \{v_6, v_7, v_8\}$. $d(v_1) \geq 5 \Rightarrow \|[v_1, Q]\| \geq 1$. Every vertex in $\{v_2, v_3, v_4, v_5\}$ has at least one lower degree neighbor in Q. Hence $\|[P, Q]\| \geq 1 + 4 = 5$, which, together with the fact that $v_6v_7v_8$ is a cycle, implies that $\sum_{v \in Q} d(v) \geq 5 + 3 \times 2 = 11$. Since $b > d(v_8)$, $b \geq 4 \Rightarrow a \geq 5 \Rightarrow d(v_1) \geq 6 \Rightarrow \|[P, Q]\| \geq 2 + 4 = 6 \Rightarrow \sum_{v \in Q} d(v) \geq 6 + 3 \times 2 = 12 \Rightarrow b \geq 5 \Rightarrow a \geq 6 \Rightarrow a = 6$ and $d(v_1) = 7 \Rightarrow \|[v_1, Q]\| \geq 3$ and $\|[v_i, Q]\| \geq 2$ for $i = 2, 3, 4, 5 \Rightarrow \|[P, Q]\| \geq 3 + 4 \times 2 = 11 \Rightarrow \sum_{v \in Q} d(v) \geq 11 + 3 \times 2 = 17 \Rightarrow b \geq 6$, contradicting $b \leq 5$.

Next we distinguish two cases.

Case 1 $d(v_5) > b$. Then $a > d(v_4) = d(v_5) > b$. Let $P = \{v_1, v_2, v_3\}$ and $Q = \{v_6, v_7, v_8\}$. Since $d(v_1) \geq d(v_8) + 4$, $d(v_2) \geq d(v_6) + 2$ and $d(v_3) \geq d(v_7) + 2$, we have $\sum_{v \in P} d(v) - \sum_{v \in Q} d(v) \geq 8$. But this is impossible. Since $v_1v_2v_3$ and $v_6v_7v_8$ are both triangles, more degrees in $\sum_{v \in P} d(v)$ than in $\sum_{v \in Q} d(v)$ can only come from edges incident to v_4 and v_5 and hence $\sum_{v \in P} d(v) - \sum_{v \in Q} d(v) \leq 3 \times 2 = 6$.

Case 2 $d(v_5) = b$. Then $d(v_4) = a$ since $d(v_4) > b$, $\{v_2, v_3, v_4\} \subseteq N(v_1)$ and $\{v_5, v_6, v_7\} \subseteq N(v_8)$. Let $P = \{v_1, v_2, v_3, v_4\}$ and $Q = \{v_5, v_6, v_7, v_8\}$. Since $d(v_1) \geq d(v_8) + 3$, $d(v_2) \geq d(v_5) + 1$, $d(v_3) \geq d(v_6) + 1$ and $d(v_4) \geq d(v_7) + 1$, we have $\sum_{v \in P} d(v) - \sum_{v \in Q} d(v) \geq 6$. On the other hand, since each of the induced subgraphs $G[P]$ and $G[Q]$ has size either 5 or 6, $\sum_{v \in P} d(v) - \sum_{v \in Q} d(v) \leq 2$, a contradiction. This completes the proof. □

Corollary 2 follows from Theorem 1 immediately. We will repeatedly use the following lemma which follows from the definition.

Lemma 4 *Let s', s, r, t', w and w' be a VS, S, R, VT, W and VW vertex respectively in a graph. Then*

$$s' \nleftrightarrow s, \quad w' \nleftrightarrow w, \quad r \nleftrightarrow t', \quad r \nleftrightarrow s', \quad r \nleftrightarrow w'.$$

Proof of Theorem 3. In Fig. 6 we give a pantypical graph of order 9 and size 11.

For $n > 9$, we attach a path of order $n - 8$ to the vertex of maximum degree in Fig. 6 to obtain a pantypical graph of order n.

Conversely we need to prove that there is no pantypical graph of order ≤ 8. Obviously a pantypical graph must have order at least 7. Since the proof for the case $n = 7$ is similar but easier, we present only the proof for the case $n = 8$.

To the contrary, suppose that there is a pantypical graph G of order 8. Since a VT, T, W or VW vertex has a higher degree neighbor, it cannot have the maximum degree. Hence G has at most 4 vertices of the maximum degree. Let Δ be the maximum degree of G. Then $4 \leq \Delta \leq 7$. We distinguish four cases according to the values of Δ.

$$\Delta = 7$$

By considering a vertex of degree 7 and a regular vertex we conclude that $G = K_8$ which is not pantypical, a contradiction.

$$\Delta = 6$$

Let v be a vertex of G with degree 6. Then v is VS or S. If v is VS, then v has a neighbor which is S or R, contradicting Lemma 4. If v is S, then the only vertex $\notin N[v]$ must be VS and consequently a regular vertex is adjacent to v. It follows that G has at least 7 vertices of the maximum degree 6, which is impossible.

$$\Delta = 5$$

Let v be a vertex of G with degree 5, and let r be a regular vertex. Then $r \notin N(v)$, since otherwise G would have at least 6 vertices with the maximum degree 5, which is impossible. Note that v must be either S or VS.

Case 1 v is S. Let s' be a VS vertex. Then $s' \nleftrightarrow v$. By definition, v has a neighbor x with $d(x) = 5$. Then x is S. Let $N(v) = \{x, y, z, t, t'\}$ where t is T and t' is VT. Clearly $N(x) = \{v, y, z, t, t'\}$. Since $d(t) \geq 3$ and t already has two neighbors v and x of the maximum degree 5, $d(t) = 4$. It is easy to see that $t \nleftrightarrow r$ and $t \nleftrightarrow s'$. Now $v, t, x, r \notin N(s')$ and hence y, z, t' are the only possible neighbors of $s' \Rightarrow d(s') \leq 3$. Since $v, x \in N(y) \cap N(z) \cap N(t')$, s' cannot have any lower degree neighbors, a contradiction.

Case 2 v is VS. In this case, among the two non-neighbors of v, one is R, denoted r and the other is S, denoted s. Let $N(v) = \{t, t', a, b, c\}$ where t is T and t' is VT. We have $3 \leq d(t) \leq 4$. Note that $d(t') \geq 3$, since any lower degree neighbor of t' has degree ≥ 2.

Subcase 1 $d(t) = 3$. Since $d(t') \geq 3$ and t already has a higher degree neighbor v, $t \nleftrightarrow t'$. The lower degree neighbor of t is in $\{a, b, c\}$, say $t \leftrightarrow a$ and $d(a) = 2$. Clearly t cannot be adjacent to both r and s. Thus we have three cases. (1) $t \leftrightarrow r$. If $r \leftrightarrow s$, then $d(r) = d(s) = d(t) = 3$. Since $d(t') \geq 3$, s has at least one lower degree neighbor in $\{b, c\}$, say $s \leftrightarrow b$ and $d(b) = 2$. Then $d(r) = 3 \Rightarrow r \leftrightarrow c$. But now t' has no lower degree neighbor, a contradiction. If $r \nleftrightarrow s$, then $d(r) = 3 \Rightarrow r \leftrightarrow b$ and $r \leftrightarrow c$. Now t' can only have a lower degree neighbor in $\{b, c\}$ which has degree $3 \Rightarrow d(t') = 4 \Rightarrow t' \leftrightarrow b$, $t' \leftrightarrow c$ and $t' \leftrightarrow s \Rightarrow d(s) = 1$, a contradiction. (2) $t \leftrightarrow s$. Now $d(s) = 3 \Rightarrow s$ has at least one lower degree neighbor in $\{b, c\}$, say $s \leftrightarrow b$ and $d(b) = 2$. Consequently $N(t') \subseteq \{v, c, s\} \Rightarrow d(t') \leq 3 \Rightarrow d(t') = 3$. But $d(s) = 3 \Rightarrow t' \nleftrightarrow s$. Then $d(t') \leq 2$, a contradiction. (3) $t \nleftrightarrow r$ and $t \nleftrightarrow s$. In this case, the neighbor of t with degree 3 can only be b or c, say $t \leftrightarrow b$ and $d(b) = 3$. We have $N(t') \subseteq \{v, b, c, s\} \Rightarrow d(t') = 3$ or $4 \Rightarrow t' \leftrightarrow s \Rightarrow d(s) \geq 4$. Since $N(s) \subseteq \{r, b, c, t'\}$ we have $d(s) = 4$ and $N(s) = \{r, b, c, t'\} \Rightarrow d(t') = 3 \Rightarrow t' \leftrightarrow c \Rightarrow d(c) \geq 3$, but now t' has no lower degree neighbor, a contradiction.

Subcase 2 $d(t) = 4$. Suppose $t \leftrightarrow r$. Then $d(r) = 4$. But r is not adjacent to any VW or VT vertex by Lemma 4. Without loss of generality, suppose c is VW. Then $N(r) = \{t, a, b, s\}$, and r, t, a, b, s all have degree 4. Now t' can only have c as its lower degree neighbor $\Rightarrow c \leftrightarrow t'$ and $d(c) \geq 2$. But then t and s cannot both have a lower degree neighbor, a contradiction.

Now suppose $t \not\leftrightarrow r$. (1) Suppose $t \leftrightarrow t'$. Then $d(t') = 3$ and t' has a lower degree neighbor in $\{a, b, c\}$, say c, of degree 2. The vertex t has a neighbor w in $\{a, b, s\}$ of degree 4, which has possible neighbors in $\{t, a, b, r, s\}$ except itself. Hence $w \leftrightarrow r$ and $d(r) = 4$. But r has only the three possible neighbors a, b, s, a contradiction. (2) Suppose $t \not\leftrightarrow t'$. If $t \not\leftrightarrow s$, then $N(t) = \{v, a, b, c\} \Rightarrow$ At least one of $\{a, b, c\}$ has degree 4 and any lower degree neighbor of t' has degree ≥ 3. Hence $d(t') = 4 \Rightarrow t' \leftrightarrow s \Rightarrow d(s) = 5 \Rightarrow N(s) = \{r, a, b, c, t'\} \Rightarrow d(r) = 5$, contradicting the fact that $N(r) \subseteq \{s, a, b, c\}$ now. If $t \leftrightarrow s$, then $d(s) \geq 4$. Clearly $r \not\leftrightarrow s$, since otherwise $a, b, c \in N(r)$ and they all have degree ≥ 4, implying that t has no lower degree neighbor. If $s \not\leftrightarrow t'$, then $N(s) = \{t, a, b, c\}$, implying that t' has no lower degree neighbor. Hence $s \leftrightarrow t' \Rightarrow d(t') \neq d(s) \Rightarrow d(t') \leq 3$, since $d(s) \geq 4$ and $d(t') \leq \Delta - 1 = 4 \Rightarrow d(t') = 3$, since $d(t') \geq 3$. Consequently t' has a lower degree neighbor in $\{a, b, c\}$, say c, of degree $2 \Rightarrow s \not\leftrightarrow c$, $t \leftrightarrow a$ and $t \leftrightarrow b \Rightarrow N(s) = \{t, t', a, b\} \Rightarrow r \not\leftrightarrow a$ and $r \not\leftrightarrow b \Rightarrow d(a) = d(b) = 3$. Finally G has three VW vertices a, b, c, a contradiction.

$$\Delta = 4$$

Let v be a vertex of G with degree 4. Then v is S or VS.

Case 1 v is S. v has a neighbor x with $d(x) = 4$. It is easy to see that an R or VS vertex is not in $N(v) \cup N(x)$. Since G has two S vertices v and x, G has exactly one vertex of each of the remaining six types. Let t, r, s' be the T, R, VS vertex of G respectively.

Subcase 1 $t \in N(v)$. Then $d(t) = 3$, $t \not\leftrightarrow x$ and $t \not\leftrightarrow s'$. Let $N(v) = \{x, y, z, t\}$ and $V(G) \setminus N[v] = \{a, r, s'\}$. Note that $N(x) = \{v, a, y, z\}$. Now a is the lower degree neighbor of $t \Rightarrow t \leftrightarrow a$ and $d(a) = 2$. If $t \leftrightarrow r$, then $d(r) = 3 \Rightarrow r \leftrightarrow y$ and $r \leftrightarrow z \Rightarrow d(y) = d(z) = 3$. But now s' has no neighbors, a contradiction. Thus, $t \not\leftrightarrow r$.

As a T vertex, t must have a neighbor of the same degree and a neighbor of lower degree. We deduce that $t \leftrightarrow a$, $d(a) = 2$ and that t is adjacent to one of y and z, say $z : t \leftrightarrow z$. Hence $d(z) = 3$. But now y is the only possible neighbor of $s' \Rightarrow d(s') \leq 1$, a contradiction.

Subcase 2 $t \notin N(v)$. We have $V(G) \setminus N[v] = \{t, r, s'\}$. Let $N(v) = \{x, y, z, b\}$. Suppose $x \leftrightarrow t$. Then $d(t) = 3$. If $t \leftrightarrow r$, then $d(r) = 3$. Since among the three vertices y, z, b, one is VW and one is VT, neither of which is adjacent to r. Also $r \not\leftrightarrow s'$. Hence $d(r) \leq 2$, a contradiction. This shows $t \not\leftrightarrow r$. Without loss of generality, suppose besides v and t, the other two neighbors of x are y and z. Note that $t \nleftrightarrow s'$ since $d(t) = 3$ and t already has a higher degree neighbor x. Then $t \leftrightarrow b$ and t is adjacent to exactly one of y and z, say z. Now $N(t) = \{x, z, b\} \Rightarrow d(z) = 3$ and $d(b) = 2$. Then s' has y as the only possible neighbor $\Rightarrow d(s') \leq 1$, a contradiction. Hence $x \nleftrightarrow t$. Now $N(x) = \{v, y, z, b\}$. One of y, z and b is VT with degree ≤ 3, but it has no lower degree neighbor now, a contradiction.

Case 2 v is VS. By Lemma 4, an R or S vertex is not adjacent to v. We distinguish two subcases according as whether v has a neighbor which is VT.

Subcase 1 v has a neighbor t' which is VT. Let t be a T vertex. Clearly $d(t) = 3$. First suppose $d(t') = 2$. Then t' has a neighbor w' of degree 1, and $V(G)\backslash N[v] = \{w', s, r\}$ where s is strong and r is regular. Let $N(v) = \{t', a, b, t\}$. Now both t and s can only have lower degree neighbors in $\{a, b\}$. Hence $d(s) \geq 3$. Consequently $s \leftrightarrow t$ and $s \leftrightarrow r$. But r can have s as the only neighbor $\Rightarrow d(r) = 1$, which is a contradiction since r is regular, $r \leftrightarrow s$ and $d(s) \geq 3$.

Next suppose $d(t') = 3$. Since $d(t) = 3$, $t' \nleftrightarrow t$. If $t \nleftrightarrow v$, let $N(v) = \{t', a, b, c\}$. Both t' and t have a lower degree neighbor in $\{a, b, c\}$. Without loss of generality, suppose $t' \leftrightarrow a$ and $t \leftrightarrow b$. We have $d(a) = d(b) = 2$. Now s is the only possible higher degree neighbor of t. Hence $s \leftrightarrow t$ and $d(s) = 4 \Rightarrow s \leftrightarrow t'$, $s \leftrightarrow c$ and $s \leftrightarrow r \Rightarrow d(r) = 4$. On the other hand, $N(r) \subseteq \{s, t, c\} \Rightarrow d(r) \leq 3$, a contradiction.

If $t \leftrightarrow v$, let $N(v) = \{t', t, x, y\}$ and $V(G) \setminus N[v] = \{z, r, s\}$ where r is R and s is S. We first assert that $t' \nleftrightarrow s$. Otherwise $t' \leftrightarrow s \Rightarrow d(s) = 4$, $s \nleftrightarrow t$, since $d(t) = 3$ and t already has a higher degree neighbor v. By definition, s has a neighbor with degree 4, which can only be r or z. If $s \leftrightarrow r$, then $d(r) = 4$. But r has no so many neighbors of the same degree. If $s \leftrightarrow z$ and $d(z) = 4$, then $z \nleftrightarrow t \Rightarrow z \leftrightarrow x$, $z \leftrightarrow y$, $z \leftrightarrow t'$. But then t' has no lower degree neighbor.

If $t' \leftrightarrow z$, then $t' \leftrightarrow$ one of x and y, say $t' \leftrightarrow x \Rightarrow d(x) = 2$. y and z are the only possible lower degree neighbors of t and s. But y or z cannot be the common lower degree neighbor of t and s. Hence $d(y) = d(z) = 2 \Rightarrow d(s) \geq 3 \Rightarrow s \leftrightarrow r$ and $s \leftrightarrow t \Rightarrow d(r) = d(s) \geq 3$. But now r has no further neighbors besides s, a contradiction.

The remaining case is that $t' \nleftrightarrow z$. Then $t' \leftrightarrow x$ and $t' \leftrightarrow y \Rightarrow d(x) = d(y) = 2 \Rightarrow z$ is the only possible lower degree neighbor of both t and $s \Rightarrow z \leftrightarrow t$ and $z \leftrightarrow s \Rightarrow d(z) = 2$ and $d(s) \geq 3 \Rightarrow s \leftrightarrow t$ and $s \leftrightarrow r \Rightarrow d(r) = d(s) \geq 3$. But now r has no further neighbors besides s, a contradiction.

Subcase 2 Each neighbor of v is not VT. Now $V(G) \setminus N[v] = \{t', r, s\}$ where t', r and s are VT, R and S respectively. Let $N(v) = \{a, b, c, t\}$ where t is T. We have $d(t) = 3$. First note that $t \nleftrightarrow t'$. Otherwise $d(t') = 2$. But then t' has no lower degree neighbor. Hence t can have a lower degree neighbor only in $\{a, b, c\}$. Next we distinguish three cases by considering where the third neighbor of t with degree 3 lies.

(1) $t \leftrightarrow s$. Then $d(s) = 3$. We assert that $s \nleftrightarrow t'$. Otherwise $d(t') = 2$. But then t' has no lower degree neighbor. Now $N(t') \subseteq \{a, b, c\} \Rightarrow 2 \leq d(t') \leq 3$. If $d(t') = 2$, then t' has no lower degree neighbor, a contradiction; if $d(t') = 3$, then t' has no higher degree neighbor, a contradiction again.

(2) $t \leftrightarrow r$. Then $d(r) = 3$. Since $r \not\leftrightarrow t'$, r has at least one neighbor in $\{a, b, c\}$, say $r \leftrightarrow a$. Consequently $d(a) = 3$. Then t can have a lower degree neighbor only in $\{b, c\}$, say $t \leftrightarrow b$ and $d(b) = 2$. Now t' can only have c as its lower degree neighbor $\Rightarrow$ $t' \leftrightarrow c$, $d(c) \geq 2$ and $d(t') \geq 3 \Rightarrow t' \leftrightarrow a$ and $t' \leftrightarrow s$. Then $d(t') = 4$, which is impossible since $\Delta = 4$ and t' cannot have a higher degree neighbor.

(3) $N(t) \subseteq N[v]$. Without loss of generality, suppose $N(t) = \{v, b, c\}$, $d(b) = 3$ and $d(c) = 2$. Then $t' \not\leftrightarrow b$, since otherwise $d(t') = 4 \Rightarrow t'$ has no higher degree neighbor or $d(t') = 2 \Rightarrow t'$ has no lower degree neighbor . Now $N(t') \subseteq \{a, s\} \Rightarrow d(t') = 2$. But then t' has no lower degree neighbor, a contradiction. This completes the proof. □

Remark It is not difficult to prove that the smallest size of a pantypical graph of order 9 is 10, and such a graph must have an isolated vertex. The graph in Fig. 6 has the smallest size 11 among all connected pantypical graphs of order 9.

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References

1. Hedetniemi, J.T., Hedetniemi, S.M., Hedetniemi, S.T., Lewis, T.M.: Analyzing graphs by degrees. AKCE Int. J. Graphs Comb. **10**(4), 359–375 (2013)
2. Kamath, S.S., Bhat, R.S.: On strong (weak) independent sets and vertex coverings of a graph. Discrete Math. **307**(9–10), 1136–1145 (2007)
3. Sampathkumar, E., Latha, L.P.: Strong weak domination and domination balance in a graph. Discrete Math. **161**(1–3), 235–242 (1996)