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Note

The maximum girth and minimum circumference of graphs with prescribed radius and diameter

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ABSTRACT

Ostrand posed the following two questions in 1973. (1) What is the maximum girth of a graph with radius r and diameter d? (2) What is the minimum circumference of a graph with radius r and diameter d? Question 2 has been answered by Hrnčiar who proves that if $d \leq 2r-2$ the minimum circumference is $4r-2d$. In this note we first answer Question 1 by proving that the maximum girth is $2r+1$. This improves on the obvious upper bound $2d+1$ and implies that every Moore graph is self-centered. We then prove a property of the blocks of a graph which implies Hrnčiar's result.

1. Introduction

We consider finite simple graphs. Ostrand [6, p.75] posed the following two questions in 1973.

Question 1. What is the maximum girth of a graph with radius r and diameter d?

Question 2. What is the minimum circumference of a graph with radius r and diameter d?

Question 2 has been answered by Hrnčiar [5] who proves that if $d \leq 2r-2$ the minimum circumference is $4r-2d$. In this note we first answer Question 1 by proving that the maximum girth is $2r+1$. This improves on the obvious upper bound $2d+1$ and implies that every Moore graph is self-centered. We then prove a property of the blocks of a graph which implies Hrnčiar's result.

Google shows 63 citations of Ostrand's paper [6] and MathSciNet shows 7 citations. It seems that Question 1 has not been treated.

For terminology and notations we follow the books [1,3,8]. We denote by $V(G)$ the vertex set of a graph G and by $d(u, v)$ the distance between two vertices u and v. The *eccentricity*, denoted by $e(v)$, of a vertex v in a graph G is the distance to a vertex farthest from v. Thus $e(v) = \max\{d(v, u)|u \in V(G)\}$. If $e(v) = d(v, x)$, then the vertex x is called an *eccentric vertex* of v. The *radius* of a graph G, denoted $\mathrm{rad}(G)$, is the minimum eccentricity of all the vertices in $V(G)$, whereas the diameter of G, denoted $\mathrm{diam}(G)$, is the maximum eccentricity. A vertex v is a *central vertex* of G if $e(v) = \mathrm{rad}(G)$. When H is a subgraph of a graph G and $u, v \in V(H)$, $d_H(u, v)$ and $e_H(v)$ will mean the distance and eccentricity in H respectively.

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2. Main results

Recall that a *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. Thus in a connected graph, a block is either a cut-edge or a maximal 2-connected subgraph of order at least 3. If H is a subgraph of a graph G and v is a vertex of G, then the distance between v and H, denoted $d(v, H)$, is defined as $d(v, H) = \min\{d(v, x)|x \in V(H)\}$. Throughout the symbol C_k denotes a cycle of length k, and P_t denotes a path of order t (and hence of length $t - 1$).

Lemma 1. *If B is a block of a graph G, then* $\mathrm{rad}(B) \le \mathrm{rad}(G)$.

Proof. Let c be a central vertex of G. If $c \in V(B)$, then $\mathrm{rad}(B) \le e_B(c) \le e_G(c) = \mathrm{rad}(G)$. Thus $\mathrm{rad}(B) \le \mathrm{rad}(G)$. Now suppose $c \notin V(B)$. In the block-cutvertex tree of G ([1, p.121] or [8, p.156]), from any block containing c there is a unique path to B. Hence there is a unique vertex v of B such that $d(c, B) = d(c, v)$. Note that v is a cut-vertex of G. Let u be an eccentric vertex of v in B; i.e., $e_B(v) = d(v, u)$. Then we have

$$\begin{aligned}\mathrm{rad}(G) = e_G(c) \ge d(c, u) &= d(c, v) + d(v, u)\\ &\ge 1 + e_B(v) \ge 1 + \mathrm{rad}(B),\end{aligned}$$

showing that $\mathrm{rad}(G) > \mathrm{rad}(B)$ in this case. □

We remark that in Lemma 1, if we replace the block B by a generic subgraph, the conclusion may not be true. To see this, consider the wheel graph W_n of order $n \ge 5$ with central vertex c. We have $\mathrm{rad}(W_n - c) = \lfloor (n-1)/2 \rfloor > 1 = \mathrm{rad}(W_n)$.

We make the convention that the girth of an acyclic graph is undefined. Thus whenever we speak of the girth of a graph, we have already implicitly assumed that the graph contains at least one cycle. We will need the well-known fact (e.g. [6]) that if r and d are the radius and diameter of a graph respectively, then $r \le d \le 2r$. The following result answers Question 1.

Theorem 2. *The maximum girth of a graph with radius r and diameter d is* $2r + 1$.

Proof. Let G be a graph with radius r and diameter d containing at least one cycle. Every cycle lies within one block. Let B be a block of G containing a cycle. Then B is 2-connected. Let x be a central vertex of the subgraph B. There is at least one cycle containing x, since B is 2-connected [8, p.162]. Among all the cycles containing x, we choose one of the shortest length and denote it by C. Let $s = \mathrm{rad}(B)$ and denote the length of C by q. We assert that $q \le 2s+1$. To the contrary suppose $q \ge 2s+2$. Since C is a shortest cycle containing x, C contains a vertex y such that

$$d_B(x, y) = d_C(x, y) = \lfloor q/2 \rfloor \ge s + 1,$$

implying that $s = e_B(x) \ge d_B(x, y) \ge s + 1$, a contradiction. By Lemma 1, $s \le r$. We deduce that G has girth at most q and $q \le 2s + 1 \le 2r + 1$.

Conversely, given any positive integers r and d with $r \le d \le 2r$, let $M(r, d)$ be the monocle graph which is the union of the cycle C_{2r+1} and the path P_{d-r+1}, the cycle and the path having only one common vertex which is an end vertex of the path. Then $M(r, d)$ has radius r, diameter d and girth $2r + 1$. This completes the proof. □

Theorem 2 improves on the obvious upper bound $2d+1$. A *Moore graph* was originally defined in [4] as a graph of diameter d, maximum degree Δ and the largest possible order $1 + \Delta\sum_{i=1}^{d}(\Delta - 1)^{i-1}$. An equivalent definition [2,7] of a Moore graph is a graph of diameter d and girth $2d + 1$. A graph G is said to be *self-centered* if $\mathrm{rad}(G) = \mathrm{diam}(G)$. Thus self-centered graphs are those graphs in which every vertex is a central vertex. The following result is deduced in [3, pp.100–101] by using the concept of the distance degree sequence of a vertex. Now it becomes obvious.

Corollary 3. *Every Moore graph is self-centered.*

Proof. Let G be a Moore graph of radius r, diameter d and girth $2d + 1$. By Theorem 2 and the fact $r \le d$ we have $2d + 1 \le 2r + 1 \le 2d + 1$. Hence $r = d$. □

Now we prove a property of the blocks of a graph. In the proof we will use an idea in [5].

Theorem 4. *Every graph of radius r and diameter d has a block whose diameter is at least* $2r - d$.

Proof. We first prove the following

Claim. *Let H be a graph of radius r and diameter d. If B is a block of H with* $\mathrm{diam}(B) < 2r - d$ *and u is a vertex of H such that* $d(u, B) = \max\{d(x, B)|x \in V(H)\}$, *then* $d(u, B) \ge r$.

In the block-cutvertex tree of H ([1, p.121] or [8, p.156]), from the block (necessarily an end block) containing u to B there is a unique path. Thus there is a unique vertex $v \in V(B)$ such that $d(u, B) = d(u, v)$ and v is a cut-vertex of H. Denote $d(u, v) = a$. We need to prove $a \ge r$. To the contrary, suppose $a < r$. We will show that $e(v) < r$, which is a contradiction since r is the radius of H. Let H_1 be the component of $H - v$ containing the vertex u. For $w \in V(H)$, if $w \in V(H_1)$ then $d(v, w) \le d(v, u) = a < r$. If $w \in V(B)$ then $d(v, w) \le \mathrm{diam}(B) < 2r - d \le r$ and hence $d(v, w) < r$. Finally suppose $w \in V(H) \setminus (V(H_1) \cup V(B))$. Let $z \in V(B)$ such that $d(w, z) = d(w, B)$. Then z is a cut-vertex of H and $d_B(z, v) = d_H(z, v)$. Denote $d(z, v) = b$ and $d(w, z) = c$. Note that $c \le a$ and $b \le \mathrm{diam}(B) \le 2r - d - 1$. If $b + c \ge r$, then $c \ge r - b$ and

$$\begin{aligned} d(u, w) = d(u, v) + d(v, z) + d(z, w) = a + b + c \ge 2c + b &\ge 2(r - b) + b \\ &= 2r - b \\ &\ge 2r - (2r - d - 1) \\ &= d + 1, \end{aligned}$$

which is a contradiction since d is the diameter of H. Hence $b + c < r$. Then $d(v, w) = b + c < r$. Thus we have $e(v) < r$, a contradiction. This proves the claim.

Now we prove the theorem. It is well-known (e.g. [6]) that $r \le d \le 2r$. First note that if $d = 2r$ or $d = 2r - 1$ then the theorem holds trivially. To the contrary suppose the theorem is false and let G be a counterexample of the smallest order with radius r and diameter d such that every block of G has diameter $< 2r - d$. Then $d \le 2r - 2$. Choose a central vertex p of G. We assert that G has a block containing p and of order at least 3. Otherwise each block containing p is an edge. Let $f = pq$ be an edge of G and let $u \in V(G)$ such that $d(u, f) = \max\{d(x, f)|x \in V(G)\}$. Here we regard the edge f as a subgraph of G, and the meaning of $d(u, f)$ is defined at the beginning of Section 2. By the Claim we have $d(u, f) \ge r$. Note that f is a cut-edge of G. Let G_1 be the component of $G - f$ containing q. Then $u \notin V(G_1)$, since otherwise

$$r = e(p) \ge d(p, u) = d(p, q) + d(q, u) = 1 + d(u, f) \ge 1 + r,$$

a contradiction. It follows that $d(u, f) = d(u, p)$. Let $h = pt$ be the end edge of a shortest (u, p)-path and let G_2 be the component of $G - h$ containing the vertex t. h is also a cut-edge of G. Let $v \in V(G)$ such that $d(v, h) = \max\{d(x, h)|x \in V(G)\}$. By the Claim, $d(v, h) \ge r$. As argued above, $v \notin V(G_2)$ and $d(v, h) = d(v, p)$. But then

$$d \ge d(u, v) = d(u, p) + d(p, v) \ge r + r = 2r,$$

contradicting the condition that $d \le 2r - 2$.

Let B be a block of G containing p and of order at least 3. Then $\mathrm{diam}(B) < 2r - d$ since G is a counterexample. Let $u \in V(G)$ such that $d(u, B) = \max\{d(x, B)|x \in V(G)\}$. There is a unique vertex u' of B such that $d(u, u') = d(u, B)$ and u' is a cut-vertex of G. By the Claim, $d(u, B) \ge r$. On the other hand, $r = e(p) \ge d(p, u) = d(p, u') + d(u', u) \ge d(p, u') + r$, implying that $u' = p$ and $d(p, u) = r$. Let G_1 be the component of $G - p$ containing u and denote $G_2 = G - V(G_1)$. Let $v \in V(G_2)$ such that $d(p, v) = \max\{d(p, x)|x \in V(G_2)\}$. Then the conditions $d(u, v) \le d \le 2r - 2$ and $d(u, p) = r$ imply that $d(p, v) = d(u, v) - d(u, p) \le 2r - 2 - r = r - 2$. Let Q be a shortest (p, v)-path and let L be the $\{p\}$-lobe [8, p.211] of G containing u; i.e., $L = G[V(G_1) \cup \{p\}]$. Define a new graph $G' = L \cup Q$. We will show that G' is a counterexample of a smaller order.

We first show that the graph G' has the same radius and diameter as G. It is clear that p remains a central vertex of G' with $e_{G'}(p) = d_{G'}(p, u) = r$. Let $x, y \in V(G)$ be a diametral pair of G; i.e., $d(x, y) = d$. It is impossible that both x and y lie in G_2, since otherwise

$$\begin{aligned} d(u, x) = d(u, p) + d(p, x) = r + d(p, x) &\ge r + d(x, y) - d(p, y) \\ &\ge r + d - d(p, v) \\ &\ge r + d - (r - 2) = d + 2, \end{aligned}$$

a contradiction. Hence either both x and y lie in L or one of them, say x, lies in L and the other y lies in G_2. In the former case, $d = d_G(x, y) = d_{G'}(x, y)$ and in the latter case,

$$d = d_G(x, y) = d_G(x, p) + d_G(p, y) \le d_G(x, p) + d_G(p, v) = d_G(x, v) = d_{G'}(x, v).$$

Thus $\mathrm{diam}(G') \ge \mathrm{diam}(G)$. On the other hand, since for any two vertices $s, t \in V(G')$, $d_{G'}(s, t) = d_G(s, t) \le d$, we have $\mathrm{diam}(G') \le \mathrm{diam}(G)$. Hence $\mathrm{diam}(G') = \mathrm{diam}(G)$.

G' has two kinds of blocks: the blocks in L which are also blocks of G and the edges of the path Q. Hence every block of G' has diameter $< 2r - d$. Since the block B has order ≥ 3, it is 2-connected and it contains at least one cycle. Let C be a cycle in B. Since any shortest path between two vertices cannot contain all the vertices of a cycle, at least one vertex of C does not lie in the path Q. It follows that G' has a strictly smaller order than G, contradicting the choice of G. This completes the proof. □

Next we use Theorem 4 to deduce Hrnčiar's result [5] which answers Question 2. Note that if $d = 2r$ or $d = 2r - 1$, there is a tree with radius r and diameter d. A tree has no cycle. Thus, for Question 2 it suffices to consider the case $d \le 2r - 2$.

Theorem 5 (*Hrnčiar*). *The minimum circumference of a graph of radius r and diameter d with $d \le 2r - 2$ is $4r - 2d$.*

Proof. Let G be a graph of radius r and diameter d with $d \leq 2r - 2$. By Theorem 4, G has a block B with $\mathrm{diam}(B) \geq 2r - d$. Let $u, v \in V(B)$ such that $d(u, v) = \mathrm{diam}(B)$. Since $2r - d \geq 2$, B is 2-connected. Hence there exists a cycle C containing u and v. The condition $d(u, v) \geq 2r - d$ implies that the length of C is at least $2(2r - d) = 4r - 2d$, proving that the circumference of G is at least $4r - 2d$. The sun-graph $S_{4r-2d,d-r}$ obtained by attaching a path P_{d-r+1} to each vertex of the cycle C_{4r-2d} has radius r, diameter d and circumference $4r - 2d$. □

The above proof of Theorem 5 shows that large diameter of a block implies the existence of a long cycle. The converse need not be true. For example, the complete graph of order ≥ 3 has a long cycle (Hamilton cycle) but it has a very small diameter ($= 1$). Finally, we remark that the corresponding problems of Question 1 for minimum girth and Question 2 for maximum circumference are trivial. It is easy to see that the minimum girth of a graph with radius r and diameter d is 3, and the circumference can be arbitrarily large.

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