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ORIGINAL PAPER

The Largest Graphs with Given Order and Diameter: A Simple Proof

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Abstract
A classic theorem of Ore determines the maximum size of graphs with given order and diameter. We give a very short and simple proof of this result, based on a well-known observation.

Keywords Diameter · Size · Extremal graphs

We consider simple and finite graphs. For terminologies and notation we follow the book [2]. The *order* of a graph is its number of vertices, and the *size* its number of edges. The *diameter* of a graph G is the greatest distance between two vertices of G. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G respectively. For a subset of vertices $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. We give a very short and simple proof of the following theorem of Ore [1].

Theorem (Ore) *For $d \geq 2$, the maximum size of a simple graph of order n and diameter d is $d + (n - d - 1)(n - d + 4)/2$. This size is attained by a graph G if and only if G consists of a path P of length d such that the vertices outside P form a clique and are each adjacent to the first three or last three among some three or four consecutive vertices on P.*

Proof Let G be a simple graph of order n and diameter d. Since G is of diameter d, there are vertices x and y which are at distance d. Let P be an (x, y)-path of length d and denote $S = V(G) \setminus V(P)$. To avoid bringing x and y closer, every vertex of S has at most three neighbors on P, and if there are three then they are consecutive on P. Also, $G[S]$ can have at most $\binom{n-d-1}{2}$ edges, since $|S| = n-d-1$. Counting also the edges on P, we thus have $|E(G)| \leq d+3(n-d-1)+\binom{n-d-1}{2} = d+(n-d-1)(n-d+4)/2$.

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To achieve equality and thus prove sharpness of the bound, S must be a clique, and each vertex of S must have three consecutive neighbors along P. Since the vertices of S are pairwise adjacent, their neighborhoods on P together can include only at most four consecutive vertices without providing a shorter (x, y)-path. Hence the extremal graphs are formed by choosing three or four consecutive vertices along P and making each vertex of S adjacent to the first three or the last three of them. □

This proof makes it clear why there is the term d and where the factor $n - d - 1$ comes from in the expression of the maximum size. Ore [1] proved the result by first characterizing the maximal n-vertex graphs with diameter d. Zhou, Xu and Liu [3] gave a different proof of the maximum size by considering the complement graph, but they did not treat the extremal graphs.

Ore [1] also considered k-connected graphs. One would like to generalize the above argument to a simple proof of Ore's extremal result for k-connected graphs with diameter d, but this does not work. The problem is that in applying Menger's Theorem [2, p. 167] to obtain k internally disjoint paths joining two vertices at distance d, some of the paths may have length greater than d. The simplest example is an odd cycle. Ore was able to solve the more general problem by characterizing all the diameter-critical graphs.

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