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The minimum number of Hamilton cycles in a Hamiltonian threshold graph of a prescribed order

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Abstract
We prove that the minimum number of Hamilton cycles in a Hamiltonian threshold graph of order n is $2^{\lfloor (n-3)/2 \rfloor}$ and this minimum number is attained uniquely by the graph with degree sequence $n-1, n-1, n-2, \ldots, \lceil n/2 \rceil, \lceil n/2 \rceil, \ldots, 3, 2$ of $n-2$ distinct degrees. This graph is also the unique graph of minimum size among all Hamiltonian threshold graphs of order n.

KEYWORDS
Hamiltonian graph, minimum size, number of Hamilton cycles, threshold graph

1 | INTRODUCTION

There are few results concerning the precise value of the minimum or maximum number of Hamilton cycles of graphs in a special class with a prescribed order. For example, it is known that the minimum number of Hamilton cycles in a simple Hamiltonian cubic graph of order n is 3, which follows from Smith's theorem [1, p. 493] and an easy construction [13, p. 479], but the maximum number of Hamilton cycles is not known; even the conjectured upper bound $2^{n/3}$ [3, p. 312] has not been proved. Another example is Sheehan's conjecture that every simple Hamiltonian 4-regular graph has at least two Hamilton cycles [13] (see also [1, pp. 494 and 590]), which is still unsolved.

Upper and lower bounds on the maximum number of Hamilton cycles in a graph with prescribed order and size are given in [14]. The number of Hamilton cycles in maximal planar graphs is studied in [9].

In this paper we will determine the minimum number of Hamilton cycles in a Hamiltonian threshold graph of order n, as well as determining the unique minimizing graph attaining this minimum. Threshold graphs were introduced by Chvátal and Hammer [4] in 1973. Besides the original definition, seven equivalent characterizations are given in [10].

Definition 1. A finite simple graph G is called a *threshold graph* if there exists a nonnegative real-valued function f defined on the vertex set of G, $f: V(G) \to \mathbb{R}$ and a nonnegative real number t such that for any two distinct vertices u and v, u and v are adjacent if and only if $f(u) + f(v) > t$.

It is known [10, p. 10] that a graph is a threshold graph if and only if it is $\{P_4, \ C_4, \ 2K_2\}$-free.

Threshold graphs play a special role for many reasons of which we list three in the following. First, they have geometrical significance. Let Ω_n be the convex hull of all degree sequences of the simple graphs of order n. Then the extreme points of the polytope Ω_n are exactly the degree sequences of threshold graphs of order n [8] (for another proof see [12]). Second, a nonnegative integer sequence is graphical if and only if it is majorized by the degree sequence of some threshold graph [12]. Third, a graphical sequence has a unique labeled realization if and only if it is the degree sequence of a threshold graph [10, p. 72].

For terminology and notations we follow the textbooks [10,15]. The order of a graph is its number of vertices, and the size its number of edges. For graphs we will use equality up to isomorphism, so $G = H$ means that G and H are isomorphic. $N(v)$ and $N[v]$ denote the neighborhood and closed neighborhood of a vertex v, respectively. For a real number r, $\lfloor r \rfloor$ denotes the largest integer less than or equal to r, and $\lceil r \rceil$ denotes the least integer larger than or equal to r. The notation $|\ S\ |$ denotes the cardinality of a set S.

2 | MAIN RESULTS

Let $G = (V, E)$ be a graph whose distinct positive vertex-degrees are $\delta_1 < \ \cdots < \delta_m$ and let $\delta_0 = 0$. Denote $D_i = \{v \in V \,|\, \deg(v) = \delta_i\}$ for $i = 0,1,...,m$. The sequence $D_0, D_1,..., D_m$ is called the *degree partition* of G. Each D_i is called a *degree set*. Sometimes when D_0 is empty it may be omitted. These notations will be used throughout. We will need the following characterization [10, p. 11], which describes the basic structure of a threshold graph.

Lemma 1. *G is a threshold graph if and only if for each $v \in D_k$,*

$$N(v) = \bigcup_{j=1}^{k} D_{m+1-j} \quad \text{if } k = 1,...,\lfloor m/2 \rfloor,$$
$$N[v] = \bigcup_{j=1}^{k} D_{m+1-j} \quad \text{if } k = \lfloor m/2 \rfloor + 1,...,m.$$

In other words, for $x \in D_i$ and $y \in D_j$, x is adjacent to y if and only if $i + j > m$.

Clearly, Lemma 1 not only implies another characterization that the vicinal preorder of a threshold graph is a total preorder, but also indicates that every threshold graph is determined uniquely by its degree sequence [10, p. 72].

The following lemma can be found in [10, pp. 11-13].

Lemma 2. *For any threshold graph,*

$$\delta_{k+1} = \delta_k + |D_{m-k}| \quad \text{for } k = 0,1,\ldots,m, \;\; k \neq \lfloor m/2 \rfloor,$$
$$\delta_{k+1} = \delta_k + |D_{m-k}| - 1 \quad \text{for } k = \lfloor m/2 \rfloor.$$

For two subsets S and T of the vertex set of a graph G, the notation $[S, T]$ denotes the set of edges of G with one end in S and the other end in T. Here S and T need not be disjoint. In the case $T = S$, $[S, S]$ is just the edge set of the subgraph $G[S]$ of G induced by S. Next we define a new concept, which will be used in the proofs.

Definition 2. An edge of a threshold graph G with degree partition $D_0, D_1,\ldots,D_m$ is called a *key edge* of G if it lies in $[D_k, D_{m+1-k}]$ for some k with $1 \leq k \leq \lceil m/2 \rceil$.

Thus when m is even we have only one type of key edges, and when m is odd ($m \geq 3$) we have two types of key edges. For example, if $m = 4$, then the set of key edges is $[D_1, D_4] \cup [D_2, D_3]$, whereas if $m = 5$, then the set of key edges is $[D_1, D_5] \cup [D_2, D_4] \cup [D_3, D_3]$. In the second case there are key edges which have ends with different degrees as well as key edges which have both ends of the same degree. We will need the following two lemmas concerning properties of key edges.

Lemma 3. *If e is a key edge of a threshold graph G, then $G - e$ is a threshold graph.*

Proof. Denote $G' = G - e$ and let m' be the number of distinct positive vertex-degrees of G'. Let $e = xy$. First suppose that $x \in D_j$ and $y \in D_{m+1-j}$ for some $1 \leq j \leq \lfloor m/2 \rfloor$. We write TPO for the conditions in Lemma 1 (suggesting total preorder). To prove that G' is a threshold graph, by Lemma 1 it suffices to show that the degree sets of G' satisfy TPO. The structural change of the degree partitions depends on the sizes of the two sets D_j and D_{m+1-j}. We distinguish four cases.

Case 1 $|D_j| = 1$ and $|D_{m+1-j}| = 1$. The condition $|D_j| = 1$ implies that $j = \lfloor m/2 \rfloor$ is possible only if m is odd, since if m is even then $|D_{m/2}| \geq 2$. Hence $m - j > j$, implying that D_{m-j} and D_j are two distinct sets. By Lemma 2,

$$\delta_j = \delta_{j-1} + |D_{m+1-j}| = \delta_{j-1} + 1 \quad \text{and} \quad \delta_{m+1-j} = \delta_{m-j} + |D_j| = \delta_{m-j} + 1.$$

After deleting e, the two sets D_j and D_{m+1-j} become empty, and they disappear in G'. x goes to D_{j-1} and y goes to D_{m-j}. Now $m' = m - 2$ and the adjacency relations among the vertices of G' still satisfy TPO.

Case 2 $|D_j| = 1$ and $|D_{m+1-j}| \geq 2$. As in case 1, D_{m-j} and D_j are two distinct sets. By Lemma 2, we have

$$\delta_j = \delta_{j-1} + |D_{m+1-j}| \geq \delta_{j-1} + 2 \quad \text{and} \quad \delta_{m+1-j} = \delta_{m-j} + |D_j| = \delta_{m-j} + 1.$$

When deleting e, x stays in D_j and y goes to D_{m-j}. Thus $m' = m$ and G' satisfies TPO.

Case 3 $\mid D_j \mid \geq 2$ and $\mid D_{m+1-j} \mid = 1$. We have $\delta_j = \delta_{j-1} + \mid D_{m+1-j} \mid = \delta_{j-1} + 1$. When deleting e, x goes to D_{j-1}. If m is even, $j = m/2$ and $\mid D_j \mid = 2$, then $\delta_{m+1-j} = \delta_{m/2} + \mid D_{m/2} \mid - 1 = \delta_j + 1$. When deleting e, y goes to D_j and the set D_{m+1-j} disappears. Thus $m' = m - 1$. In all other cases, we have $\delta_{m+1-j} \geq \delta_{m-j} + 2$. In fact, if m is odd or m is even and $j < m/2$, we have $\delta_{m+1-j} = \delta_{m-j} + \mid D_j \mid \geq \delta_{m-j} + 2$, whereas if m is even, $j = m/2$ and $\mid D_j \mid \geq 3$, we have $\delta_{m+1-j} = \delta_{m-j} + \mid D_j \mid - 1 \geq \delta_{m-j} + 2$. When deleting e, y remains in D_{m+1-j}. Thus $m' = m$. In each case, G' satisfies TPO.

Case 4 $\mid D_j \mid \geq 2$ and $\mid D_{m+1-j} \mid \geq 2$. We have $\delta_j = \delta_{j-1} + \mid D_{m+1-j} \mid \geq \delta_{j-1} + 2$. If m is even, $j = m/2$ and $\mid D_j \mid = 2$, then $\delta_{m+1-j} = \delta_{j+1} = \delta_j + \mid D_j \mid - 1 = \delta_j + 1$. When deleting e, x remains in D_j (but with degree $\delta_j - 1$) and a new degree set $\{y\} \cup (D_j \setminus \{x\})$ appears. Now $m' = m + 1$. In all other cases, two new degree sets appear, one containing only x and the other containing only y, so that $m' = m + 2$. In either case, G' satisfies TPO and hence it is a threshold graph.

Now suppose that m is odd and $x, y \in D_t$, where $t = \lfloor m/2 \rfloor + 1 = \lceil m/2 \rceil$. Apply Lemma 2. If $\mid D_t \mid = 2$, when deleting e, both x and y go to $D_{\lfloor m/2 \rfloor}$ and the degree set D_t disappears. Then $m' = m - 1$ and G' satisfies TPO. Otherwise $\mid D_t \mid \geq 3$. When deleting e, a new degree set $\{x, y\}$ appears, where x and y are nonadjacent. In this case $m' = m + 1$ and G' again satisfies TPO. □

Lemma 4. *Every key edge of a Hamiltonian threshold graph lies in at least one Hamilton cycle.*

Proof. Let G be a Hamiltonian threshold graph with degree partition $D_1,\ldots,D_m$. Let $e = xy$ be a key edge of G with $x \in D_j$ and $y \in D_{m+1-j}$ for some $1 \leq j \leq \lceil m/2 \rceil$. Choose any Hamilton cycle C of G. If e lies in C, we are done. Otherwise let $C = (x, s,\ldots,y, t,\ldots)$. Then s and x are adjacent, and t and y are adjacent. Applying Lemma 1 we deduce that s and t are adjacent. Now the classical cycle exchange [1, p. 485] with $x^+ = s$ and $y^+ = t$ yields a new Hamilton cycle containing the edge e. □

Different necessary and sufficient conditions for a threshold graph to be Hamiltonian are given by Golumbic [5], Harary and Peled [7], and Mahadev and Peled [11]. What we need is the following one by Golumbic [5, p. 231] whose proof can be found in [10, p. 25].

Lemma 5. *Let G be a threshold graph of order at least 3 with the degree partition $D_0, D_1,\ldots,D_m$. Then G is Hamiltonian if and only if $D_0 = \phi$,*

$$\sum_{j=1}^{k} \mid D_j \mid < \sum_{j=1}^{k} \mid D_{m+1-j} \mid, \quad k = 1,\ldots,\lfloor (m-1)/2 \rfloor$$

and if m is even, then $\sum_{j=1}^{m/2} \mid D_j \mid \leq \sum_{j=1}^{m/2} \mid D_{m+1-j} \mid$.

Definition 3. For every integer $n \geq 3$, we denote by G_n the graph with degree sequence $n-1, n-1, n-2, \ldots, \lceil n/2 \rceil, \lceil n/2 \rceil, \ldots, 3, 2$ of $n-2$ distinct degrees.

We remark that the graph G_n is well defined and it is a Hamiltonian threshold graph. First, the existence of a graph with this sequence as its degree sequence is shown in the proof of Theorem 6 below. Second, by Condition 7 in [10, p. 11] we easily verify that a graph with this degree sequence is a threshold graph, and consequently [10, p. 72] G_n is uniquely determined by the degree sequence. Third, by [10, p. 26] we deduce that G_n is Hamiltonian.

G_8 is depicted in Figure 1.

Now we are ready to prove the main results.

Theorem 6. *The minimum number of Hamilton cycles in a Hamiltonian threshold graph of order* n *is* $2^{\lfloor (n-3)/2 \rfloor}$ *and this minimum number is attained uniquely by the graph* G_n.

Proof. We first determine the minimizing graph and then count its number of Hamilton cycles. Let G be a Hamiltonian threshold graph of order n having the minimum number of Hamilton cycles. Let $D_1, \ldots, D_m$ be the degree partition of G. Note that for any threshold graph with $m \geq 1$, we have $|D_{\lceil m/2 \rceil}| \geq 2$. This follows from

$$1 \leq \delta_{\lfloor m/2 \rfloor + 1} - \delta_{\lfloor m/2 \rfloor} = |D_{\lceil m/2 \rceil}| - 1$$

by Lemma 2.

The theorem holds trivially for the case $n = 3$. Next suppose $n \geq 4$. $m = 1$ means that G is a complete graph, which is impossible. Thus $m \geq 2$. We claim that $|D_m| = 2$. Lemma 5 with $k = 1$ implies $|D_m| \geq 2$. Hence it suffices to prove $|D_m| \leq 2$. To the contrary suppose $|D_m| \geq 3$. Let e be any edge in $[D_1, D_m]$. Then e is a key edge by definition. By Lemma 3, $G - e$ is a threshold graph. Since G is a Hamiltonian threshold

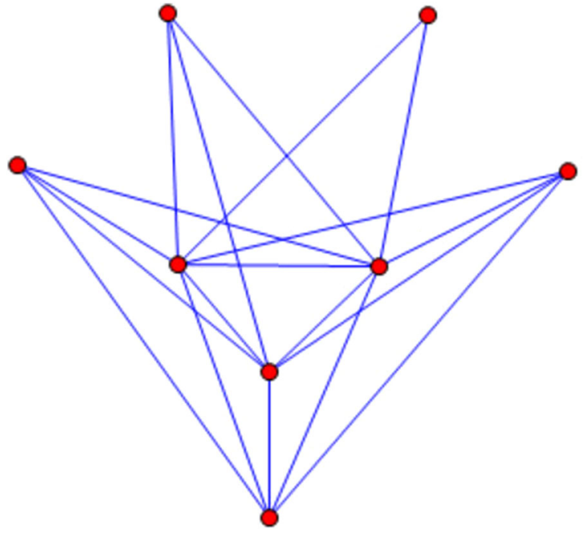

FIGURE 1 The minimizer G_8 [Color figure can be viewed at wileyonlinelibrary.com]

graph, its degree sets $D_1,...,D_m$ satisfy the inequalities in Lemma 5. Analyzing the change of degree partitions from G to $G - e$ as in the proof of Lemma 3, we see that the degree sets of $G - e$ also satisfy the inequalities in Lemma 5. Hence by Lemma 5, $G - e$ is Hamiltonian. Deleting any edge cannot increase the number of Hamilton cycles. By Lemma 4, the key edge e lies in at least one Hamilton cycle of G. It follows that $G - e$ has fewer Hamilton cycles than G, contradicting the minimum property of G. This proves $|D_m| = 2$.

If $m = 2$, then by Lemma 5 we have $|D_1| \le 2$. Since $n \ge 4$, we must have $n = 4$ and $|D_1| = 2$. Then applying Lemma 1 we deduce that G has the degree sequence 3,3, 2,2, so that $G = G_4$. Next suppose $m \ge 3$. By Lemma 5, $|D_1| < |D_m| = 2$. Hence $|D_1| = 1$. We first consider the case $m \ge 4$. (The case $m = 3$ will be treated later.) We claim that $|D_{m-1}| = 1$. Otherwise, as argued above, deleting any key edge f in $[D_2, D_{m-1}]$ would reduce the number of Hamilton cycles such that $G - f$ is still a Hamiltonian threshold graph, a contradiction. Then using the fact that $|D_1| = 1$ and $|D_m| = 2$ and applying Lemma 5 we deduce that if m is odd or if m is even and $m \ge 6$ then $|D_2| = 1$, and if $m = 4$ then $|D_2| = 2$. Continuing in this way, by successively deleting a key edge in $[D_j, D_{m+1-j}]$ for $j = 2,...,\lfloor m/2 \rfloor$ if $|D_{m+1-j}| \ge 2$ we conclude that $|D_{m+1-j}| = 1$ for each $j = 2,...,\lfloor m/2 \rfloor$. Then using the fact that $|D_1| = 1$ and $|D_m| = 2$ and applying Lemma 5, we conclude that $|D_i| = 1$ for each $i = 2,...,\lfloor m/2 \rfloor - 1$ and that if m is odd then $|D_{\lfloor m/2 \rfloor}| = 1$ and if m is even then $|D_{m/2}| = 2$. Thus, if m is even then $n = m + 2$ is even, G has the degree sequence $n - 1$, $n - 1,...,n/2$, $n/2,...,3,2$ and hence $G = G_n$.

If $m \ge 3$ and m is odd, we assert that $|D_{\lceil m/2 \rceil}| = 2$. As remarked at the beginning, we always have $|D_{\lceil m/2 \rceil}| \ge 2$. Thus it suffices to show $|D_{\lceil m/2 \rceil}| \le 2$. To the contrary suppose $|D_{\lceil m/2 \rceil}| \ge 3$. By Lemma 4, any key edge h in $G[D_{\lceil m/2 \rceil}]$ lies in at least one Hamilton cycle. With the assumption that $|D_{\lceil m/2 \rceil}| \ge 3$, applying Lemma 3 and Lemma 5 we see that $G - h$ is also a Hamiltonian threshold graph with fewer Hamilton cycles than G, a contradiction. This shows $|D_{\lceil m/2 \rceil}| = 2$. Now $n = m + 2$ is odd. Combining all the above information about G we deduce that G has the degree sequence $n - 1$, $n - 1,...,(n + 1)/2$, $(n + 1)/2,...,3,2$ and hence $G = G_n$.

Denote the number of Hamilton cycles of G_n by $f(n)$. Since $f(3) = f(4) = 1$, to prove $f(n) = 2^{\lfloor (n-3)/2 \rfloor}$ it suffices to show the following:

Claim. For every integer $k \ge 2$,

$$f(2k - 1) = f(2k) \quad \text{and} \quad f(2k + 1) = 2f(2k).$$

In G_{2k}, let $D_k = \{x, y\}$ and $D_{k+1} = \{z\}$. By Lemma 5, neither $G_{2k} - xz$ nor $G_{2k} - yz$ is Hamiltonian. Thus the path xzy must lie in every Hamilton cycle of G_{2k}. Deleting the vertex z and adding the edge xy we obtain a graph which is isomorphic to G_{2k-1} and has the same number of Hamilton cycles as G_{2k}. Hence $f(2k - 1) = f(2k)$.

In G_{2k+1}, let $D_{k+1} = \{u, v\}$. Then the edge uv lies in every Hamilton cycle of G_{2k+1}. Denote $G' = G - v$. Clearly G' is isomorphic to G_{2k} and hence G' has $f(2k)$ Hamilton cycles. Since $N(u)\setminus\{v\} = N(v)\setminus\{u\}$ in G, from each Hamilton cycle of G' we can obtain two distinct Hamilton cycles of G_{2k+1} by replacing the vertex u by the edge uv in two ways. More precisely, a Hamilton cycle $(...,s, u, t,...)$ of G' yields two Hamilton cycles $(...,s, u, v, t,...)$ and $(...,s, v, u, t,...)$ of G. Conversely every Hamilton cycle of G_{2k+1} can be obtained in such a vertex-to-edge expansion from a Hamilton cycle of G'. Hence $f(2k+1) = 2f(2k)$. This shows the claim and completes the proof. □

The above proof of Theorem 6 also proves that G_n is the unique graph that has the minimum size among all Hamiltonian threshold graphs of order n. To see this, just replace the assumption that G has the minimum number of Hamilton cycles by the one that G has the minimum size. Also note that the size of a threshold graph is easy to count, since it is a split graph with the clique $\bigcup_{j=\lfloor m/2\rfloor+1}^{m} D_j$ and the independent set $\bigcup_{j=1}^{\lfloor m/2\rfloor} D_j$. Thus we have the following result.

Theorem 7. *The minimum size of a Hamiltonian threshold graph of order n is*

$$\begin{cases} (n^2+2n-3)/4 & \text{if } n \text{ is odd,} \\ (n^2+2n-4)/4 & \text{if } n \text{ is even,} \end{cases}$$

and this minimum size is attained uniquely by the graph G_n.

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