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Formulae for sums of consecutive square roots

For a real number x, let $[x]$ denote the largest integer not exceeding x. The following result might be surprising.

Theorem 1. The following formulae hold for every positive integer n.

(i) $[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}]$
(ii) $[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}] = [\sqrt{9n+8}]$
(iii) $[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3}] = [\sqrt{16n+20}]$
(iv) $[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4}] = [\sqrt{25n+49}]$

Formula (i) is folklore; (ii) is a problem in [1]; (iii) can be found in [2, p. 274]. The purpose of this note is to prove (iv) and consider related questions.

Proof of (iv). For positive numbers $x \neq y$ we have $\sqrt{x} + \sqrt{y} < \sqrt{2(x+y)}$. Using this inequality twice we get

$$\begin{aligned} &\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} \\ &= (\sqrt{n} + \sqrt{n+4}) + (\sqrt{n+1} + \sqrt{n+3}) + \sqrt{n+2} \\ &< \sqrt{4n+8} + \sqrt{4n+8} + \sqrt{n+2} \\ &= 5\sqrt{n+2} \\ &= \sqrt{25n+50}. \end{aligned}$$

Thus

$$\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} < \sqrt{25n+50}. \quad (1)$$

Using the fact that $\sqrt{x} > \frac{2}{3}\left\{\left(x+\frac{1}{2}\right)^{\frac{3}{2}} - \left(x-\frac{1}{2}\right)^{\frac{3}{2}}\right\}$ for any real number $x \geq 1$, we obtain

$$\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} > \frac{2}{3}\left\{\left(n+\frac{9}{2}\right)^{\frac{3}{2}} - \left(n-\frac{1}{2}\right)^{\frac{3}{2}}\right\}. \quad (2)$$

Now we show that when $n \geq 12$,

$$\frac{2}{3}\left\{\left(n+\frac{9}{2}\right)^{\frac{3}{2}} - \left(n-\frac{1}{2}\right)^{\frac{3}{2}}\right\} > \sqrt{25n+49}. \quad (3)$$

Let $f(x) = \frac{2}{3}\left\{\left(x+\frac{9}{2}\right)^{\frac{3}{2}} - \left(x-\frac{1}{2}\right)^{\frac{3}{2}}\right\} - \sqrt{25x+49}$. Then $f(12) > 0$, $\lim_{x\to\infty} f(x) = 0$, $f(x)$ is increasing on $[12, 14841/400]$ and decreasing on $[14841/400, \infty)$. So $f(x)$ is positive on $[12, \infty)$ and (3) is proved. Combining (1), (2), and (3), we deduce that when $n \geq 12$,

$$\sqrt{25n+49} < \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} < \sqrt{25n+50}.$$

Since no integer lies strictly between $\sqrt{25n+49}$ and $\sqrt{25n+50}$, we conclude that *(iv)* is valid for the case $n \geq 12$. The cases $n = 1, 2, \ldots, 11$ are verified by the computer software Matlab. This completes the proof. □

In view of Theorem 1, it is natural to suspect that for any positive integer k there is a constant c depending on k such that

$$[\sqrt{n} + \sqrt{n+1} + \cdots + \sqrt{n+k-1}] = [\sqrt{k^2 n + c}] \quad (4)$$

holds for all positive integers n. This is not the case. It is shown in [2, pp. 725–727] that for sufficiently large k no such c exists. Our next result shows that 6 is the first k for which (4) cannot hold for all n.

Theorem 2. For any real number c, there is a positive integer n such that

$$[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} + \sqrt{n+5}] \neq [\sqrt{36n+c}].$$

Proof. Let $s(n) = [\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} + \sqrt{n+5}]$. Using Matlab we find that

$$s(1) = 10 < 11 = [\sqrt{36 \times 1 + 85}],$$
$$s(11) = 22 > 21 = [\sqrt{36 \times 11 + 85}].$$

Therefore, when $c \geq 85$, $s(1) < [\sqrt{36 \times 1 + c}]$, and when $c \leq 85$, $s(11) > [\sqrt{36 \times 11 + c}]$. □

Prompted by the evidence in Theorems 1 and 2, I pose the following conjecture.

Conjecture. For any positive integer $k \geq 6$, no constant c depending only on k exists such that (4) is valid for all positive integers n.

We also have the following related question.

Question. For which positive integers k does there exist an integer c such that

$$\left|\sqrt{n} + \sqrt{n+1} + \cdots + \sqrt{n+k-1} - \sqrt{k^2 n + c}\right| < 1$$

holds for all positive integers n? When such a c exists, determine it.

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