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Completion of a partial integral matrix to a unimodular matrix ☆

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Abstract

We first characterize submatrices of a unimodular integral matrix. We then prove that if n entries of an $n \times n$ partial integral matrix are prescribed and these n entries do not constitute a row or a column, then this matrix can be completed to a unimodular matrix. Consequently an $n \times n$ partial integral matrix with $n - 1$ prescribed entries can always be completed to a unimodular matrix.

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1. Introduction and statement of results

For simplicity of presentation, we will consider only matrices over the rational integers $\mathbb{Z}$, but all the results have obvious generalizations to matrices over a principal ideal ring. Let $M_n(\mathbb{Z})$ be the ring of $n \times n$ matrices over $\mathbb{Z}$. A matrix $A \in M_n(\mathbb{Z})$ is called *unimodular* if $\det A = \pm 1$. Being units of $M_n(\mathbb{Z})$, such matrices are used to define the equivalence relation between integral quadratic forms [1, p. 127]. It is known [7, p. 15] that if $a_1, \ldots, a_n$ are relatively prime integers then there is a unimodular matrix with $(a_1, \ldots, a_n)$ as any prescribed row. This result plays a key role in the proofs of the Hermite normal form and the Smith normal form [7]. A row is a special case of submatrices. We will prove the following more general result.

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Theorem 1. *Let r, s, n be positive integers with $r, s \leqslant n$. An $r \times s$ integral matrix A is a submatrix of some unimodular matrix of order n if and only if A has at least $r + s - n$ invariant factors equal to* 1.

Here we make the convention that if $r + s \leqslant n$ then all $r \times s$ matrices satisfy the condition in Theorem 1. To be definite, throughout we take invariant factors and determinantal divisors to be nonnegative. A partial matrix is one in which some entries are prescribed and the other entries are to be chosen. Since row or column permutations do not change unimodularity, we can put the submatrix A in any prescribed place and Theorem 1 may be regarded as a matrix completion result.

Any n integers not relatively prime cannot be a row or a column of a unimodular matrix of order n. Our next result shows that these are the only two cases which should be excluded.

Theorem 2. *Let $a_1, \ldots, a_n$ be prescribed integers. Let $(i_t, j_t), t = 1, \ldots, n$, be prescribed different positions in an $n \times n$ matrix and these positions do not constitute a row or a column. Then there exists a unimodular matrix of order n with the entry a_t in the position (i_t, j_t) for $t = 1, \ldots, n$.*

Note that the number n of prescribed entries in Theorem 2 is best possible. Consider the following partial matrix:

$$\begin{pmatrix} 2 & 2 & ? \\ 2 & 2 & ? \\ ? & ? & ? \end{pmatrix},$$

where the ?'s are free entries. Any of the prescribed $4 = 3 + 1$ entries do not constitute one row or one column. Since the determinant of this matrix is always even, no matter what the ?'s are, it cannot be completed to a unimodular matrix. Theorem 1 will be needed in the proof of Theorem 2.

An immediate consequence of Theorem 2 is the following corollary.

Corollary 3. *Let $a_1, \ldots, a_{n-1}$ be prescribed integers and (i_t, j_t), $t = 1, \ldots, n-1$, be prescribed different positions in an $n \times n$ matrix. Then there exists a unimodular matrix of order n with the entry a_t in the position (i_t, j_t) for $t = 1, \ldots, n-1$.*

We remark that there are results for eigenvalues of a flavor similar to that of Theorem 2 and Corollary 3 [2–4,6,8,9].

2. Proofs

Denote by $M_{r,s}(\mathbb{Z})$ the set of $r \times s$ integral matrices. For $A \in M_{r,s}(\mathbb{Z})$ we denote the jth determinantal divisor of A by $d_j(A)$, $j = 1, \ldots, \min\{r, s\}$, and the jth invariant factor by $s_j(A)$, $j = 1, \ldots, \operatorname{rank} A$. We will need the following lemma.

Lemma 4. *Let $A \in M_{r,s}(\mathbb{Z})$ be a submatrix of $B \in M_n(\mathbb{Z})$ and $r + s - n \geqslant 1$. Then $d_{r+s-n}(A) |$* det B.

Proof. Denote $p = \min\{r, s\}$, $k = r + s - n$. Then $p \geqslant k$. Without loss of generality (by row or column permutations if necessary), we may assume that A is in the left-upper corner of B. Let $UAV = \mathrm{diag}(s_1, \ldots, s_p)_{r\times s} = D$ be the Smith normal form where U, V are unimodular matrices, $s_1, \ldots, s_m$ are the invariant factors and $s_{m+1} = \cdots = s_p = 0$, $m = \operatorname{rank} A$. Throughout we denote by I_t the identity matrix of order t. Denote

$$\widetilde{B} = \mathrm{diag}(U, I_{n-r})B\,\mathrm{diag}(V, I_{n-s}) = \begin{pmatrix} D & B_1 \\ B_2 & B_3 \end{pmatrix} = (b_{ij})_{n\times n}.$$

Since $r + s = n + k$, by König's theorem [5, p. 73] every diagonal $(b_{1,\sigma(1)}, \ldots, b_{n,\sigma(n)})$ of $\widetilde{B}$ contains at least k entries of D where σ is a permutation of $1, \ldots, n$. Therefore if $\prod_{i=1}^n b_{i,\sigma(i)} \neq 0$, then $\prod_{i=1}^n b_{i,\sigma(i)} = w\prod_{t=1}^k s_{i_t}$ for some integer w and $1 \leqslant i_1 < \cdots < i_k \leqslant p$. Since $s_j|s_{j+1}$, $j = 1, \ldots, p-1$, we have $s_t|s_{i_t}$, $t = 1, \ldots, k$. Hence

$$d_k(A) = \left(\prod_{t=1}^k s_t\right)\Bigg|\left(w\prod_{t=1}^k s_{i_t}\right)$$

and we have $d_k(A)|\prod_{i=1}^n b_{i,\sigma(i)}$ for any σ. So $d_k(A)|\det \widetilde{B} = \pm\det B$. □

Proof of Theorem 1. Suppose $A \in M_{r,s}(\mathbb{Z})$ is a submatrix of a unimodular matrix $B \in M_n(\mathbb{Z})$. If $r + s > n$, by Lemma 4

$$d_{r+s-n}(A)|\det B = \pm 1.$$

Thus $d_{r+s-n}(A) = 1$. Since $\prod_{j=1}^{r+s-n} s_j(A) = d_{r+s-n}(A)$, we must have $s_1(A) = \cdots = s_{r+s-n}(A) = 1$. So A has at least $r + s - n$ invariant factors equal to 1.

Conversely let $A \in M_{r,s}(\mathbb{Z})$. Using transpose we need consider only the case $r \geqslant s$. If $r + s \leqslant n$ we show that A can always be completed to a unimodular matrix of order n. Let $A = U\begin{pmatrix} D_s \\ 0 \end{pmatrix}V$ be the Smith normal form decomposition with U, V unimodular and D_s diagonal of order s. Consider the matrix

$$B = \begin{pmatrix} D_s & 0 & -I_s & 0 \\ 0 & I_{r-s} & 0 & 0 \\ I_s - D_s & 0 & I_s & 0 \\ 0 & 0 & 0 & I_{n-r-s} \end{pmatrix} \in M_n(\mathbb{Z}),$$

where the zero blocks are of appropriate sizes and a matrix with zero column or row number does not appear. By adding the third block row to the first block row we see that B is unimodular. Thus

$$\mathrm{diag}(U, I_{n-r})B\,\mathrm{diag}(V, I_{n-s})$$

is unimodular and has A as a submatrix.

If $r + s > n$, let $k = r + s - n$. Suppose A has at least k invariant factors equal to 1. Then A has the following Smith normal form decomposition:

$$A = U\begin{pmatrix} I_k & 0 \\ 0 & D_{n-r} \\ 0 & 0 \end{pmatrix}V$$

with U, V unimodular and D_{n-r} diagonal of order $n - r$. Consider the matrix

$$G = \begin{pmatrix} I_k & 0 & 0 & 0 \\ 0 & D_{n-r} & 0 & -I_{n-r} \\ 0 & 0 & I_{r-s} & 0 \\ 0 & I_{n-r} - D_{n-r} & 0 & I_{n-r} \end{pmatrix} \in M_n(\mathbb{Z}).$$

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By adding the fourth block row to the second block row we see that G is unimodular. Thus

$$\operatorname{diag}(U, I_{n-r})G \operatorname{diag}(V, I_{n-s})$$

is unimodular and has A as a submatrix. This completes the proof. □

Proof of Theorem 2. We use induction on the order n. The case $n = 2$ is easily checked to be true. We assume that the assertion holds for matrices of order $n-1$ and prove the result for matrices of order $n \geqslant 3$. There are two cases.

Case (i). There are $n-1$ prescribed positions in one row or column.

We consider the row case. The column case is similar. By row or column permutations if necessary, without loss of generality we may assume that $(i_t, j_t) = (1, t)$, $t = 2, \ldots, n$. Then apart from row and column permutations there are essentially the following two situations.

Subcase i(1). $(i_1, j_1) = (2, 1)$. The matrix

$$\begin{pmatrix} a_1a_2+1 & a_2 & a_3 & \cdots & a_n \\ a_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is unimodular and has the prescribed entries in the prescribed positions.

Subcase i(2). $(i_1, j_1) = (2, 2)$. We choose any $n-2$ prime numbers $p_1, \ldots, p_{n-2}$ each of which is greater than $\max_{1 \leqslant j \leqslant n} |a_j|$. Let

$$B = \begin{pmatrix} a_2 & a_3 & \cdots & a_n \\ a_1 & p_1 & \cdots & p_{n-2} \end{pmatrix}.$$

Then $s_1(B) = d_1(B) = 1$. By Theorem 1, the $2 \times (n-1)$ matrix B can be completed to a unimodular matrix of order n. So there exist column vectors α, β and matrix F such that

$$\begin{pmatrix} \alpha & B \\ \beta & F \end{pmatrix} \in M_n(\mathbb{Z})$$

is a unimodular matrix, which satisfies the requirement.

Case (ii). Every row or column of a matrix of order n contains at most $n-2$ prescribed positions.

There is at least one column, say, the last column which contains at most one prescribed position. We distinguish the following two cases.

Subcase ii(1). The last column has no prescribed positions.

If the last row has at least one prescribed position, then there are at most $n-1$ prescribed positions (i_t, j_t) with $i_t \leqslant n-1$ and $j_t \leqslant n-1$. No row or column of a matrix of order $n-1$ is fully prescribed by the general assumption of Case (ii). Thus by the induction hypothesis there exists a unimodular matrix A' of order $n-1$ with the prescribed entries a_t in the positions (i_t, j_t) for all (i_t, j_t) with $i_t \leqslant n-1$ and $j_t \leqslant n-1$. Let β be the $(n-1)$-dimensional row vector whose j_tth component is equal to a_t for those $i_t = n$ and whose other entries are zero. Then the matrix $\begin{pmatrix} A' & 0 \\ \beta & 1 \end{pmatrix}$ is unimodular and satisfies the requirement.

If the last row has no prescribed positions, then by the induction hypothesis these exists a unimodular matrix $H' = (h_{ij})$ of order $n-1$ with the $n-1$ prescribed entries a_t in the positions (i_t, j_t), respectively, for $t = 1, \ldots, n-1$. Note that the position (i_n, j_n) is also in H'. Let γ be the $(n-1)$-dimensional column vector with the i_nth component equal to $a_n - h_{i_n, j_n}$ and with other components equal to zero. Now consider the matrix

$$H = \begin{pmatrix} H' & \gamma \\ 0 & 1 \end{pmatrix} \in M_n(\mathbb{Z}).$$

Let G be the matrix obtained from H by adding the last column to the j_nth column. Then G is unimodular and has the entries a_t in the positions (i_t, j_t) for $t = 1, \ldots, n$.

Subcase ii(2). The last column has exactly one prescribed position.

By row permutations if necessary, we may assume that $(i_n, j_n) = (n, n)$. Then there are at most $n - 1$ prescribed positions (i_t, j_t) with $i_t \leqslant n - 1$ and $j_t \leqslant n - 1$. By the induction hypothesis there is a unimodular matrix Q of order $n - 1$ with a_t in (i_t, j_t) for all t satisfying $i_t \leqslant n - 1$ and $j_t \leqslant n - 1$. There is at least one position (k, n) which is not prescribed. We will show that there exist an $(n - 1)$-dimensional row vector β and an $(n - 1)$-dimensional column vector γ such that the matrix

$$W = \begin{pmatrix} Q & \gamma \\ \beta & a_n \end{pmatrix} \in M_n(\mathbb{Z})$$

is unimodular and has the prescribed entries in the prescribed positions. For the latter property we need only set $\beta(j_t) = a_t$ if $i_t = n$ and $1 \leqslant j_t \leqslant n - 1$, set $\beta(k) = x$ to be chosen, and set other components of β to be zero. By Schur complement we have $\det W = (\det Q)(a_n - \beta Q^{-1}\gamma)$. So to make W unimodular it suffices to require $a_n - \beta Q^{-1}\gamma = 1$. We first show that we can choose x such that the components of βQ^{-1} are relatively prime. Note that Q^{-1} is also unimodular and hence the components $q_1, \ldots, q_{n-1}$ of the kth row of Q^{-1} are relatively prime. Let

$$\beta Q^{-1} = x(q_1, \ldots, q_{n-1}) + (c_1, \ldots, c_{n-1}).$$

There are integers $u_1, \ldots, u_{n-1}$ such that $\sum_{i=1}^{n-1} u_i q_i = 1$. Set $x = 1 - \sum_{i=1}^{n-1} u_i c_i$. Then $\sum_{i=1}^{n-1} u_i(xq_i + c_i) = 1$. So the components of βQ^{-1} are relatively prime. Consequently there exists an integral column vector γ such that $\beta Q^{-1}\gamma = a_n - 1$. Thus W is unimodular. This completes the proof. □

Proof of Corollary 3. We can always prescribe one more position (i_n, j_n) such that the prescribed n positions do not constitute one row or one column, and prescribe 1 in (i_n, j_n). Then apply Theorem 2. □

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