

# Extremal numbers of positive entries of imprimitive nonnegative matrices

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**Abstract.** We determine the maximum and minimum numbers of positive entries of imprimitive nonnegative matrices with a given imprimitivity index. One application of the results is to estimate the imprimitivity index by the number of positive entries. The proofs involve the study of a cyclic quadratic form. This completes a research initiated by M. Lewin in 1990.

**Key words.** nonnegative matrix, imprimitive matrix, quadratic form

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## 1. Introduction

A square nonnegative matrix  $A$  is said to be primitive if  $A^p$  is a positive matrix for some positive integer  $p$ ; otherwise  $A$  is called imprimitive. A primitive matrix is necessarily irreducible. The imprimitivity index  $k$  of  $A$  is the number of eigenvalues of  $A$  whose moduli are equal to the spectral radius of  $A$ .  $A$  is primitive if  $k = 1$ , and imprimitive if  $k > 1$ .

Let  $A$  be an  $n \times n$  irreducible nonnegative matrix with imprimitivity index  $k$ . Then  $A$  is permutation similar to

$$\begin{pmatrix} 0 & A_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{k-1} \\ A_k & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad (1)$$

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where the zero blocks along the diagonal are square (see [1, p.32] or [2, p.71]). The matrix in (1) is called the canonical form of  $A$ .

In [3] M. Lewin proved that if an irreducible nonnegative matrix has more positive entries than zero entries then it is primitive. R.A. Brualdi (see reference 2 in [4]) observed that this result follows immediately from the canonical form (1). Let  $\sigma(A)$  denote the number of positive entries of a nonnegative matrix  $A$ . For a real number  $x$ , denote by  $\lfloor x \rfloor$  the largest integer not exceeding  $x$ . Suppose  $A$  is an  $n \times n$  irreducible nonnegative matrix with imprimitivity index  $k$ . M. Lewin [4] proved that if  $k \leq 4$  then  $\sigma(A) \leq \lfloor n^2/k \rfloor$  and this bound is sharp, and if  $k \geq 5$  then  $\sigma(A) \leq \lfloor n^2/4 \rfloor$ . The case  $n = 7$  and  $k = 5$  shows that the bound  $\lfloor n^2/4 \rfloor$  is not sharp. In fact by the canonical form (1) it is easy to see that if a nonnegative matrix  $A$  is  $7 \times 7$  and of imprimitivity index 5 then  $\sigma(A) \leq 10$ , while  $\lfloor 7^2/4 \rfloor = 12$ .

The purpose of this note is to determine the sharp upper and lower bounds for  $\sigma(A)$ , and consider related applications. One interesting thing here is that the rule for the maximum  $\sigma(A)$  changes when  $k$  turns from 4 to 5.

## 2. Main results

Let  $A$  be an  $n \times n$  irreducible nonnegative matrix of imprimitivity index  $k$  whose canonical form is the matrix in (1). Suppose the orders of the zero blocks on the diagonal of (1) are  $x_1, x_2, \dots, x_k$  respectively. Since the positive entries of  $A$  are contained in the submatrices  $A_j$ ,  $j = 1, \dots, k$ ,

$$\sigma(A) \leq x_1x_2 + x_2x_3 + \dots + x_{k-1}x_k + x_kx_1. \quad (2)$$

Thus to determine the maximum  $\sigma(A)$  it suffices to determine the maximum value of the cyclic quadratic form

$$f(x_1, x_2, \dots, x_k) = \sum_{i=1}^k x_i x_{i+1} \quad (3)$$

under the condition that  $x_1, \dots, x_k$  are positive integers and  $\sum_{i=1}^k x_i = n$ . Here  $x_{k+1} = x_1$  and throughout we read the subscripts of  $x_i$  modulo  $k$ . We will also determine the minimum value of this quadratic form under the same restriction on its variables.

**Theorem 1.** Given positive integers  $n \geq k \geq 2$ , let  $x_1, \dots, x_k$  be positive integers with  $\sum_{i=1}^k x_i = n$ . Then

$$2n - k \leq \sum_{i=1}^k x_i x_{i+1}. \quad (4)$$

For any  $n$  and  $k$  this lower bound can be attained. If  $k \leq 4$  then

$$\sum_{i=1}^k x_i x_{i+1} \leq \lfloor n^2/k \rfloor. \quad (5)$$

If  $k \geq 5$  then

$$\sum_{i=1}^k x_i x_{i+1} \leq 2n - k + \lfloor (n - k)^2/4 \rfloor. \quad (6)$$

For any  $n$  and  $k$  the upper bounds in (5) and (6) can be attained.

*Proof.* Our strategy is to change successively a pair of variables without changing their sum to make the value of the quadratic form smaller or bigger. We first prove inequality (4). If  $k = 2$ , then

$$2x_1x_2 = 2x_1(n - x_1) \geq 2n - 2.$$

Let  $k = 3$ . Use the notation (3). There exists some  $i$  such that  $x_i \leq x_{i+1}$ . Since the value of  $f$  is invariant under cyclic permutations of  $x_1, x_2, x_3$ , we may suppose  $x_1 \leq x_2$ . Then

$$\begin{aligned} f(x_1 - d, x_2 + d, x_3) &= f(x_1, x_2, x_3) - d(x_2 - x_1) - d^2 \\ &\leq f(x_1, x_2, x_3) \end{aligned}$$

for any nonnegative number  $d$ . Taking  $d = x_1 - 1$  we get

$$f(1, x_2 + x_1 - 1, x_3) \leq f(x_1, x_2, x_3).$$

Thus the minimum value of  $f$  can be attained at some  $(x_1, x_2, x_3)$  with  $x_1 = 1$ . Then we have

$$\begin{aligned} f(1, x_2, x_3) = x_2 + x_3 + x_2x_3 &= n - 1 + x_2x_3 \\ &\geq n - 1 + n - 2 \\ &= 2n - 3. \end{aligned}$$

Let  $k = 4$ . Since  $f(x_1, x_2, x_3, x_4) = f(x_1 - d, x_2, x_3 + d, x_4)$  for any number  $d$ , we may suppose  $x_1 = 1$ . Then  $x_2 + x_3 + x_4 = n - 1$  and

$$\begin{aligned} f(1, x_2, x_3, x_4) &= x_2 + x_2x_3 + x_3x_4 + x_4 = (x_2 + x_4)(x_3 + 1) \\ &= (n - 1 - x_3)(x_3 + 1) \\ &\geq 2n - 4. \end{aligned}$$

Now let  $k \geq 5$ . We use induction on  $k$ . Suppose inequality (4) holds for  $k - 1$ . Since

$$f(x_1, x_2 + d, x_3, x_4 - d, x_5, \dots, x_k) - f(x_1, x_2, \dots, x_k) = d(x_1 - x_5),$$

by choosing  $d = 1 - x_2$  if  $x_1 > x_5$  and choosing  $d = x_4 - 1$  if  $x_1 \leq x_5$  we see that the minimum value of  $f$  is attained at some  $(x_1, \dots, x_k)$  with  $x_2 = 1$  or  $x_4 = 1$ . By cyclic permutations of the variables it suffices to consider the case  $x_1 = 1$ . We have

$$f(1, x_2 + x_4 - 1, x_3, 1, x_5, \dots, x_k) \leq f(1, x_2, x_3, x_4, \dots, x_k).$$

Thus we need consider only  $f(1, x_2, x_3, 1, x_5, \dots, x_k)$ . But

$$f(1, x_2 + x_3 - 1, 1, 1, x_5, \dots, x_k) \leq f(1, x_2, x_3, 1, x_5, \dots, x_k)$$

if  $x_3 < x_2$  and

$$f(1, 1, x_2 + x_3 - 1, 1, x_5, \dots, x_k) \leq f(1, x_2, x_3, 1, x_5, \dots, x_k)$$

if  $x_3 \geq x_2$ . Therefore in any case the minimum value of  $f$  can be attained when two consecutive variables are equal to 1. By cyclic permutations if necessary we may suppose  $x_1 = x_2 = 1$ . Then by the inductive hypothesis we have

$$\begin{aligned} f(1, 1, x_3, x_4, \dots, x_k) &= 1 + f(1, x_3, x_4, \dots, x_k) \\ &\geq 1 + 2(n - 1) - (k - 1) \\ &= 2n - k. \end{aligned}$$

This completes the proof of inequality (4). For any  $n$  and  $k$  the lower bound can be attained at  $x_1 = \dots = x_{k-1} = 1, x_k = n - k + 1$ .

Inequality (5) and its sharpness are proved in [4]. We sketch the proof here. When  $k \leq 4$ , for any positive real numbers  $x_1, \dots, x_k$  with  $\sum_{i=1}^k x_i = n$  it is not

hard to show  $\sum_{i=1}^k x_i x_{i+1} \leq n^2/k$ . Thus if  $x_1, \dots, x_k$  are positive integers we have (5). By considering the division  $n = km + r$  with  $0 \leq r < k$  we can show that  $\lfloor n^2/k \rfloor$  can be attained for each of  $k = 2, 3, 4$ .

Next we suppose  $k \geq 5$  and prove (6). It is easy to verify that

$$f(\dots, x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \dots) \leq f(\dots, x_{i-2}, 1, x_i, x_{i+1} + x_{i-1} - 1, x_{i+2}, \dots) \quad (7)$$

if  $x_{i-2} \leq x_{i+2}$  and

$$f(\dots, x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \dots) \leq f(\dots, x_{i-2}, x_{i-1} + x_{i+1} - 1, x_i, 1, x_{i+2}, \dots) \quad (8)$$

if  $x_{i-2} > x_{i+2}$ . Applying (7) or (8) with  $i = 3$  we see that the maximum value of  $f$  can be attained when  $x_2 = 1$  or  $x_4 = 1$ . By cyclic permutations of the variables we may suppose  $x_1 = 1$ . Now using (7) repeatedly we have

$$\begin{aligned} f(1, x_2, \dots, x_k) &\leq f(1, 1, x_3, x_4 + x_2 - 1, x_5, \dots, x_k) \\ &\leq f(1, 1, 1, x_4 + x_2 - 1, x_5 + x_3 - 1, \dots, x_k) \\ &\leq \dots \\ &\leq f(1, 1, \dots, 1, a, b) \\ &= n - 1 + ab \end{aligned}$$

where  $a$  and  $b$  are positive integers and  $a + b = n - k + 2$ . Then

$$ab \leq \left( \frac{a+b}{2} \right)^2 = \frac{(n-k+2)^2}{4}.$$

But  $ab$  is an integer, so

$$f(1, x_2, \dots, x_k) \leq n - 1 + ab \leq n - 1 + \lfloor (n - k + 2)^2/4 \rfloor = 2n - k + \lfloor (n - k)^2/4 \rfloor.$$

This proves (6). The upper bound in (6) can be attained at  $x_1 = \dots = x_{k-2} = 1$ ,  $x_{k-1} = x_k = (n - k + 2)/2$  when  $n - k$  is even and at  $x_1 = \dots = x_{k-2} = 1$ ,  $x_{k-1} = (n - k + 1)/2$ ,  $x_k = (n - k + 3)/2$  when  $n - k$  is odd.  $\square$

**Theorem 2.** *Let  $\Gamma(n, k)$  be the set of  $n \times n$  irreducible nonnegative matrices with imprimitivity index  $k$ . Then*

$$\max\{\sigma(A) | A \in \Gamma(n, k)\} = \begin{cases} \lfloor n^2/k \rfloor & \text{if } 1 \leq k \leq 4, \\ 2n - k + \lfloor (n - k)^2/4 \rfloor & \text{if } k \geq 5. \end{cases}$$

*Proof.* The case  $k = 1$  is trivial. Let  $k \geq 2$ . These upper bounds follow from (2) and Theorem 1. On the other hand, let  $A_i$  be the  $x_i \times x_{i+1}$  matrix with all entries equal to 1, where  $x_1, \dots, x_k$  are positive integers such that  $\sum_{i=1}^k x_i x_{i+1}$  attains its upper bound in (5) or (6) according as  $k \leq 4$  or  $k \geq 5$ . Then the matrix in (1) is in  $\Gamma(n, k)$  and has the maximum number of positive entries, which is just the bound in Theorem 2.  $\square$

We denote by  $\text{ind}(A)$  the imprimitivity index of an irreducible nonnegative matrix  $A$ . One application of Theorem 2 is to estimate  $\text{ind}(A)$  in terms of  $\sigma(A)$ .

**Corollary 3.** *Let  $A$  be an  $n \times n$  irreducible nonnegative matrix whose number of positive entries is  $\sigma(A)$ . If  $n^2 < 4\sigma(A)$  then*

$$\text{ind}(A) \leq \lfloor n^2/\sigma(A) \rfloor.$$

If  $n^2 \geq 4\sigma(A)$  then

$$\text{ind}(A) \leq \lfloor n + 2 - 2\sqrt{\sigma(A) - n + 1} \rfloor.$$

*Proof.* Denote

$$g(n, k) = \begin{cases} \lfloor n^2/k \rfloor & \text{if } 1 \leq k \leq 4, \\ 2n - k + \lfloor (n - k)^2/4 \rfloor & \text{if } k \geq 5. \end{cases}$$

It is not hard to verify that for a given  $n$ ,  $g(n, k)$  is strictly decreasing in  $k$  for  $1 \leq k \leq n$ . By Theorem 2 and the strict monotonicity of  $g(n, k)$  in  $k$ , we deduce that if  $\sigma(A) > g(n, k)$  then  $\text{ind}(A) \leq k - 1$ .

When  $n^2 < 4\sigma(A)$ , we have  $k_1 \equiv \lfloor n^2/\sigma(A) \rfloor + 1 \leq 4$  and

$$\sigma(A) > \lfloor n^2/k_1 \rfloor.$$

Thus  $\text{ind}(A) \leq k_1 - 1 = \lfloor n^2/\sigma(A) \rfloor$ . Since  $A$  is irreducible,  $\sigma(A) \geq n$ . If  $\sigma(A) = n$ , the conclusion holds trivially. When  $n^2 \geq 4\sigma(A)$  and  $\sigma(A) \geq n + 1$ , we have

$$n \geq k_2 \equiv \lfloor n + 2 - 2\sqrt{\sigma(A) - n + 1} \rfloor + 1 \geq 5$$

and

$$\sigma(A) > 2n - k_2 + \lfloor (n - k_2)^2/4 \rfloor.$$

Thus  $\text{ind}(A) \leq k_2 - 1 = \lfloor n + 2 - 2\sqrt{\sigma(A) - n + 1} \rfloor$ . This completes the proof.  $\square$

We remark that Corollary 3 includes Lewin's result that if  $\sigma(A) > n^2/2$  then  $\text{ind}(A) = 1$ , i.e.,  $A$  is primitive.

The matrix of the quadratic form  $f$  in (3) is the  $k \times k$  matrix  $G_k/2$  where

$$G_k = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

By the way we point out two interesting properties of  $G_k$ :

- (i)  $\det G_{2m-1} = 2$  and  $\det G_{2m} = 2((-1)^m - 1)$ . Thus if  $4|k$ ,  $G_k$  is singular and otherwise  $G_k$  is nonsingular.
- (ii) When  $k \geq 3$  the maximum eigenvalue of  $G_k$  is always 2.

Finally we determine the minimum number of positive entries of an irreducible nonnegative matrix with a given imprimitivity index.

**Theorem 4.** *Let  $\Gamma(n, k)$  be the set of  $n \times n$  irreducible nonnegative matrices with imprimitivity index  $k$ . Then*

$$\min\{\sigma(A) | A \in \Gamma(n, k)\} = \begin{cases} n + 1 & \text{if } k < n, \\ n & \text{if } k = n. \end{cases}$$

*Proof.* Let  $A \in \Gamma(n, k)$ . Since  $A$  is irreducible,  $A$  cannot have zero rows or zero columns. Thus  $\sigma(A) \geq n$ . The case  $k = 1$ , i.e., when  $A$  is a primitive matrix, is easy

and well-known:  $\min\sigma(A) = n + 1$ . Now suppose  $k \geq 2$  and consider the matrix

$$A = \begin{pmatrix} 0 & A_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{k-1} \\ A_k & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad (9)$$

which is in its canonical form. If  $k = n$ , each  $A_i$  is forced to be  $1 \times 1$  and  $A$  with each  $A_i = (1)$  satisfies  $A \in \Gamma(n, n)$  and  $\sigma(A) = n$ .

Next we assume  $2 \leq k \leq n - 1$ . An  $n \times n$  irreducible nonnegative matrix with exact  $n$  positive entries has the same sign pattern as that of a permutation matrix, and hence has imprimitivity index  $n$ . Therefore

$$\min\{\sigma(A) | A \in \Gamma(n, k)\} \geq n + 1$$

for  $k < n$ . It remains to exhibit an  $A \in \Gamma(n, k)$  which attains this lower bound.

First we consider the case  $k|n$ . Let  $n = km$  with  $m \geq 2$ . Denote by  $I_t$  the identity matrix of order  $t$ . Define  $A$  to be the matrix in (9) with  $A_1 = A_2 = \cdots = A_{k-1} = I_m$  and with  $A_k$  being the matrix of order  $m$  whose only  $m + 1$  nonzero entries are  $A_k(1, 2) = A_k(2, 3) = \cdots = A_k(m - 1, m) = A_k(m, 1) = A_k(m, 2) = 1$ . Then the digraph of  $A$  is strongly connected, and hence  $A$  is irreducible. We have

$$A^k = \text{diag}(A_k, A_k, \cdots, A_k).$$

Since  $A_k$  is a primitive matrix (the Wielandt matrix),  $\text{ind}(A_k) = 1$ . Hence  $\text{ind}(A) = k$ . So  $A \in \Gamma(n, k)$  and  $\sigma(A) = n + 1$ .

Then we consider the case  $k \nmid n$ . Let  $n = km + r$  with  $1 \leq r < k$ . Denote  $e_m = (0, \cdots, 0, 1)^T \in \mathbb{R}^m$ . Define  $A$  to be the matrix in (9) with  $A_1 = A_2 = \cdots = A_{r-1} = I_{m+1}$ ,  $A_{r+1} = \cdots = A_{k-1} = I_m$ ,

$$A_r = \begin{pmatrix} I_m \\ e_m^T \end{pmatrix}, \quad A_k = (e_m, I_m).$$

The digraph of  $A$  with vertices  $1, 2, \dots, n$  is represented by the following arcs:

$$1 \rightarrow (m+1)+1 \rightarrow \cdots \rightarrow r(m+1)+1 \rightarrow r(m+1)+m+1 \rightarrow \cdots \rightarrow (k-1)m+r+1 \rightarrow 2$$

$$\begin{array}{c}
2 \rightarrow (m+1)+2 \rightarrow \cdots \rightarrow r(m+1)+2 \rightarrow r(m+1)+m+2 \rightarrow \cdots \rightarrow (k-1)m+r+2 \rightarrow 3 \\
\cdots \quad \cdots \quad \cdots \\
m \rightarrow \cdots \rightarrow r(m+1)-1 \rightarrow r(m+1)+m \rightarrow r(m+1)+2m \rightarrow \cdots \rightarrow km+r \rightarrow 1 \\
m+1 \rightarrow \cdots \rightarrow r(m+1) \rightarrow r(m+1)+m \\
km+r \rightarrow m+1.
\end{array}$$

This digraph contains two closed paths going through all the vertices. Thus it is strongly connected and  $A$  is irreducible. We also have

$$A^k = \text{diag}(B_1, B_2, \dots, B_k)$$

where  $B_1 = A_1 A_2 \cdots A_k$ ,  $B_2 = A_2 A_3 \cdots A_k A_1$ ,  $\dots$ ,  $B_k = A_k A_1 \cdots A_{k-1}$ . Obviously all the  $B_j$ ,  $j = 1, \dots, k$ , have the same nonzero eigenvalues. Observe that

$$B_k = A_k A_r$$

is irreducible and has a positive trace. So  $B_k$  is primitive and hence  $\text{ind}(B_k) = 1$ . Thus each  $B_j$ ,  $j = 1, \dots, k$  has exactly one eigenvalue with modulus equal to the spectral radius, and consequently  $\text{ind}(A) = k$ . We conclude that  $A \in \Gamma(n, k)$  and evidently  $\sigma(A) = n + 1$ . This completes the proof.  $\square$

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