

On the span of Hadamard products of vectors

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Abstract. Let A_1, \dots, A_k be positive semidefinite matrices and B_1, \dots, B_k arbitrary complex matrices of order n . We show that

$$\text{span}\{(A_1x) \circ (A_2x) \circ \dots \circ (A_kx) \mid x \in \mathbb{C}^n\} = \text{range}(A_1 \circ A_2 \circ \dots \circ A_k)$$

and

$$\text{span}\{(B_1x_1) \circ (B_2x_2) \circ \dots \circ (B_kx_k) \mid x_j \in \mathbb{C}^n\} = \text{range}((B_1B_1^*) \circ (B_2B_2^*) \circ \dots \circ (B_kB_k^*))$$

where \circ means the Hadamard product. This generalizes two recent results of Sun, Du and Liu.

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1. Introduction

For two $m \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$, their Hadamard product (entrywise product) is defined to be $A \circ B = (a_{ij}b_{ij})$. Note that when $n = 1$ the matrices are column vectors. Given a positive integer k , the k -th Hadamard power of A is $A^{(k)} = (a_{ij}^k)$. The book [4] contains many results on the Hadamard product.

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Throughout we consider complex matrices and denote by A^* the conjugate transpose of A . We regard an $n \times n$ matrix A as a linear operator on \mathbb{C}^n , so that $\text{range}(A)$ is the image of A . Sun, Du and Liu [5] have proved the following results.

Theorem 1. *For any $n \times n$ positive semidefinite matrix A and any positive integer k ,*

$$\text{span}\{(Ax)^{(k)} \mid x \in \mathbb{C}^n\} = \text{range}(A^{(k)}).$$

Theorem 2. *For any $n \times n$ complex matrix B and any positive integer k ,*

$$\text{span}\{(Bx)^{(k)} \mid x \in \mathbb{C}^n\} = \text{range}((BB^*)^{(k)}).$$

Theorem 2 follows immediately from Theorem 1. These two results were conjectured by Gorni and Tuta-j-Gasinska [2] in their study related to the well-known Jacobian conjecture which states that if $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial map and the determinant of the Jacobian matrix of f is a nonzero constant, then f is bijective.

In this note we will generalize Theorems 1 and 2 to the case of Hadamard product of different matrices. The basic ideas in our proof are similar to those in [5], but the proof here is simpler.

2. Main results

We need the following fact, which is known as the principal submatrix rank property [3]. For the sake of completeness, we give a short proof.

Lemma 3. *Let A, B, C be complex matrices such that*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

is positive semidefinite. Then $\text{range}(B) \subseteq \text{range}(A)$.

Proof. Since the given block matrix is positive semidefinite, there exists a contraction W such that $B = A^{1/2}WC^{1/2}$ [6, p.15]. Thus $\text{range}(B) \subseteq \text{range}(A^{1/2}) = \text{range}(A)$, where we used the fact that $\text{range}(G) = \text{range}(GG^*)$ for any complex matrix G . \square

Theorem 4. *Let $A_j, j = 1, \dots, k$, be $n \times n$ positive semidefinite matrices. Then*

$$\text{span}\{(A_1x) \circ (A_2x) \circ \dots \circ (A_kx) \mid x \in \mathbb{C}^n\} = \text{range}(A_1 \circ A_2 \circ \dots \circ A_k).$$

Proof. Let e_i be the vector in \mathbb{C}^n whose only nonzero component is the i -th component which is equal to 1. Then the i -th column of $A_1 \circ \cdots \circ A_k$ is

$$(A_1 \circ \cdots \circ A_k)e_i = (A_1 e_i) \circ \cdots \circ (A_k e_i),$$

$i = 1, \dots, n$. Therefore

$$\text{range}(A_1 \circ A_2 \circ \cdots \circ A_k) \subseteq \text{span}\{(A_1 x) \circ (A_2 x) \circ \cdots \circ (A_k x) \mid x \in \mathbb{C}^n\}. \quad (1)$$

It remains for us to prove the reversed inclusion relation

$$\text{span}\{(A_1 x) \circ (A_2 x) \circ \cdots \circ (A_k x) \mid x \in \mathbb{C}^n\} \subseteq \text{range}(A_1 \circ A_2 \circ \cdots \circ A_k). \quad (2)$$

Let

$$A_j = (a_1^{[j]}, a_2^{[j]}, \dots, a_n^{[j]}), \quad j = 1, \dots, k$$

and $x = (x_1, x_2, \dots, x_n)^T$. Then

$$(A_1 x) \circ (A_2 x) \circ \cdots \circ (A_k x) = \sum x_{i_1} x_{i_2} \cdots x_{i_k} a_{i_1}^{[1]} \circ a_{i_2}^{[2]} \circ \cdots \circ a_{i_k}^{[k]}$$

where the summation is taken over all tuples (i_1, i_2, \dots, i_k) with $1 \leq i_t \leq n$. Hence, to prove (2) it suffices to show

$$a_{i_1}^{[1]} \circ a_{i_2}^{[2]} \circ \cdots \circ a_{i_k}^{[k]} \in \text{range}(A_1 \circ A_2 \circ \cdots \circ A_k) \quad (3)$$

for all tuples (i_1, i_2, \dots, i_k) with $1 \leq i_t \leq n$. For each t with $1 \leq t \leq k$, there is a permutation matrix P_t such that $a_{i_t}^{[t]}$ is the first column of $A_t P_t$. So $a_{i_1}^{[1]} \circ a_{i_2}^{[2]} \circ \cdots \circ a_{i_k}^{[k]}$ is the first column of $(A_1 P_1) \circ (A_2 P_2) \circ \cdots \circ (A_k P_k)$. Now, (3) will follow from

$$\text{range}((A_1 P_1) \circ (A_2 P_2) \circ \cdots \circ (A_k P_k)) \subseteq \text{range}(A_1 \circ A_2 \circ \cdots \circ A_k). \quad (4)$$

Next we prove (4). Let I be the identity matrix. Choose an arbitrary but fixed real number r such that r is bigger than the spectral radius of A_j for all $1 \leq j \leq k$. Then by the Schur complement criterion [6, p.5] we see that

$$\begin{pmatrix} A_j & A_j P_j \\ P_j^* A_j & rI \end{pmatrix}$$

is positive semidefinite. The Schur product theorem ([1, p.23] or [6, p.8]) asserts that the Hadamard product of two positive semidefinite matrices is positive semidefinite.

So

$$\begin{aligned} & \begin{pmatrix} A_1 \circ A_2 \circ \cdots \circ A_k & (A_1 P_1) \circ (A_2 P_2) \circ \cdots \circ (A_k P_k) \\ (P_1^* A_1) \circ (P_2^* A_2) \circ \cdots \circ (P_k^* A_k) & r^k I \end{pmatrix} \\ &= \begin{pmatrix} A_1 & A_1 P_1 \\ P_1^* A_1 & rI \end{pmatrix} \circ \begin{pmatrix} A_2 & A_2 P_2 \\ P_2^* A_2 & rI \end{pmatrix} \circ \cdots \circ \begin{pmatrix} A_k & A_k P_k \\ P_k^* A_k & rI \end{pmatrix} \end{aligned}$$

is positive semidefinite. Applying Lemma 3 we obtain (4). This completes the proof.

□

The relations (1) and (3) in the proof of Theorem 4 yield the following result.

Theorem 5. *Let $A_j, j = 1, \dots, k$, be $n \times n$ positive semidefinite matrices. Then*

$$\text{span}\{(A_1 x_1) \circ (A_2 x_2) \circ \cdots \circ (A_k x_k) \mid x_j \in \mathbb{C}^n\} = \text{range}(A_1 \circ A_2 \circ \cdots \circ A_k).$$

Combining Theorem 4 and Theorem 5 we get the following interesting conclusion:

If $A_j, j = 1, \dots, k$, are $n \times n$ positive semidefinite matrices then

$$\text{span}\{(A_1 x_1) \circ (A_2 x_2) \circ \cdots \circ (A_k x_k) \mid x_j \in \mathbb{C}^n\} = \text{span}\{(A_1 x) \circ (A_2 x) \circ \cdots \circ (A_k x) \mid x \in \mathbb{C}^n\}.$$

The direct generalization of Theorem 2 would be

$$\text{span}\{(B_1 x) \circ (B_2 x) \circ \cdots \circ (B_k x) \mid x \in \mathbb{C}^n\} = \text{range}((B_1 B_1^*) \circ (B_2 B_2^*) \circ \cdots \circ (B_k B_k^*))$$

for $n \times n$ complex matrices $B_j, j = 1, \dots, k$. We point out that this is not true in general. Consider the example $n = 2$,

$$B_1 = I, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\text{span}\{(B_1 x) \circ (B_2 x) \mid x \in \mathbb{C}^2\} = \{(\alpha, \alpha)^T \mid \alpha \in \mathbb{C}\} \neq \mathbb{C}^2 = \text{range}((B_1 B_1^*) \circ (B_2 B_2^*)).$$

This example also shows that the condition that $A_j, j = 1, \dots, k$, be positive semidefinite in Theorems 4 and 5 cannot be removed. The correct extension of Theorem 2 seems to be the following result.

Theorem 6. *Let $B_j, j = 1, \dots, k$, be $n \times n$ complex matrices. Then*

$$\text{span}\{(B_1x_1) \circ (B_2x_2) \circ \dots \circ (B_kx_k) \mid x_j \in \mathbb{C}^n\} = \text{range}((B_1B_1^*) \circ (B_2B_2^*) \circ \dots \circ (B_kB_k^*)).$$

Proof. By Theorem 5 and the fact that $\text{range}(BB^*) = \text{range}(B)$ for any complex matrix B , we have

$$\begin{aligned} & \text{range}((B_1B_1^*) \circ (B_2B_2^*) \circ \dots \circ (B_kB_k^*)) \\ &= \text{span}\{(B_1B_1^*y_1) \circ (B_2B_2^*y_2) \circ \dots \circ (B_kB_k^*y_k) \mid y_j \in \mathbb{C}^n\} \\ &= \text{span}\{(B_1x_1) \circ (B_2x_2) \circ \dots \circ (B_kx_k) \mid x_j \in \mathbb{C}^n\}. \end{aligned}$$

This completes the proof. \square

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