On the span of Hadamard products of vectors

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Abstract. Let $A_1, \ldots, A_k$ be positive semidefinite matrices and $B_1, \ldots, B_k$ arbitrary complex matrices of order n. We show that

$$\text{span}\{(A_1x) \circ (A_2x) \circ \cdots \circ (A_kx) |\, x \in \mathbb{C}^n\} = \text{range}(A_1 \circ A_2 \circ \cdots \circ A_k)$$

and

$$\text{span}\{(B_1x_1) \circ (B_2x_2) \circ \cdots \circ (B_kx_k) |\, x_j \in \mathbb{C}^n\} = \text{range}((B_1B_1^*) \circ (B_2B_2^*) \circ \cdots \circ (B_kB_k^*))$$

where $\circ$ means the Hadamard product. This generalizes two recent results of Sun, Du and Liu.

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1. Introduction

For two $m \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$, their Hadamard product (entrywise product) is defined to be $A \circ B = (a_{ij}b_{ij})$. Note that when $n = 1$ the matrices are column vectors. Given a positive integer k, the k-th Hadamard power of A is $A^{(k)} = (a_{ij}^k)$. The book [4] contains many results on the Hadamard product.

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Throughout we consider complex matrices and denote by A^* the conjugate transpose of A. We regard an $n \times n$ matrix A as a linear operator on $\mathbb{C}^n$, so that range(A) is the image of A. Sun, Du and Liu [5] have proved the following results.

Theorem 1. *For any $n \times n$ positive semidefinite matrix A and any positive integer k,*

$$\text{span}\{(Ax)^{(k)} | \, x \in \mathbb{C}^n\} = \text{range}(A^{(k)}).$$

Theorem 2. *For any $n \times n$ complex matrix B and any positive integer k,*

$$\text{span}\{(Bx)^{(k)} | \, x \in \mathbb{C}^n\} = \text{range}((BB^*)^{(k)}).$$

Theorem 2 follows immediately from Theorem 1. These two results were conjectured by Gorni and Tutaj-Gasinska [2] in their study related to the well-known Jacobian conjecture which states that if $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map and the determinant of the Jacobian matrix of f is a nonzero constant, then f is bijective.

In this note we will generalize Theorems 1 and 2 to the case of Hadamard product of different matrices. The basic ideas in our proof are similar to those in [5], but the proof here is simpler.

2. Main results

We need the following fact, which is known as the principal submatrix rank property [3]. For the sake of completeness, we give a short proof.

Lemma 3. *Let A, B, C be complex matrices such that*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

is positive semidefinite. Then range(B) $\subseteq$ range(A).

Proof. Since the given block matrix is positive semidefinite, there exists a contraction W such that $B = A^{1/2}WC^{1/2}$ [6, p.15]. Thus range(B) $\subseteq$ range($A^{1/2}$) $=$ range(A), where we used the fact that range(G) $=$ range(GG^*) for any complex matrix G. $\square$

Theorem 4. *Let $A_j, j = 1, \ldots, k$, be $n \times n$ positive semidefinite matrices. Then*

$$\text{span}\{(A_1x) \circ (A_2x) \circ \cdots \circ (A_kx) | \, x \in \mathbb{C}^n\} = \text{range}(A_1 \circ A_2 \circ \cdots \circ A_k).$$

Proof. Let e_i be the vector in $\mathbb{C}^n$ whose only nonzero component is the i-th component which is equal to 1. Then the i-th column of $A_1 \circ \cdots \circ A_k$ is

$$(A_1 \circ \cdots \circ A_k)e_i = (A_1 e_i) \circ \cdots \circ (A_k e_i),$$

$i = 1, \ldots, n$. Therefore

$$\mathrm{range}(A_1 \circ A_2 \circ \cdots \circ A_k) \subseteq \mathrm{span}\{(A_1 x) \circ (A_2 x) \circ \cdots \circ (A_k x) |\, x \in \mathbb{C}^n\}. \qquad (1)$$

It remains for us to prove the reversed inclusion relation

$$\mathrm{span}\{(A_1 x) \circ (A_2 x) \circ \cdots \circ (A_k x) |\, x \in \mathbb{C}^n\} \subseteq \mathrm{range}(A_1 \circ A_2 \circ \cdots \circ A_k). \qquad (2)$$

Let

$$A_j = (a_1^{[j]}, a_2^{[j]}, \ldots, a_n^{[j]}), \quad j = 1, \ldots, k$$

and $x = (x_1, x_2, \ldots, x_n)^T$. Then

$$(A_1 x) \circ (A_2 x) \circ \cdots \circ (A_k x) = \sum x_{i_1} x_{i_2} \cdots x_{i_k} a_{i_1}^{[1]} \circ a_{i_2}^{[2]} \circ \cdots \circ a_{i_k}^{[k]}$$

where the summation is taken over all tuples $(i_1, i_2, \ldots, i_k)$ with $1 \le i_t \le n$. Hence, to prove (2) it suffices to show

$$a_{i_1}^{[1]} \circ a_{i_2}^{[2]} \circ \cdots \circ a_{i_k}^{[k]} \in \mathrm{range}(A_1 \circ A_2 \circ \cdots \circ A_k) \qquad (3)$$

for all tuples $(i_1, i_2, \ldots, i_k)$ with $1 \le i_t \le n$. For each t with $1 \le t \le k$, there is a permutation matrix P_t such that $a_{i_t}^{[t]}$ is the first column of $A_t P_t$. So $a_{i_1}^{[1]} \circ a_{i_2}^{[2]} \circ \cdots \circ a_{i_k}^{[k]}$ is the first column of $(A_1 P_1) \circ (A_2 P_2) \circ \cdots \circ (A_k P_k)$. Now, (3) will follow from

$$\mathrm{range}((A_1 P_1) \circ (A_2 P_2) \circ \cdots \circ (A_k P_k)) \subseteq \mathrm{range}(A_1 \circ A_2 \circ \cdots \circ A_k). \qquad (4)$$

Next we prove (4). Let I be the identity matrix. Choose an arbitrary but fixed real number r such that r is bigger than the spectral radius of A_j for all $1 \le j \le k$. Then by the Schur complement criterion [6, p.5] we see that

$$\begin{pmatrix} A_j & A_j P_j \\ P_j^* A_j & rI \end{pmatrix}$$

is positive semidefinite. The Schur product theorem ([1, p.23] or [6, p.8]) asserts that the Hadamard product of two positive semidefinite matrices is positive semidefinite. So

$$\begin{pmatrix} A_1 \circ A_2 \circ \cdots \circ A_k & (A_1P_1) \circ (A_2P_2) \circ \cdots \circ (A_kP_k) \\ (P_1^*A_1) \circ (P_2^*A_2) \circ \cdots \circ (P_k^*A_k) & r^kI \end{pmatrix}$$
$$= \begin{pmatrix} A_1 & A_1P_1 \\ P_1^*A_1 & rI \end{pmatrix} \circ \begin{pmatrix} A_2 & A_2P_2 \\ P_2^*A_2 & rI \end{pmatrix} \circ \cdots \circ \begin{pmatrix} A_k & A_kP_k \\ P_k^*A_k & rI \end{pmatrix}$$

is positive semidefinite. Applying Lemma 3 we obtain (4). This completes the proof. □

The relations (1) and (3) in the proof of Theorem 4 yield the following result.

Theorem 5. *Let $A_j, j = 1, \ldots, k$, be $n \times n$ positive semidefinite matrices. Then*

$$\mathrm{span}\{(A_1x_1) \circ (A_2x_2) \circ \cdots \circ (A_kx_k) |\, x_j \in \mathbb{C}^n\} = \mathrm{range}(A_1 \circ A_2 \circ \cdots \circ A_k).$$

Combining Theorem 4 and Theorem 5 we get the following interesting conclusion: If $A_j, j = 1, \ldots, k$, are $n \times n$ positive semidefinite matrices then

$$\mathrm{span}\{(A_1x_1)\circ(A_2x_2)\circ\cdots\circ(A_kx_k) |\, x_j \in \mathbb{C}^n\} = \mathrm{span}\{(A_1x)\circ(A_2x)\circ\cdots\circ(A_kx) |\, x \in \mathbb{C}^n\}.$$

The direct generalization of Theorem 2 would be

$$\mathrm{span}\{(B_1x) \circ (B_2x) \circ \cdots \circ (B_kx) |\, x \in \mathbb{C}^n\} = \mathrm{range}((B_1B_1^*) \circ (B_2B_2^*) \circ \cdots \circ (B_kB_k^*))$$

for $n \times n$ complex matrices $B_j, j = 1, \ldots, k$. We point out that this is not true in general. Consider the example $n = 2$,

$$B_1 = I, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\mathrm{span}\{(B_1x) \circ (B_2x) |\, x \in \mathbb{C}^2\} = \{(\alpha, \alpha)^T |\, \alpha \in \mathbb{C}\} \neq \mathbb{C}^2 = \mathrm{range}((B_1B_1^*) \circ (B_2B_2^*)).$$

This example also shows that the condition that $A_j, j = 1, \ldots, k$, be positive semidefinite in Theorems 4 and 5 cannot be removed. The correct extension of Theorem 2 seems to be the following result.

Theorem 6. *Let $B_j, j = 1, \dots, k$, be $n \times n$ complex matrices. Then*

$$\mathrm{span}\{(B_1x_1)\circ(B_2x_2)\circ\cdots\circ(B_kx_k)|\, x_j \in \mathbb{C}^n\} = \mathrm{range}((B_1B_1^*)\circ(B_2B_2^*)\circ\cdots\circ(B_kB_k^*)).$$

Proof. By Theorem 5 and the fact that $\mathrm{range}(BB^*) = \mathrm{range}(B)$ for any complex matrix B, we have

$$\begin{aligned} & \mathrm{range}((B_1B_1^*)\circ(B_2B_2^*)\circ\cdots\circ(B_kB_k^*)) \\ = \;& \mathrm{span}\{(B_1B_1^*y_1)\circ(B_2B_2^*y_2)\circ\cdots\circ(B_kB_k^*y_k)|\, y_j \in \mathbb{C}^n\} \\ = \;& \mathrm{span}\{(B_1x_1)\circ(B_2x_2)\circ\cdots\circ(B_kx_k)|\, x_j \in \mathbb{C}^n\}. \end{aligned}$$

This completes the proof. □

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References

[1] R. Bhatia, Matrix Analysis, GTM 169, Springer, New York, 1997.

[2] G. Gorni and H. Tutaj-Gasinska, On the entrywise powers of matrices, Comm. Algebra, 32(2004), no.2, 495-520.

[3] D. Hershkowitz and H. Schneider, Lyapunov diagonal semistability of real H-matrices, Linear Algebra Appl., 71(1985), 119-149.

[4] R.A. Horn and C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.

[5] X. Sun, X. Du and D. Liu, On the range of a Hadamard power of a positive semidefinite matrix, Linear Algebra Appl., 416(2006), 868-871.

[6] X. Zhan, Matrix Inequalities, LNM 1790, Springer, Berlin, 2002.