Extremal digraphs whose walks with the same initial and terminal vertices have distinct lengths

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Digraphs here allow loops but do not allow multiple arcs. The number of vertices in a digraph is called its order and the number of arcs its size. For digraphs, cycles and walks will mean directed cycles and directed walks respectively.

For a given positive integer n, let $\Theta(n)$ denote the set of digraphs of order n in which any two walks with the same initial vertex and the same terminal vertex have distinct lengths.

Thus, for a digraph D on the vertices $1, 2, \ldots, n$, $D \in \Theta(n)$ if and only if for every pair of vertices i, j and for every positive integer k there is at most one walk of length k from i to j. Let $\theta(n)$ denote the maximum size of a digraph in $\Theta(n)$.

We consider the following

Problem 1 For a given positive integer n, determine $\theta(n)$ and the digraphs in $\Theta(n)$ that attain the size $\theta(n)$.

The motivation for studying Problem 1 is to explore the relation between the size and the walks of a digraph. Intuitively $\theta(n)$ cannot be very large compared with n^2, while the structure of the extremal digraphs attaining $\theta(n)$ seems unclear.

Problem 1 has an equivalent matrix version.

For a given positive integer n, denote by

$M_n\{0,1\}$ the set of $n \times n$ 0-1 matrices,

$\Gamma(n) = \{A \in M_n\{0,1\} | A^k \in M_n\{0,1\} \text{ for every positive integer } k\}$,

$f(A)$: the number of 1's in a matrix A, and

$\gamma(n) = \max\{f(A) | A \in \Gamma(n)\}$.

We denote by $J_{r,t}$ the $r \times t$ matrix with each entry equal to 1.

For $A \in M_n\{0,1\}$ and a given positive integer k, $A^k \in M_n\{0,1\}$ if and only if in the digraph of A, for every pair of vertices i, j there is at most one walk of length k from i to j.

Thus, considering the adjacency matrix of a digraph we see that Problem 1 is equivalent to the following

Problem 2 For a given positive integer n, determine $\gamma(n)$ and the matrices in $\Gamma(n)$ that attain $\gamma(n)$.

Theorem 1 Let n be a positive integer. Then

$$\theta(n) = \begin{cases} \frac{(n+1)^2}{4} & \text{if } n \text{ is odd,} \\ \frac{n(n+2)}{4} & \text{if } n \text{ is even.} \end{cases}$$

A digraph $D \in \Theta(n)$ has size $\theta(n)$ if and only if the adjacency matrix of D is permutation similar to

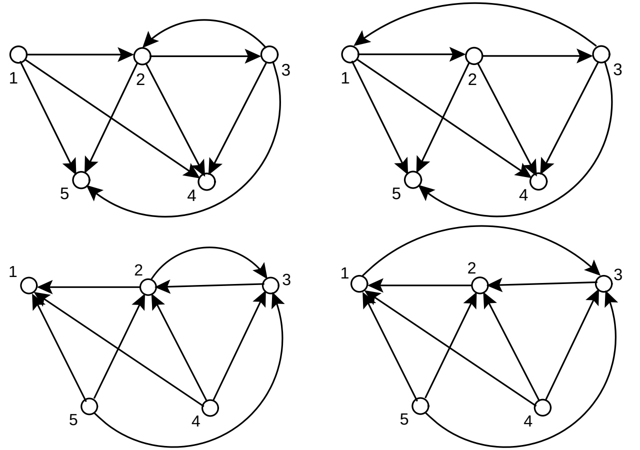
$$\begin{pmatrix} U & E & J_{r,t} \\ 0 & P & J_{s,t} \\ 0 & 0 & 0 \end{pmatrix}$$

or its transpose, where P is a permutation matrix and it does appear, U is a strictly upper triangular matrix, there is exactly one entry 1 in each row of (U, E), $t = (n-1)/2$ if n is odd and $t = n/2 - 1$ or $n/2$ if n is even.

The first assertion of Theorem 1 can be interpreted as a Ramsey type result:

If a digraph of order n has size larger than $\theta(n)$, then there exist two walks of the same length with the same initial vertex and the same terminal vertex.

The extremal loopless digraphs of order 5:



A related problem and some results

For given integers n and k, denote

$$\begin{aligned}\Delta(n,k) &= \{A | A \in M_n\{0,1\} \text{ and } A^k \in M_n\{0,1\}\},\\ \delta(n,k) &= \max\{f(A) | A \in \Delta(n,k)\}.\end{aligned}$$

Problem 3 For given positive integers n and k, determine $\delta(n,k)$ as well as the matrices in $\Delta(n,k)$ that attain $\delta(n,k)$.

This problem has also a graphic version.

Several solved cases of Problem 3:

Theorem 2[Wu, 2010]

$$\delta(n,2)=\begin{cases}\frac{n^2+4n-1}{4}, & \text{if } n \text{ is odd},\\ \frac{n^2+4n-4}{4}, & \text{if } n \text{ is even and } n\neq 4,\\ 8, & \text{if } n=4\end{cases}$$

Wu also determined the matrices in $\Delta(n,2)$ attaining $\delta(n,2)$.

Theorem 3 Let n, k be given integers with $n \geq 5$ and $k \geq n-1$. Then $\delta(n,k) = n(n-1)/2$ and a matrix $A \in \Delta(n,k)$ satisfies $f(A) = n(n-1)/2$ if and only if A is permutation similar to

$$\begin{pmatrix} 0 & 1 & \cdots & 1 \\ & \ddots & \ddots & \vdots \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}.$$

Theorem 4 If $n \geq 6$ then

$$\delta(n, n-2) = \frac{n(n-1)}{2} - 1.$$

If $n \geq 7$ then

$$\delta(n, n-3) = \frac{n(n-1)}{2} - 2.$$

In view of Theorems 3, 4 above, one might conjecture that for $2 \leq k \leq n-2$,

$$\gamma(n,k) = \frac{n(n-1)}{2} - (n-k-1). \qquad (*)$$

This is not the case. Wu's Theorem 2 on squares already indicates that (*) is false for $k=2$. In fact, there are other values of $k>2$ for which (*) is false. We have proved that at least one of $\gamma(10,4)$ and $\gamma(11,4)$ does not satisfy (*).

Problem 3 is still open in general.

References

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Thank you!