Open Problems in Matrix Theory

Xingzhi Zhan

zhan@math.ecnu.edu.cn

East China Normal University

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Here I can describe only the problems. For their background and current state please see the Proceedings.

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1, Existence of Hadamard matrices

A Hadamard matrix is a square matrix with entries equal to ± 1 whose rows and hence columns are mutually orthogonal. In other words, a Hadamard matrix of order n is a $\{1, -1\}$-matrix A satisfying

$$AA^T = nI$$

where I is the identity matrix.

In 1867 Sylvester proposed a method to construct Hadamard matrices of order 2^k.

In 1933 Paley stated that the order n $(n \geq 4)$ of any Hadamard matrix is divisible by 4. This is easy to prove. The converse has been a long-standing conjecture.

Conjecture 1 *For every positive integer n, there exists a Hadamard matrix of order $4n$.*

2, Characterization of the eigenvalues of nonnegative matrices

The *nonnegative inverse eigenvalue problem* is

Problem 2 (Suleimanova, 1949) *Determine necessary and sufficient conditions for a set of* n *complex numbers to be the eigenvalues of a nonnegative matrix of order* n*.*

The *real nonnegative inverse eigenvalue problem* is
Problem 3 (Suleimanova, 1949) *Determine necessary and sufficient conditions for a set of* n *real numbers to be the eigenvalues of a nonnegative matrix of order* n*.*

The *symmetric nonnegative inverse eigenvalue problem* is
Problem 4 (Fiedler, 1974) *Determine necessary and sufficient conditions for a set of* n *real numbers to be the eigenvalues of a symmetric nonnegative matrix of order* n*.*

3, The permanental dominance conjecture

S_n: the symmetric group on $\{1, 2, \ldots, n\}$
M_n: the set of complex matrices of order n.
Suppose G is a subgroup of S_n and χ is a character of G. The *generalized matrix function* $d_\chi : M_n \to \mathbf{C}$ is defined by

$$d_\chi(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $A = (a_{ij})$.

Let χ be an irreducible character of G and e be the identity permutation in S_n. The normalized generalized matrix function is defined as

$$\bar{d}_\chi(A) = d_\chi(A)/\chi(e).$$

Conjecture 5 (Lieb, 1966) (The permanental dominance conjecture) *Suppose G is a subgroup of S_n and χ is an irreducible character of G. Then for any positive semidefinite matrix A of order n,*

$$\mathsf{per} A \geq \bar{d}_\chi(A).$$

This conjecture can be traced back to Schur's work in 1918.

We order the elements of S_n lexicographically to obtain a sequence L_n. For $A = (a_{ij}) \in M_n$ the *Schur power* of A, denoted by $\Pi(A)$, is the matrix of order $n!$ whose rows and columns are indexed by L_n and whose $(\sigma, \tau)-$entry is $\prod_{i=1}^{n} a_{\sigma(i),\tau(i)}$. Since $\Pi(A)$ is a principal submatrix of $\otimes^n A$, if A is positive semidefinite then so is $\Pi(A)$. It is not difficult to see that both $\mathsf{per} A$ and $\det A$ are eigenvalues of $\Pi(A)$. A result of Schur asserts that if A is positive semidefinite then $\det A$ is the smallest eigenvalue of $\Pi(A)$.

Conjecture 6(Soules, 1966) (The "permanent on top" conjecture) *If the matrix A is positive semidefinite, then* $\mathsf{per} A$ *is the largest eigenvalue of* $\Pi(A)$.

Conjecture 6, if true, implies Conjecture 5.

4, The Marcus-de Oliveira conjecture

S_n: the symmetric group on $\{1, 2, \ldots, n\}$
$\mathsf{co}\Omega$: the convex hull of a set Ω in the complex plane.

Conjecture 7 (Marcus, 1973 and de Oliveira, 1982) *Let A, B be normal complex matrices of order n with eigenvalues $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively. Then*

$$\mathsf{det}(A + B) \in \mathsf{co}\left\{\prod_{i=1}^{n}\left(x_i + y_{\sigma(i)}\right) : \sigma \in S_n\right\}.$$

5, Permanents of Hadamard matrices

Question 8(Wang, 1974) *Can the permanent of a Hadamard matrix of order n vanish for $n > 2$?*

Wanless showed that the answer is negative for $2 < n < 32$.

6, The Bessis-Moussa-Villani trace conjecture

In 1975, while studying partition functions of quantum mechanical systems, Bessis, Moussa and Villani formulated the conjecture that if A, B are Hermitian matrices of the same order with B positive semidefinite then the function

$$f(t) = \mathsf{Tr}\exp(A - tB)$$

is the Laplace transform of a positive measure on $[0, \infty)$, where t is a real variable and Tr means trace.

In 2004 Lieb and Seiringer proved that this conjecture is equivalent to the following

Conjecture 9 (Bessis-Moussa-Villani) *Let A, B be positive semidefinite matrices of order n and let k be a positive integer. Then the polynomial $p(t) = \mathsf{Tr}\,(A + tB)^k$ has all nonnegative coefficients.*

An S-matrix of order n is a 0-1 matrix formed by taking a Hadamard matrix of order $n+1$ in which the entries in the first row and column are 1, changing 1's to 0's and -1's to 1's, and deleting the first row and column. Let $\|\cdot\|_F$ denote the Frobenius norm.

Conjecture 10 (Sloane and Harwit, 1976) *If A is a nonsingular matrix of order n all of whose entries are in the interval $[0, 1]$, then*

$$\|A^{-1}\|_F \geq \frac{2n}{n+1}.$$

Equality holds if and only if A is an S-matrix.

8, Foregger's conjecture on minimum values of permanents

A fully indecomposable square matrix A is called *nearly decomposable* if whenever a nonzero entry of A is replaced with a 0, the resulting matrix is partly decomposable.

Conjecture 11 (Foregger, 1980) *If A is a nearly decomposable doubly stochastic matrix of order n, then*

$$\mathrm{per} A \geq 2^{1-n}.$$

Note that this lower bound can be attained at $A = (I + P)/2$ where P is the permutation matrix corresponding to the permutation cycle $(1234 \cdots n)$.

9, The Brualdi-Li conjecture on tournament matrices

A tournament matrix is a square 0-1 matrix A satisfying $A + A^T = J - I$ where J is the all ones matrix. Such matrices arise from the results of round robin competitions.

T_n: the set of $n \times n$ tournament matrices.

$A \in T_n$ is called *regular* if each of the row sums of A is $(n-1)/2$.

It is known that for odd n the regular tournament matrices maximize the Perron root over T_n.

Let U_k be the strictly upper triangular matrix of order k with ones above the main diagonal.

Conjecture 12 (Brualdi and Li, 1983) *For even n, the matrix*

$$\begin{bmatrix} U_{n/2} & U_{n/2}^T \\ U_{n/2}^T + I & U_{n/2} \end{bmatrix}$$

maximizes the Perron root over T_n.

10, A possible generalization of the Perron-Frobenius theorem

Let $A = (A_{ij})_{n \times n}$ be a block matrix of order nm, where each A_{ij} is a positive semidefinite matrix of order m. Let us call such matrices *block positive semidefinite (BPSD)*. Note that when $m = 1$, A is a nonnegative matrix, while when $n = 1$, A is a positive semidefinite matrix. Thus BPSD matrices interpolate two familiar classes of matrices. Both nonnegative matrices and positive semidefinite matrices have the Perron-Frobenius property: The spectral radius is an eigenvalue.

Numerical experiments show that some BPSD matrices have the Perron-Frobenius property while some others do not.

Problem 13 (Horn, 1988) *Let A be a BPSD matrix. Give necessary and/or sufficient conditions on A such that the spectral radius $\rho(A)$ is an eigenvalue of A. More generally, study the properties of the eigenvalues and eigenvectors of BPSD matrices.*

An $n \times n$ real matrix A is called *completely positive* (CP) if, for some m, there exists an $n \times m$ nonnegative matrix B such that $A = BB^T$. The smallest such m is called the *CP-rank* of A. CP matrices have applications in block designs.

Conjecture 14 (Drew, Johnson and Loewy, 1994) *If A is a CP matrix of order $n \geq 4$, then*

$$\textit{CP-rank}(A) \leq \lfloor n^2/4 \rfloor.$$

It is known that for each $n \geq 4$ the conjectured upper bound $\lfloor n^2/4 \rfloor$ can be attained.

12, Bhatia-Kittaneh's question on singular values

Denote by $s_1(X) \geq s_2(X) \geq \cdots$ the ordered singular values of a complex matrix X.

Question 15 (Bhatia and Kittaneh, 2000) *Let A, B be positive semidefinite matrices of order n. Is it true that*

$$s_j^{1/2}(AB) \leq \frac{1}{2} s_j(A+B), \quad j = 1, 2, \ldots, n?$$

Since the square function $f(t) = t^2$ is operator convex on $\mathbf{R}$, this inequality is stronger than the known inequality

$$2s_j(XY^*) \leq s_j(X^*X + Y^*Y), \quad j = 1, 2, \ldots, n$$

for any complex matrices X, Y of order n due to the same authors.

13, Convergence of the iterated Aluthge transforms

Every square complex matrix A has the polar decomposition $A = UP$ where U is unitary and P is positive semidefinite. The *Aluthge transform* of A is

$$\Delta(A) = P^{1/2} U P^{1/2}.$$

Though the unitary factor in the polar decomposition is not unique when A is singular, the Aluthge transform is well defined, that is, it does not depend on the choice made for the unitary factor.

Let $B(H)$ be the algebra of bounded linear operators on a Hilbert space H. The Aluthge transform can also be defined for operators in $B(H)$. In 2000, Jung, Ko and Pearcy conjectured that for any $T \in B(H)$, the sequence $\{\Delta^m(T)\}_{m=1}^{\infty}$ is norm convergent to an operator. Here $\Delta^1(T) = \Delta(T)$ and $\Delta^m(T) = \Delta(\Delta^{m-1}(T))$, $m = 2, 3, \ldots$.

However Cho, Jung and Lee showed that this conjecture is false for infinite dimensional Hilbert spaces.

So there remains the possibility that it holds in finite dimensions:

Conjecture 16 *Let A be a square complex matrix. Then the sequence $\{\Delta^m(A)\}_{m=1}^{\infty}$ converges.*

14, Expressing real matrices as linear combinations of orthogonal matrices

In 2002, Li and Poon proved that every square real matrix is a linear combination of 4 orthogonal matrices, i.e., given a square real matrix A, there exist real orthogonal matrices Q_i and real numbers r_i, $i = 1, 2, 3, 4$ (depending on A, of course) such that

$$A = r_1 Q_1 + r_2 Q_2 + r_3 Q_3 + r_4 Q_4.$$

They asked the following

Question 17 *Is the number 4 of the terms in the above expression least possible?*

15, Sign patterns

Research on sign patterns of matrices is active now and there are many open problems in that field.

Let $f(A)$ be the number of positive entries of a nonnegative matrix A.

Problem 18 (Z, 2005) *Characterize those sign patterns of square nonnegative matrices A such that the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is nondecreasing.*

Sidak observed in 1964 that there exists a primitive nonnegative matrix A of order 9 satisfying

$$18 = f(A) > f(A^2) = 16.$$

This is the motivation.

16, Monotonicity of a geometric mean of positive definite matrices

$\mathbf{P}_n$: the set of positive definite matrices of order n.

The distance $\delta(A, B)$ between A, $B \in \mathbf{P}_n$ is the infimum of lengths of curves in $\mathbf{P}_n$ that connect A to B. It can be proved that $\delta(A, B) = \|\log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})\|_F$.

Given $A_i \in \mathbf{P}_n$, $i = 1, 2, \ldots, k$, there is a unique matrix in $\mathbf{P}_n$, denoted $G(A_1, \ldots, A_k)$, that minimizes the function

$$f(X) = \sum_{i=1}^{k} \delta^2(A_i, X).$$

$G(A_1, \ldots, A_k)$ is called the *geometric mean* of $A_1, \ldots, A_k$. This geometric mean is symmetric, invariant under congruence, and continuous.

$\leq$: the Loewner partial order.

Conjecture 19 (Bhatia and Holbrook, 2006) *The mean G is monotone with respect to its arguments, i.e., if A_i, $B_i \in \mathbf{P}_n$ satisfy $A_i \leq B_i$, $i = 1, \ldots, k$, then*

$$G(A_1, \ldots, A_k) \leq G(B_1, \ldots, B_k).$$

17, Eigenvalues of real symmetric matrices

$S_n[a,b]$: the set of $n \times n$ real symmetric matrices whose entries are in the interval $[a, b]$.
For an $n \times n$ real symmetric matrix A, we always denote the eigenvalues of A in decreasing order by $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$.

Problem 20 (Z, 2006) *For a given j with $2 \leq j \leq n-1$, determine*

$$\max\{\lambda_j(A) : A \in S_n[a,b]\},$$

$$\min\{\lambda_j(A) : A \in S_n[a,b]\}$$

and determine the matrices that attain the maximum and the matrices that attain the minimum.

The cases $j = 1, n$ are solved.

18, Sharp constant in spectral variation

Let α_j and β_j, $j = 1, \ldots, n$, be the eigenvalues of $n \times n$ complex matrices A and B, respectively, and denote

$$\mathsf{Eig}A = \{\alpha_1, \ldots, \alpha_n\}, \quad \mathsf{Eig}B = \{\beta_1, \ldots, \beta_n\}.$$

The *optimal matching distance* between the spectra of A and B is

$$d(\mathsf{Eig}A, \mathsf{Eig}B) = \min_{\sigma} \max_{1 \le j \le n} |\alpha_j - \beta_{\sigma(j)}|$$

where σ varies over all permutations of the indices $\{1, 2, \ldots, n\}$.

Let $\| \cdot \|$ be the spectral norm. It is known that there exists a number c with $1 < c < 3$ such that

$$d(\mathsf{Eig}A, \mathsf{Eig}B) \le c\|A - B\|$$

for any normal matrices A, B of any order.

Problem 21 (Bhatia, 2007) *Determine the best possible constant c such that*

$$d(\mathsf{Eig}A,\, \mathsf{Eig}B) \le c\|A - B\|$$

for any normal matrices A, B of any order.

Thank you!