Extremal eigenvalues of real symmetric matrices with entries in an interval

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We determine the sharp lower and upper bounds for the smallest and largest eigenvalues of real symmetric matrices of a given order whose entries are in a given interval. The maximizing and minimizing matrices are specified.

We also consider the maximal spread of such matrices.

Finally we pose some open problems.

Let $S_n[a, b]$ denote the set of $n \times n$ real symmetric matrices whose entries are in the interval $[a,\ b]$. For an $n \times n$ real symmetric matrix A, we always denote the eigenvalues of A in decreasing order by $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$.

In 1985 G. Constantine proved that if $A \in S_n[0, b]$, then

$$\lambda_n(A) \geq \begin{cases} -nb/2 & \text{if } n \text{ is even,} \\ -\sqrt{n^2-1}b/2 & \text{if } n \text{ is odd.} \end{cases}$$

These bounds are sharp. Constantine's proof techniques are graph-theoretic. In 1989 R. Roth gave another proof of this result by analysis of eigenvectors.

We will determine the smallest and largest values of both $\lambda_n(A)$ and $\lambda_1(A)$ when $A \in S_n[a, b]$ for generic $a < b$, thus generalizing Constantine's result.

Lower bounds for the smallest eigenvalue

Denote by $J_{r,s}$ the $r \times s$ matrix with all entries equal to 1, and write J_r for $J_{r,r}$.

Theorem 1. Let $A \in S_n[a,b]$ with $n \geq 2$ and $a < b$.
(i) If $|a| < b$, then

$$\lambda_n(A) \geq \begin{cases} n(a-b)/2 & \text{if } n \text{ is even,} \\ \left(na - \sqrt{a^2 + (n^2-1)b^2}\right)/2 & \text{if } n \text{ is odd.} \end{cases}$$

If n is even, equality holds if and only if A is permutation similar to

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \otimes J_{\frac{n}{2}}.$$

If n is odd, equality holds if and only if A is permutation similar to

$$\begin{bmatrix} aJ_{\frac{n-1}{2}} & bJ_{\frac{n-1}{2},\frac{n+1}{2}} \\ bJ_{\frac{n+1}{2},\frac{n-1}{2}} & aJ_{\frac{n+1}{2}} \end{bmatrix}.$$

(ii) If $|a| \geq b$, then $\lambda_n(A) \geq na$. If $|a| > b$, equality holds if and only if $A = aJ_n$. If $|a| = b$, equality holds if and only if A is permutation similar to

$$\begin{bmatrix} aJ_k & bJ_{k,n-k} \\ bJ_{n-k,k} & aJ_{n-k} \end{bmatrix}$$

for some k with $1 \leq k \leq n$.

Upper bounds for the largest eigenvalue

Corollary 2. Let $A \in S_n[a,b]$ with $n \geq 2$ and $a < b$.
(i) If $a < -|b|$, then

$$\lambda_1(A) \leq \begin{cases} n(b-a)/2 & \text{if } n \text{ is even,} \\ \left(nb + \sqrt{b^2 + (n^2-1)a^2}\right)/2 & \text{if } n \text{ is odd.} \end{cases}$$

If n is even, equality holds if and only if A is permutation similar to

$$\begin{bmatrix} b & a \\ a & b \end{bmatrix} \otimes J_{\frac{n}{2}}.$$

If n is odd, equality holds if and only if A is permutation similar to

$$\begin{bmatrix} bJ_{\frac{n-1}{2}} & aJ_{\frac{n-1}{2},\frac{n+1}{2}} \\ aJ_{\frac{n+1}{2},\frac{n-1}{2}} & bJ_{\frac{n+1}{2}} \end{bmatrix}.$$

(ii) If $a \geq -|b|$, then $\lambda_1(A) \leq nb$. If $a > -|b|$, equality holds if and only if $A = bJ_n$. If $a = -|b|$, equality holds if and only if A is permutation similar to

$$\begin{bmatrix} bJ_k & aJ_{k,n-k} \\ aJ_{n-k,k} & bJ_{n-k} \end{bmatrix}$$

for some k with $1 \leq k \leq n$.

Upper bounds for the smallest eigenvalue and lower bounds for the largest eigenvalue

Theorem 3. Let $A \in S_n[a, b]$ with $n \geq 2$ and $a < b$.
(i) Let $0 < a < b$. Then

$$\lambda_n(A) \leq b - a. \tag{1}$$

Equality in (1) holds if and only if $A = aJ + (b - a)I$.

$$\lambda_1(A) \geq na. \tag{2}$$

Equality in (2) holds if and only if $A = aJ$.
(ii) Let $a \leq 0 < b$. Then

$$\lambda_n(A) \leq b. \tag{3}$$

Equality in (3) holds if and only if $A = bI$.

$$\lambda_1(A) \geq a. \tag{4}$$

Equality in (4) holds if and only if $A = aI$.

The maximal spread

The spread of an $n \times n$ real symmetric matrix A is defined as $s(A) = \lambda_1(A) - \lambda_n(A)$. This quantity has applications in combinatorial optimization problems

We treat only the case when the interval is symmetric about the origin. Of course we may use the upper bound on λ_1 in Corollary 2 and the lower bound on λ_n in Theorem 1 to give an upper bound on the spread $\lambda_1 - \lambda_n$, but that bound is not sharp. This is because the upper bound on λ_1 and the lower bound on λ_n cannot be simultaneously attained at one common matrix.

Two matrices A, B of the same order are said to be *sign-permutation similar* if there exist a permutation matrix P and a diagonal matrix D with diagonal entries equal to 1 or -1 such that $DP^TAPD = B$. It is clear that sign-permutation similarity is an equivalence relation.

Theorem 4. Let $A \in S_n[-a,a]$ with $n \geq 2$ and $a > 0$. Then

$$s(A) \leq \begin{cases} \sqrt{2}na & \text{if } n \text{ is even,} \\ \sqrt{2n^2-1}a & \text{if } n \text{ is odd.} \end{cases}$$

If n is even, equality holds if and only if A is sign-permutation similar to

$$a \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes J_{\frac{n}{2}}.$$

If n is odd, equality holds if and only if A is sign-permutation similar to

$$\pm a \begin{bmatrix} J_{\frac{n+1}{2}} & J_{\frac{n+1}{2},\frac{n-1}{2}} \\ J_{\frac{n-1}{2},\frac{n+1}{2}} & -J_{\frac{n-1}{2}} \end{bmatrix}.$$

To appear in **SIAM J. Matrix Anal. Appl.**

Open problems

1. For a given j with $2 \le j \le n-1$, determine

$$\max\{\lambda_j(A) : A \in S_n[a,b]\},$$

$$\min\{\lambda_j(A) : A \in S_n[a,b]\}$$

and determine those matrices which attain the maximum and those matrices which attain the minimum.

2. Determine

$$\max\{s(A) : A \in S_n[a,b]\}$$

and determine those matrices which attain the maximum.

3. Let $z(n)$ be the largest number z such that every $n \times n$ complex matrix is unitarily similar to a matrix with at least z zero entries. Determine $z(n)$.

Remark: We have

$$\frac{(n-1)n}{2} \leq z(n) \leq \left[\frac{n^2-1}{2}\right].$$

The LHS follows from Schur's triangularization theorem. The RHS can be proved by the weak Sard theorem in differential topology.

4. Let $f(A)$ be the number of nonzero entries of the matrix A. Characterize those sign patterns of (entry-wise) nonnegative matrices A such that the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is nondecreasing. The same problem with "nondecreasing" replaced by "nonincreasing".

Remark: Sidak observed in 1964 that there exists a primitive matrix A of order 9 satisfying

$$18 = f(A) > f(A^2) = 16.$$