Extremal sparsity property of the Jordan canonical form

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1, Main result

One of the central results in linear algebra is the Jordan canonical form theorem which states that every square complex matrix A is similar to a Jordan matrix

$$J(A) = \text{diag}(J_{n_1}(\lambda_1), \dots, J_{n_k}(\lambda_k))$$

where

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

is a Jordan block of order n_i. This Jordan canonical form $J(A)$ is unique up to permutations of the diagonal Jordan blocks.

Two matrices are similar if and only if they have a common Jordan canonical form.

We use $J(A)$ to mean any of the Jordan canonical forms of A.

$\mathcal{S}(A):$ the set of all complex matrices that are similar to A.

It seems that $J(A)$ has the simplest form among the matrices in $\mathcal{S}(A)$. But in what sense? Structure? Sparsity? In 2005 one of the authors asked whether for any matrix A, $J(A)$ has the largest number of zero entries among all the matrices in $\mathcal{S}(A)$.

The answer is no, as shown by the following example due to Chi-Kwong Li:

$$A = \begin{bmatrix} 0 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

A has 11 zero entries while $J(A)$ has 10 zero entries.

The correct direction is to consider off-diagonal entries.

We show that for any complex matrix A, $J(A)$ has the largest number of *off-diagonal* zero entries among all the matrices in $\mathcal{S}(A)$, and we characterize the matrices in $\mathcal{S}(A)$ that attain this largest number.

A square complex matrix is called a *monomial matrix* if it has exactly one nonzero entry in each row and each column. Let Γ_n be the set of monomial matrices of order n. Then $M \in \Gamma_n$ if and only if there exist a permutation matrix P and a nonsingular diagonal matrix D such that $M = PD$ if and only if there exist a permutation matrix Q and a nonsingular diagonal matrix E such that $M = EQ$. Obviously, Γ_n is a multiplicative group.

$\phi(A)$: the number of off-diagonal nonzero entries of a matrix A.

Theorem *Let A be a square complex matrix and $B \in \mathcal{S}(A)$. Then*

$$\phi(B) \geq \phi(J(A)).$$

Equality holds if and only if there exists a monomial matrix M such that

$$M^{-1}BM = J(A).$$

The Theorem shows that up to permutation similarity, $J(A)$ is the unique zero-nonzero pattern among the matrices in $\mathcal{S}(A)$ that attains the largest number of off-diagonal zero entries.

This gives a combinatorial characterization of the Jordan canonical form.

2, Key lemmas in the proof

The following purely combinatorial lemma is the basis of our analysis.

Lemma 1 *Let n, k be positive integers with $1 \leq k \leq n$. If a matrix A of order n satisfies $\phi(A) \leq n - k$, then there exists a permutation matrix P such that*

$$P^T AP = diag(A_1, A_2, \ldots, A_k)$$

where A_j is square and non-void for $j = 1, \ldots, k$.

Proof. With $A = (a_{ij})$ we associate a graph G with vertex set $V = \{1, 2, \ldots, n\}$ where there is an edge between vertices i and j if and only if $i \neq j$, and $a_{ij} \neq 0$ or $a_{ji} \neq 0$.

Another ingredient is

Lemma 2 *Let*

$$A = \begin{bmatrix} a & x^T \\ 0 & B \end{bmatrix}$$

be a complex matrix of order n where B has order $n-1$. If $J(A)$ has only one Jordan block, then $J(B)$ has only one Jordan block.

Proof. To the contrary, suppose $J(B)$ has at least two Jordan blocks. Then we can get a contradiction.

3, The real case

For real numbers a, b denote

$$C(a,b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

and denote by $C_k(a,b)$ the matrix of order $2k$:

$$C_k(a,b) = \begin{bmatrix} C(a,b) & I & 0 & \cdots & 0 \\ 0 & C(a,b) & I & \cdots & 0 \\ 0 & 0 & C(a,b) & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C(a,b) \end{bmatrix}$$

where I is the identity matrix.

It is known that every real square matrix A is similar via a real matrix to a block diagonal real matrix of the form

$$\mathrm{diag}(C_{n_1}(a_1, b_1), \ldots, C_{n_t}(a_t, b_t), J_{m_1}(\mu_1), \ldots, J_{m_s}(\mu_s))$$

where $a_j + ib_j$ are the nonreal eigenvalues of A, $j = 1, \ldots, t$ and μ_q are the real eigenvalues of A, $q = 1, \ldots, s$.

This is called the *real Jordan canonical form* of A.

A referee asked whether a conclusion similar to that in Theorem holds for the real Jordan canonical form, that is, if A is a real square matrix, does the real Jordan canonical form of A have the largest number of off-diagonal zero entries among all *real* matrices in $\mathcal{S}(A)$?

That this is not so is illustrated by the following example:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then A is a matrix in real Jordan canonical form with eigenvalues $1 \pm i$, each of multiplicity 2, and

$$T^{-1}AT = \begin{bmatrix} 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

We have $\phi(A) = 6 > 5 = \phi(T^{-1}AT)$.

On the other hand, we can prove that
the real Jordan canonical form of a diagonalizable real matrix contains the largest number of off-diagonal zero entries among all the real matrices similar to A.

In contrast to the Jordan canonical form, there are cases of equality other than the type given in Theorem. For example, let

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then the characteristic polynomial of A is $\lambda^4 + 1$, and it has four distinct, non-real eigenvalues. Thus A is diagonalizable over the complex number field, and the real Jordan canonical form B of A has four off-diagonal nonzero entries, the same number as A. But clearly $M^{-1}AM \neq B$ for any monomial matrix M.

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Three problems solved

1. Let $z(n)$ be the largest integer z such that every $n \times n$ complex matrix is unitarily similar to a matrix with at least z zero entries. Schur's unitary triangularization theorem implies that

$$z(n) \geq (n-1)n/2.$$

At the 5th China Matrix Theory and Applications Conference, Shanghai, August 2002 and the 12th ILAS Conference, Regina, Canada, June 2005, I asked whether

$$z(n) = (n-1)n/2?$$

D.Z. Dokovic and C.R. Johnson have answered this question affirmatively by using a dimension argument from differential manifolds in their paper *Unitarily achievable zero patterns and traces of words in A and A^**,
Linear Algebra Appl.,421(2007).

2. Denote by $s_1(G) \geq s_2(G) \geq \cdots$ the singular values of a matrix G in decreasing order. In 2000 I posed the following

Conjecture. Let A, B be positive semidefinite matrices of the same order and $0 \leq t \leq 1$. Then

$$s_j(A^t B^{1-t} + A^{1-t} B^t) \leq s_j(A + B), \quad j = 1, 2, \ldots$$

K.M.R. Audenaert has proved this conjecture in *A singular value inequality for Heinz means*,
Linear Algebra Appl., 422(2007).

3. For every positive integer n we have
(i) $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor$;
(ii) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} \rfloor = \lfloor \sqrt{9n+8} \rfloor$;
(iii) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} \rfloor = \lfloor \sqrt{16n+20} \rfloor$;
(iv) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} \rfloor = \lfloor \sqrt{25n+49} \rfloor$.

In the paper *Formulae for sums of consecutive square roots,* **Math. Intelligencer, 27(2005), no.4,** I proved (iv) and made the following

Conjecture. For any $k \geq 6$, there exists no constant c depending only on k such that

$$\lfloor \sum_{i=0}^{k-1} \sqrt{n+i} \rfloor = \lfloor \sqrt{k^2 n + c} \rfloor$$

holds for all positive integers n.

P.W. Saltzman and P.Z. Yuan have proved this conjecture in their paper *On sums of consecutive integral roots,* **Amer. Math. Monthly, March 2008.**

A new book:

X. Zhan, Matrix Theory (in Chinese),

Higher Education Press, Beijing, June 2008

Thank you!