Sparsity of Matrix Canonical Forms

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I. Extremal sparsity of the companion matrix of a polynomial

Joint work with Chao Ma

The companion matrix of a monic polynomial

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

over a field is defined to be

$$C(p) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}.$$

It is well known that the characteristic polynomial of $C(p)$ is $p(x)$.

Because of this relation, companion matrices can be used to
1) locate the roots of a complex polynomial,
2) prove constructively that the algebraic numbers form a field,
3) prove constructively that the algebraic integers form a ring.

The companion matrix $C(p)$ is very sparse, i.e., it has many zero entries. If we regard the coefficients $a_1, \ldots, a_n$ of $p(x)$ as distinct indeterminates, then $C(p)$ has $2n - 1$ nonzero entries. We will show that the companion matrix is the sparsest in a sense to be described below.

Let F be a field and $x_1, \dots, x_n$ be distinct indeterminates. We denote by $F[x_1, \dots, x_n]$ the ring of polynomials in $x_1, \dots, x_n$ over F, and by $F(x_1, \dots, x_n)$ the field of rational functions in $x_1, \dots, x_n$ over F :

$$F(x_1, \dots, x_n) = \left\{ \frac{f}{g} \,\middle|\, f, g \in F[x_1, \dots, x_n],\ g \neq 0 \right\}.$$

$M_n(E)$: the set of $n \times n$ matrices whose entries are elements of a given field E.

Theorem 1 Let F be a field, let $a_1, \dots, a_n$ be distinct indeterminates, and let $A \in M_n(F(a_1, \dots, a_n))$. If the characteristic polynomial of A is

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n,$$

then A has at least $2n - 1$ nonzero entries.

Tools from algebra and graph theory in the proof

Let $F \subseteq K$ be a field extension. A transcendence basis of K over F is a subset S of K which is algebraically independent over F and is maximal with respect to set-theoretic inclusion in the set of all algebraically independent subsets of K.

The transcendence degree of K over F is the cardinality of any transcendence basis of K over F.

A branching is an oriented tree having a root of in-degree 0 and all other vertices of in-degree 1. A spanning branching of a digraph is a branching that includes all vertices of the digraph.

Key lemmas

Lemma 2 The polynomial

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

is irreducible over $F(a_1, \ldots, a_n)$.

Lemma 3 In a strongly connected digraph, every vertex is the root of a spanning branching.

Lemma 4 Let E be a field. If the digraph of a matrix $A \in M_n(E)$ has a spanning branching whose arcs are $(i_1, j_1), \ldots, (i_{n-1}, j_{n-1})$, then there exists a nonsingular diagonal matrix $G \in M_n(E)$ such that $GAG^{-1}(i_k, j_k) = 1$ for $k = 1, \ldots, n-1$.

II. Extremal sparsity of the Jordan canonical form

Joint work with Richard A. Brualdi and Pei Pei

One of the central results in linear algebra is the Jordan canonical form theorem which states that every square complex matrix A is similar to a Jordan matrix

$$J(A) = \operatorname{diag}(J_{n_1}(\lambda_1), \ldots, J_{n_k}(\lambda_k))$$

where

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

is a Jordan block of order n_i. This Jordan canonical form $J(A)$ is unique up to permutations of the diagonal Jordan blocks.

Two matrices are similar if and only if they have a common Jordan canonical form.

We use $J(A)$ to mean any of the Jordan canonical forms of A.

$\mathcal{S}(A):$ the set of all complex matrices that are similar to A.

It seems that $J(A)$ has the simplest form among the matrices in $\mathcal{S}(A)$. But in what sense? Structure? Sparsity? In 2005 one of the authors asked whether for any matrix A, $J(A)$ has the largest number of zero entries among all the matrices in $\mathcal{S}(A)$.

The answer is no, as shown by the following example due to Chi-Kwong Li:

$$A = \begin{bmatrix} 0 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

A has 11 zero entries while $J(A)$ has 10 zero entries.

The correct direction is to consider off-diagonal entries.

We show that for any complex matrix A, $J(A)$ has the largest number of *off-diagonal* zero entries among all the matrices in $\mathcal{S}(A)$, and we characterize the matrices in $\mathcal{S}(A)$ that attain this largest number.

A square complex matrix is called a *monomial matrix* if it has exactly one nonzero entry in each row and each column. Let Γ_n be the set of monomial matrices of order n. Then $M \in \Gamma_n$ if and only if there exists a permutation matrix P and a nonsingular diagonal matrix D such that $M = PD$ if and only if there exists a permutation matrix Q and a nonsingular diagonal matrix E such that $M = EQ$. Obviously, Γ_n is a multiplicative group.

$\phi(A)$: the number of off-diagonal nonzero entries of a matrix A.

Theorem 5 Let A be a square complex matrix and $B \in \mathcal{S}(A)$. Then

$$\phi(B) \geq \phi(J(A)).$$

Equality holds if and only if there exists a monomial matrix M such that

$$M^{-1}BM = J(A).$$

The theorem shows that up to permutation similarity, $J(A)$ is the unique zero-nonzero pattern among the matrices in $\mathcal{S}(A)$ that attains the largest number of off-diagonal zero entries.

This gives a combinatorial characterization of the Jordan canonical form.

Key lemmas in the proof

The following purely combinatorial lemma is the basis of our analysis.

Lemma 6 Let n, k be positive integers with $1 \leq k \leq n$. If a matrix A of order n satisfies $\phi(A) \leq n - k$, then there exists a permutation matrix P such that

$$P^T AP = \mathrm{diag}(A_1, A_2, \ldots, A_k)$$

where A_j is square and non-void for $j = 1, \ldots, k$.

Proof. With $A = (a_{ij})$ we associate a graph G with vertex set $V = \{1, 2, \ldots, n\}$ where there is an edge between vertices i and j if and only if $i \neq j$, and $a_{ij} \neq 0$ or $a_{ji} \neq 0$.

Another ingredient is

Lemma 7 Let

$$A = \begin{bmatrix} a & x^T \\ 0 & B \end{bmatrix}$$

be a complex matrix of order n where B has order $n-1$. If $J(A)$ has only one Jordan block, then $J(B)$ has only one Jordan block.

Proof. To the contrary, suppose $J(B)$ has at least two Jordan blocks. Then we can get a contradiction.

References

[1] C. Ma and X. Zhan, Extremal sparsity of the companion matrix of a polynomial, *Linear Algebra Appl.*, 438(2013), 621-625.

[2] R.A. Brualdi, P. Pei and X. Zhan, An extremal sparsity property of the Jordan canonical form, *Linear Algebra Appl.*, 429(2008), 2367-2372.

Matrix theory is a classical topic of algebra that had originated, in its current form, in the middle of the 19th century. It is remarkable that for more than 150 years it continues to be an active area of research full of new discoveries and new applications.

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