Digraphs that have at most one walk of a given length with the same endpoints

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Digraphs here allow loops but do not allow multiple arcs. The number of the vertices of a digraph is called its *order* and the number of the arcs its *size*. For digraphs, walks will mean directed walks.

For given positive integers n, k, let $\Theta(n,k)$ denote the set of the digraphs D on vertices $1, 2, \ldots, n$ such that for any i, j with $1 \le i, j \le n$, D has at most one walk of length k from i to j. Let $\theta(n,k)$ denote the maximum size of a digraph in $\Theta(n,k)$.

In 2007 I posed the following problem at a seminar:

Problem 1. *For given positive integers n, k, determine $\theta(n,k)$ and determine the digraphs in $\Theta(n,k)$ that attain the size $\theta(n,k)$.*

Note that the possible sizes of the digraphs in $\Theta(n, k)$ are the integers in the interval $[0, \theta(n, k)]$.

The motivation for studying Problem 1 is to explore the relation between the size and the walks of a digraph. Intuitively digraphs in $\Theta(n, k)$ cannot have very large sizes.

A 0-1 matrix interpretation of Problem 1.

The case $k = 2$ of Problem 1 has been solved by H. Wu whose result is

$$\theta(n, 2) = \begin{cases} \frac{n^2+4n-1}{4}, & \text{if } n \text{ is odd,} \\ \frac{n^2+4n-4}{4}, & \text{if } n \text{ is even and } n \neq 4, \\ 8, & \text{if } n = 4 \end{cases}$$

and the digraphs attaining this largest size are also determined.

A digraph is said to be *transitive* if for every three distinct vertices v_i, v_j, v_k the condition that (v_i, v_j) and (v_j, v_k) are arcs implies that (v_i, v_k) is an arc. It is clear that a tournament of order n is transitive if and only if its vertices can be labeled as $1, 2 \ldots, n$ such that (i, j) is an arc if and only if $i < j$.

Theorem 1. *Let n, k be given integers with $n \geq 5$ and $k \geq n-1$. Then $\theta(n,k) = n(n-1)/2$ and a digraph $D \in \Theta(n,k)$ has size $n(n-1)/2$ if and only if D is a transitive tournament.*

A 0-1 matrix interpretation of Theorem 1

Theorem 2. *Let $n \geq 6$. Then*

$$\theta(n, n-2) = \frac{n(n-1)}{2} - 1.$$

Theorem 3. *Let $n \geq 7$. Then*

$$\theta(n, n-3) = \frac{n(n-1)}{2} - 2.$$

Our proofs use both digraphs and 0-1 matrices

In view of Theorems 2,3, one might conjecture that for $2 \le k \le n-2$,

$$\theta(n,k) = \frac{n(n-1)}{2} - (n-k-1). \tag{1}$$

This is not the case. We have proved that at least one of $\theta(10,4)$ and $\theta(11,4)$ does not satisfy (1).

Problem 1 is open for the cases $3 \le k \le n-4$

References

[1] Z. Huang and X. Zhan, Digraphs that have at most one walk of a given length with the same endpoints, Discrete Math., to appear
[2] H. Wu, On the 0-1 matrices whose squares are 0-1 matrices, Linear Algebra Appl. 432(2010), 2909-2924.

Thank you!