# Extremal numbers of positive entries of imprimitive nonnegative matrices

Xingzhi Zhan
East China Normal University
Shanghai, China
e-mail: zhan@math.ecnu.edu.cn

A square nonnegative matrix $A$ is said to be primitive if $A^p$ is a positive matrix for some positive integer $p$; otherwise $A$ is called imprimitive. A primitive matrix is necessarily irreducible. The *imprimitivity index* $k$ of an irreducible nonnegative matrix $A$ is the number of eigenvalues of $A$ whose moduli are equal to the spectral radius of $A$. $A$ is primitive if $k = 1$, and imprimitive if $k > 1$.

Let $\sigma(A)$ denote the number of positive entries of a nonnegative matrix $A$. For a real number $x$, denote by $\lfloor x \rfloor$ the largest integer not exceeding $x$.

Suppose $A$ is an $n \times n$ irreducible nonnegative matrix with imprimitivity index $k$. In 1993 M. Lewin proved that if $k \leq 4$ then $\sigma(A) \leq \lfloor n^2/k \rfloor$ and this bound is sharp, and if $k \geq 5$ then $\sigma(A) \leq \lfloor n^2/4 \rfloor$. The case $n = 16$ and $k = 15$ shows that the bound $\lfloor n^2/4 \rfloor$ is not sharp. In fact, it is easy to see that if $A$ is $16 \times 16$ and of imprimitivity index 15 then $\sigma(A) \leq 17$, while $\lfloor 16^2/4 \rfloor = 64$.

We will determine the sharp upper and lower bounds for $\sigma(A)$, and consider related applications. One interesting thing here is that the rule for the maximum $\sigma(A)$ changes when $k$ turns from 4 to 5.

**Theorem 1.** *Let $\Gamma(n,k)$ be the set of $n \times n$ irreducible nonnegative matrices with imprimitivity index $k$. Then*

$$\max\{\sigma(A) | A \in \Gamma(n,k)\} = \begin{cases} \lfloor n^2/k \rfloor & \text{if} \quad 1 \le k \le 4, \\ 2n - k + \lfloor (n-k)^2/4 \rfloor & \text{if} \quad k \ge 5. \end{cases}$$

Denote by $\mathrm{ind}(A)$ the imprimitivity index of $A$. The next result estimates this index by the number of positive entries.

**Corollary 2.** *Let $A$ be an $n \times n$ irreducible nonnegative matrix. If $n^2 < 4\sigma(A)$ then*

$$\mathrm{ind}(A) \le \lfloor n^2/\sigma(A) \rfloor.$$

*If $n^2 \ge 4\sigma(A)$ then*

$$\mathrm{ind}(A) \le \lfloor n + 2 - 2\sqrt{\sigma(A) - n + 1} \rfloor.$$

We remark that Corollary 2 includes Lewin's result that if $\sigma(A) > n^2/2$ then $\mathrm{ind}(A) = 1$, i.e., $A$ is primitive.

**Theorem 3.** *Let* $\Gamma(n,k)$ *be the set of* $n \times n$ *irreducible nonnegative matrices with imprimitivity index* $k$*. Then*

$$\min\{\sigma(A) | A \in \Gamma(n,k)\} = \begin{cases} n+1 & \text{if} \quad k < n, \\ n & \text{if} \quad k{=}n. \end{cases}$$

The proofs of the above results involve study of a cyclic quadratic form and construction of special graphs.

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## Two problems solved

1. Let $z(n)$ be the largest integer $z$ such that every $n \times n$ complex matrix is unitarily similar to a matrix with at least $z$ zero entries. Schur's unitary triangularization theorem implies that

$$z(n) \geq (n-1)n/2.$$

At the 5th China Matrix Theory and Applications Conference, Shanghai, August 2002 and the 12th ILAS Conference, Regina, Canada, June 2005, I asked whether

$$z(n) = (n-1)n/2?$$

D.Z. Dokovic and C.R. Johnson have answered this question affirmatively by using a dimension argument from differential manifolds in their paper Unitarily achievable zero patterns and traces of words in $A$ and $A^*$, which is to appear in Linear Algebra Appl.

2. For every positive integer $n$ we have

(*i*) $\lfloor\sqrt{n}+\sqrt{n+1}\rfloor=\lfloor\sqrt{4n+1}\rfloor$;

(*ii*) $\lfloor\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\rfloor=\lfloor\sqrt{9n+8}\rfloor$;

(*iii*) $\lfloor\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}\rfloor=\lfloor\sqrt{16n+20}\rfloor$;

(*iv*) $\lfloor\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}+\sqrt{n+4}\rfloor=\lfloor\sqrt{25n+49}\rfloor$.

In the paper Formulae for sums of consecutive square roots, Math. Intelligencer, 27(2005), no.4, I proved (iv) and made the following

**Conjecture.** For any $k\geq 6$, there exists no constant $c$ depending only on $k$ such that

$$\lfloor\sum_{i=0}^{k-1}\sqrt{n+i}\rfloor=\lfloor\sqrt{k^2n+c}\rfloor$$

holds for all positive integers $n$.

I also mentioned this conjecture at the 6th China Matrix Theory and Applications Conference, Harbin, July 2004.

P.W. Saltzman and P.Z. Yuan have proved this conjecture in their paper On sums of consecutive integral roots, which is to appear in Amer. Math. Monthly.

## Two open problems

Let $S_n[a, b]$ denote the set of $n \times n$ real symmetric matrices whose entries are in the interval $[a,\ b]$. For an $n \times n$ real symmetric matrix $A$, we denote the eigenvalues of $A$ in decreasing order by $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. The *spread* of such an $A$ is defined to be $s(A) = \lambda_1(A) - \lambda_n(A)$.

**Problem 1.** For a given integer $j$ with $2 \leq j \leq n - 1$, determine

$$\max\{\lambda_j(A) : A \in S_n[a, b]\},$$
$$\min\{\lambda_j(A) : A \in S_n[a, b]\}$$

and determine those matrices which attain the maximum and those matrices which attain the minimum. The cases $j = 1, n$ are solved in [1].

**Problem 2.** For generic $a < b$ determine

$$\max\{s(A) : A \in S_n[a, b]\}$$

and determine those matrices which attain the maximum. The case $a = -b$ is solved in [1].

[1], X. Zhan, Extremal eigenvalues of real symmetric matrices with entries in an interval. SIAM J. Matrix Anal. Appl. 27(2005), no.3.