Minimum spans of no-hole 2-distant colourings of Hamming graphs

Gerard J. Chang\textsuperscript{a,\ast}, Changhong Lu\textsuperscript{b,†}, Sanming Zhou\textsuperscript{c,‡}

\textsuperscript{a} Department of Mathematics, National Taiwan University, Taipei 106, Taiwan
\textsuperscript{b} Department of Mathematics, East China Normal University, Shanghai 200062, P.R. China
\textsuperscript{c} Department of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia

March 14, 2003

Abstract

A no-hole 2-distant colouring of a graph $G$ is an assignment $c$ of a non-negative integer to each vertex of $G$ such that $|c(u) - c(v)| \geq 2$ whenever $u, v$ are adjacent, and the integers used are consecutive. The difference between the largest and the smallest integers used is called the span of $c$. Practically, such a colouring corresponds to an assignment of channels to the transmitters in a communication network, and the concern is to find the minimum span $\text{nsp}(G)$ over all no-hole 2-distant colourings of $G$. In this paper we show that all Hamming graphs, namely the Cartesian products $H_{q_1, q_2, \ldots, q_d} = K_{q_1} \square K_{q_2} \cdots \square K_{q_d}$ of complete graphs (where $d \geq 2$ and all $q_i \geq 2$), other than $H_{2,2}$ admit no-hole 2-distant colourings. Moreover, we prove that $\text{nsp}(H_{q,q}) = 2q$ $(q \geq 3)$ and $\text{nsp}(H_{q_1, q_2, \ldots, q_d}) = 2 \max_{1 \leq i \leq d} (q_i - 1)$ whenever $d \geq 3$ or $q_1, q_2, \ldots, q_d$ are not all the same, and construct explicitly optimal no-hole 2-distant colourings in each case.

Keywords: channel assignment; no-hole 2-distant colouring; $T$-colouring; Hamming graph

1 Introduction

In a communication network one wishes to assign a channel, represented by a non-negative integer, to each transmitter such that interfering transmitters receive channels with separation not in a given forbidden set $T$ with $0 \in T$. This can be formulated as a colouring problem for the interference graph $G$, which is defined to have vertices representing transmitters in which

\textsuperscript{\ast}Email: gjchang@math.ntu.edu.tw. Suppose in part by the National Science Council under the grant NSC91-2115-M002-001.

\textsuperscript{†}Email: chlu@math.ecnu.edu.cn. Supported by the National Science Council of the Republic of China under a postdoctoral fellowship program from March 2001 to October 2002.

\textsuperscript{‡}Email: smzhou@ms.unimelb.edu.au. Supported by the Australian Academy of Science and the National Science Council of the Republic of China under the scheme “Scientific Visits to Taiwan”, hosted by the National Taiwan University in June 2002, and the Discovery Project Grant DP0344803 from the Australian Research Council.
Theorem 1.1 is the product $K_q$ only if they differ in exactly one coordinate, where the smallest integer used is 0, so that the span of $c$, denoted $sp(G; c)$, is equal to the largest integer used by $c$. A $T$-colouring with $T = \{0, 1\}$ is usually called a 2-distance colouring of $G$. If the channels used, namely the colours $c(v)$ for $v \in V(G)$, consist of a set of consecutive integers, then $c$ is said to have no-hole. If $G$ admits a no-hole 2-distance colouring, then we define

$$nsp(G) = \min_c sp(G; c)$$

to be the minimum span taking over all such colourings of $G$; otherwise we set $nsp(G) = \infty$ simply for convenience. Note that the latter possibility can occur indeed, that is, not every graph admits a no-hole 2-distance colouring. For example, the complete graph $K_q$ on $q \geq 2$ vertices has no such a colouring. The reader is referred to [2, 3, 4, 7, 9, 10, 11, 12] for recent research on no-hole 2-distance colourings and the minimum span $nsp$.

In general it is very hard to determine the minimum span $nsp(G)$. In this paper we will show, however, that this problem can be solved with satisfaction for Hamming graphs. The importance of such graphs lies in their extensive applications [6] in Communication and Coding Theory, and their roles as distance-transitive graphs [1, Section 9.2] with nice properties. By definition a Hamming graph is the Cartesian product $H_{q_1, q_2, \ldots, q_d} = K_{q_1} \square K_{q_2} \square \cdots \square K_{q_d}$ of complete graphs $K_{q_i}$, where $d \geq 2$ and all $q_i \geq 2$. More explicitly, the vertex set of $H_{q_1, q_2, \ldots, q_d}$ is the Cartesian product $Z_{q_1} \times Z_{q_2} \times \cdots \times Z_{q_d}$ and two vertices $(i_1, i_2, \ldots, i_d), (j_1, j_2, \ldots, j_d)$ are adjacent if and only if they differ in exactly one coordinate, where $Z_{q_i} = \{0, 1, \ldots, q_i - 1\}$ for each $i$. Whenever $q_1 = q_2 = \cdots = q_d = q$, we usually write $H(d, q)$ in place of $H_{q_1, q_2, \ldots, q_d}$. So in particular $H(d, 2)$ is the hypercube $Q_d$ of dimension $d$. The main result of this paper is the following theorem.

**Theorem 1.1** The Hamming graph $H_{q_1, q_2, \ldots, q_d}$, where $d \geq 2$ and $q_i \geq 2$, admits a no-hole 2-distance colouring if and only if it is not $Q_2$. Moreover, we have

$$nsp(H_{q_1, q_2, \ldots, q_d}) = \begin{cases} 2q, & d = 2 \text{ and } q_1 = q_2 = q \geq 3; \\ 2 \max_{1 \leq i \leq d} q_i - 1, & d \geq 3 \text{ or } q_1, q_2, \ldots, q_d \text{ not all the same}. \end{cases}$$

Indeed, one can see easily that $Q_2 \cong C_4$ does not admit any no-hole 2-distance colouring. The proof of Theorem 1.1 consists of the proofs of Lemmas 2.2, 2.3 and 2.4, which will be given in the next section. During the proof we will construct explicitly optimal no-hole 2-distance colourings of $H_{q_1, q_2, \ldots, q_d}$ by devising different colouring schemes for the three cases above.

## 2 Proof of Theorem 1.1

For each vertex $(i_1, i_2, \ldots, i_d)$ of $H_{q_1, q_2, \ldots, q_d}$, the set $\{(j, i_2, \ldots, i_d) : j \in Z_{q_1}\}$ induces a subgraph of $H_{q_1, q_2, \ldots, q_d}$ isomorphic to $K_{q_1}$, which we call the $K_{q_1}$-copy containing $(i_1, i_2, \ldots, i_d)$. We prove
first the following lower bound for \( \text{nsp}(H_{q_1, q_2, \ldots, q_d}) \).

**Lemma 2.1** For any Hamming graph \( H_{q_1, q_2, \ldots, q_d} \), where \( d \geq 2 \) and \( q_i \geq 2 \), we have

\[
\text{nsp}(H_{q_1, q_2, \ldots, q_d}) \geq 2 \max_{1 \leq i \leq d} q_i - 1.
\]

**Proof** For Hamming graphs not admitting no-hole 2-distant colourings, the lower bound is true trivially. In the following we suppose that \( H_{q_1, q_2, \ldots, q_d} \) admits a no-hole 2-distant colouring. Without loss of generality we may assume \( q_1 \geq q_2 \geq \cdots \geq q_d \geq 2 \), so that \( \max_{1 \leq i \leq d} q_i = q_1 \).

The colours of any two vertices in the same \( K_{q_1} \)-copy must differ by at least 2 under any 2-distant colouring of \( H_{q_1, q_2, \ldots, q_d} \). So the span of such a colouring is at least \( 2q_1 - 2 \), which implies that \( \text{nsp}(H_{q_1, q_2, \ldots, q_d}) \geq 2q_1 - 2 \). If \( \text{nsp}(H_{q_1, q_2, \ldots, q_d}) = 2q_1 - 2 \), let, say, \( c \) be a no-hole 2-distant colouring of \( H_{q_1, q_2, \ldots, q_d} \) with span \( 2q_1 - 2 \). Then \( c \) uses as colours all integers in the integer interval \([0, 2q_1 - 2]\). Since \([0, 2, \ldots, 2q_1 - 2]\) is the unique \( q_1 \)-subset of \([0, 2q_1 - 2]\) of which any two members differ by at least 2, the vertices of each \( K_{q_1} \)-copy of \( H_{q_1, q_2, \ldots, q_d} \) must be coloured \( 0, 2, \ldots, 2q_1 - 2 \) under \( c \). However, each vertex of \( H_{q_1, q_2, \ldots, q_d} \) is contained in some \( K_{q_1} \)-copy of \( H_{q_1, q_2, \ldots, q_d} \). So \( c \) uses only even integers in \([0, 2q_1 - 2]\). This contradiction shows that \( \text{nsp}(H_{q_1, q_2, \ldots, q_d}) \geq 2q_1 - 1 \). \(\square\)

The proof above did not touch upon when a Hamming graph admits a no-hole 2-distant colouring. We will make this clear in the following by giving explicitly such colourings for all \( H_{q_1, q_2, \ldots, q_d} \neq Q_2 \). Hamming graphs of the form \( H(d, q) \) need special treatment, and we discuss them first. For \( d = 2 \) and \( d \geq 3 \), the values of \( \text{nsp}(H(2, q)) \) are different, as shown in the following two lemmas.

**Lemma 2.2** Let \( q \geq 3 \). Then

\[
\text{nsp}(H(2, q)) = 2q.
\]

**Proof** Recall that \( H(2, q) \) has vertex set \( \mathbb{Z}_q \times \mathbb{Z}_q \). We think of \( H(2, q) \) as drawing on the plane in the usual way, so we can talk about its rows and columns: the \((i + 1)\)-th row consists of those vertices with the first coordinate \( i \), and the \((j + 1)\)-th column consists of vertices with the second coordinate \( j \), for \( 0 \leq i, j \leq q - 1 \). The vertices in the same row/column induce a complete subgraph \( K_q \) of \( H(2, q) \), and hence they must receive colours with mutual difference at least 2 under any 2-distant colouring.

Let us prove first that \( \text{nsp}(H(2, q)) \geq 2q \). Suppose otherwise, then \( \text{nsp}(H(2, q)) = 2q - 1 \) by Lemma 2.1, and \( H(2, q) \) has a no-hole 2-distant colouring \( c \) with span \( 2q - 1 \). Since \( c \) is no-hole, the integer \( 2q - 2 \) must appear in some row \( R \) of \( H(2, q) \), and hence both \( 2q - 3 \) and \( 2q - 1 \) do not appear in \( R \). Since \([0, 2, \ldots, 2q - 2]\) is the unique \( q \)-subset of \([0, 2q - 2]\) of which any two members differ by at least 2, the vertices in \( R \) must receive colours \( 0, 2, 4, \ldots, 2q - 2 \). Also, the integer 1 must appear in some column \( C \) of \( H(2, q) \). This implies that both 0 and 2 do not
appear in column $C$. Again, since \{1, 3, \ldots, 2q - 1\} is the unique $q$-subset of [1, 2q - 1] of which any two members differ by at least 2, the colours used in column $C$ are 1, 3, 5, \ldots, 2q - 1. Since $d = 2$, there is a unique common vertex of row $R$ and column $C$. From the discussion above this vertex must be coloured by an odd integer, as well as an even integer. This is a contradiction and hence we have $\text{nsp}(H(2, q)) \geq 2q$.

![Figure 1: A no-hole 2-distant colouring of $H(2, 6)$.](image)

It remains to prove that $2q$ is an upper bound for $\text{nsp}(H(2, q))$. This is achieved by giving explicitly the following no-hole 2-distant colouring of $H(2, q)$ with span $2q$. Define

$$c(i, j) = \begin{cases} 0, & (i, j) = (0, q - 1), (1, q - 2); \\ 2, & (i, j) = (1, q - 1); \\ (2i + 2j + 4) \mod (2q + 1), & (i, j) \neq (0, q - 1), (1, q - 2), (1, q - 1). \end{cases}$$

(This is illustrated in Figure 1, where the two special colours 0 and 2 are italicized.) Under this colouring $c$, the vertices in the first row are coloured 4, 6, 8, \ldots, 2q - 2, 2q, 0, and hence the mutual differences of these colours are at least 2. Similarly, the colours of the vertices in the second row are 6, 8, 10, \ldots, 2q, 0, 2, which differ pairwise by at least 2. The vertices in the last and second last columns receive colours 0, 2, 5, \ldots, 2q - 5, 2q - 3, 2q - 1 and 2q, 0, 3, \ldots, 2q - 7, 2q - 5, 2q - 3, respectively, and hence they satisfy the 2-distant condition as well. For all other vertices $(i, j)$, where $2 \leq i \leq q - 1$ and $0 \leq j \leq q - 3$, we have $c(i, j) = (2i + 2j + 4) \mod (2q + 1)$, and hence two such vertices in the same row or column receive colours with difference at least 2. Thus, $c$ is a 2-distant colouring of $H(2, q)$. Noting that $q \geq 3$, we have $c(q - 1, j) = 2j + 1$, which takes values 1, 3, 5, \ldots, 2q - 1 when $j$ runs from 0 to $q - 1$. Also, $c(i, 0) = 2i + 4 = 4, 6, \ldots, 2q$ when $i$ runs from 0 to $q - 2$. In addition, we have $c(0, q - 1) = 0$ and $c(1, q - 1) = 2$ by definition. So $c$ is a no-hole 2-distant colouring with span $2q$, and the proof is complete. \qed
Lemma 2.3 Let \( d \geq 3 \) and \( q \geq 2 \). Then

\[
\text{nsp}(H(d, q)) = 2q - 1.
\]

Proof By Lemma 2.1 it suffices to show that \( \text{nsp}(H(d, q)) \leq 2q - 1 \). We prove this by constructing the following no-hole 2-distant colouring \( c \) of \( H(d, q) \) with span \( 2q - 1 \). First, we define

\[
c(0, \ldots, 0, i_{d-1}, i_d) = \begin{cases} 
0, & i_{d-1} = i_d = 0; \\
(2(i_{d-1} + i_d) + 1) \mod 2q, & \text{otherwise.} 
\end{cases}
\]

(1)

\[
c(0, \ldots, 0, i_{d-2}, i_{d-1}, i_d) = (2i_{d-2} + c(0, \ldots, 0, i_{d-1}, i_d)) \mod (2q + 1).
\]

(2)

For \( H(3, 6) \) we display the colours of the vertices \((0, i_2, i_3)\) in Figure 2 below.

![Figure 2: The colours of the vertices of the form \((0, i_2, i_3)\) in \( H(3, 6) \).](image)

If \( d = 3 \), then all vertices have been thus coloured. In this case, for any two adjacent vertices \((i_1, i_2, i_3)\) and \((j_1, j_2, j_3)\), there is exactly one subscript \( t \), \( 1 \leq t \leq 3 \), with \( i_t \neq j_t \). For each possibility of \( t \) one can show from (2) (which applies to those vertices with the first coordinate 0 as well) that \( c(i_1, i_2, i_3) - c(j_1, j_2, j_3) = 2(i_t - j_t) \), which is at least 2 in absolute. So \( c \) is a 2-distant colouring of \( H(3, q) \). Moreover, \( c \) is no-hole. In fact, we have \( c(0, 1, i_3) = (2i_3 + 3) \mod 2q \), which takes all odd integers in \([0, 2q - 1]\) when \( i_3 \) runs from 0 to \( q - 1 \). Also, since \( c(0, 0, 0) = 0 \), by (2) we have \( c(i_1, 0, 0) = 2i_1 \) for \( i_1 = 0, 1, \ldots, q - 1 \), and hence all even integers in \([0, 2q - 1]\) are used by \( c \) as well. Thus, in the case where \( d = 3 \), \( c \) is a no-hole 2-distant colouring of \( H(3, q) \). Since \((0, \ldots, 0, i_{d-1}, i_d)\) is coloured by 0 if \( i_{d-1} = i_d = 0 \), and by an odd integer between 1 and \( 2q - 1 \) otherwise, it follows from (2) that the largest colour used by \( c \) is \( 2q - 1 \). Hence \( c \) has span \( 2q - 1 \), and \( \text{nsp}(H(3, q)) = 2q - 1 \) is proved.

Now we suppose \( d \geq 4 \). Then each of the vertices of \( H(d, q) \) not yet coloured has the form \((0, \ldots, 0, i_t, i_{t+1}, \ldots, i_{d-2}, i_{d-1}, i_d)\), for some \( t \) with \( 1 \leq t \leq d - 3 \) and \( i_t > 0 \). Based on (1) and
(2) we construct the colouring $c$ recursively by the following rule:

$$c(0, \ldots, 0, i_t, i_{t+1}, i_{t+2}, \ldots, i_{d-2}, i_{d-1}, i_d) = c(0, \ldots, 0, i_t - 1, (i_{t+1} + 1) \mod q, i_{t+2}, \ldots, i_{d-2}, i_{d-1}, i_d).$$

Geometrically, this means that we colour $H(d, q)$ layer by layer, with colours of the vertices in the present “layer” determined by colours of the vertices in the previous “layer”. Using (3) all vertices of $H(d, q)$ are coloured, and by induction we can prove that

$$c(0, \ldots, 0, i_t, i_{t+1}, i_{t+2}, \ldots, i_{d-2}, i_{d-1}, i_d) = c\left(0, \ldots, 0, \sum_{t=1}^{d-2} i_t \mod q, i_{d-1}, i_d\right).$$

By a similar argument as in the previous paragraph one can verify that under $c$ adjacent vertices receive colours with difference at least 2. Also, $c(0, \ldots, 0, 0, 1, i_d) = (2i_d + 3) \mod 2q$ takes all odd integers in $[0, 2q - 1]$ when $i_d$ runs from 0 to $q - 1$; and $c(0, \ldots, 0, i_{d-2}, 0, 0) = 2i_{d-2}$ takes all even integers in $[0, 2q - 1]$ when $i_{d-2}$ runs from 0 to $q - 1$. Thus, $c$ is a no-hole 2-distant colouring of $H(d, q)$ with span $2q - 1$. This completes the proof.

Now we come to the general case which is, interestingly, relatively easier to handle than $H(d, q)$.

![Figure 3: A no-hole 2-distant colouring of $H_{6,5}$.](image)

**Lemma 2.4** For any $d \geq 2$ integers $q_1, q_2, \ldots, q_d$ no less than 2 and not all the same, we have

$$\text{nsp}(H_{q_1, q_2, \ldots, q_d}) = 2 \max_{1 \leq i \leq d} q_i - 1.$$ 

**Proof** As before we suppose without loss of generality that $q_1 \geq q_2 \geq \cdots \geq q_d \geq 2$. Then $\max_{1 \leq i \leq q_i} = q_1$, and $q_1 > q_d$ by our assumption. By Lemma 2.1, it suffices to show
Recall that the vertices of $H_{q_1,q_2,\ldots,q_d}$ are of the form $(i_1, i_2, \ldots, i_d) \in \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \cdots \times \mathbb{Z}_{q_d}$. Define
\[
c(i_1, i_2, \ldots, i_d) = \begin{cases} 
\left( \frac{2}{d} \sum_{t=1}^d i_t \right) \mod 2q_1, & i_2 \neq q_2 - 1; \\
\left( \frac{2}{d} \sum_{t=1}^d i_t + 1 \right) \mod 2q_1, & i_2 = q_2 - 1. 
\end{cases}
\] (4)

for $0 \leq i_t \leq q_t - 1$ and $1 \leq t \leq d$. See Figure 3 for a demonstration of this colouring for $H_{6,5}$. A routine checking ensures that, under this colouring $c$, adjacent vertices receive colours that differ by at least 2. We have $c(i_1, 0, \ldots, 0) = 2i_1$, which takes values $0, 2, \ldots, 2q_1 - 2$ when $i_1$ runs from 0 to $q_1 - 1$. Also, $c(i_1, q_2 - 1, 0, \ldots, 0) = (2i_1 + 2q_2 - 1) \mod 2q_1$, which takes values $2q_2 - 1, 2q_2 + 1, \ldots, 2q_1 - 1, 1, 3, \ldots, 2q_2 - 3$ when $i_1$ runs from 0 to $q_1 - 1$. So $c$ is a no-hole 2-distinct colouring of $H_{q_1,q_2,\ldots,q_d}$ with span $2q_1 - 1$. Thus, $\text{nsp}(H_{q_1,q_2,\ldots,q_d}) \leq 2q_1 - 1$ and the proof is complete.

Combining Lemmas 2.2, 2.3 and 2.4 we get Theorem 1.1 immediately.

3 Concluding remarks

Note that, in the case where all $q_1, q_2, \ldots, q_d$ are the same, the colouring $c$ defined by (4) is not a 2-distinct colouring of $H_{q_1,q_2,\ldots,q_d}$. Thus, the colouring scheme (4) for the general case does not apply to $H(d, q)$. In fact, if, say, $q_1 = q_2$, then $c(1, 0, \ldots, 0) = 2$ and $c(1, q_1 - 1, \ldots, 0) = 1$, violating the 2-distinct condition since $(1, 0, \ldots, 0)$ and $(1, q_1 - 1, \ldots, 0)$ are adjacent vertices. Similarly, our colouring scheme for $H(d, q)$, $d \geq 3$, does not apply to $H(2, q)$, and vice versa. Note also that in general optimal no-hole 2-distinct colouring of $H_{q_1,q_2,\ldots,q_d}$ is not necessarily unique. For example, in Figure 4 we give such a colouring for $H(2, 6)$ that is different from the one in Figure 1.

As shown in [13], some Hamming graphs admit no-hole 2-distinct colouring with the additional property that vertices distance 2 apart receive distinct colours. Such colourings are no-hole $L(2,1)$-labellings in terms of [8]. The minimum span of such labellings is much larger than $\text{nsp}(H_{q_1,q_2,\ldots,q_d})$, at least $q_1q_2 - 1$, with the bound attained under certain circumstances [13] (assuming $q_1 \geq q_2 \geq \cdots \geq q_d \geq 2$ as before). In general we may ask the following questions.

Question 3.1 Does every Hamming graph other than $Q_2$ admit a no-hole $L(2,1)$-labelling? And, for those Hamming graphs which do admit, what is the minimum span of such labellings?

This is related to Question 4.5 of [13], which asks whether the minimum span of $L(2,1)$-labellings of any Hamming graph $H_{q_1,q_2,\ldots,q_d}$, other than the hypercube $Q_d$, is always $q_1q_2 - 1$. If this is true, then the answer to the question above is affirmative for such Hamming graphs and the minimum span of no-hole $L(2,1)$-labellings of $H_{q_1,q_2,\ldots,q_d}$ is $q_1q_2 - 1$ as well.
Figure 4: Another no-hole 2-distant colouring of $H(2, 6)$.

References


