Pairing Problem of Generators in Affine Kac-Moody Lie Algebras *

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Abstract

In this paper, we discuss the pair problem of generators in affine Kac-Moody Lie algebras. For any affine Kac-Moody algebra $g(A)$ of $X_l^{(k)}$ type and arbitrary nonzero imaginary root vector $x$, we prove that there exists some $y \in g(A)$, such that $g'(A)$ is contained in the Lie algebra generated by $x$ and $y$.

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§ 0. Introduction

In the field of Lie algebras, it is a basic and interesting problem to discuss the minimal number of generators and describe the properties of the minimal set of generators. In [1], it is proved that any finite dimensional semisimple Lie algebra over a field $F$ with Char$F = 0$ can be generated by two elements. In [2], this result is extended to general Lie algebra $g(A)$, where $A$ is an arbitrary $n$–index complex matrix. Assume $\text{Rank}(A) = l$, then $g(A)$ can be generated by two elements if and only if $l \geq n - 2$. When a Lie algebra $L$ can be generated by two elements, does there exist $y \in L$ such that $L$ can be generated by $x$ and $y$ for a given $x$? In the case of finite dimension and complex field, it was proved in [3] in 1976 that:

For any given non-zero $x$ in a simple Lie algebra $L$, one can find an element $y \in L$ such that $x$ and $y$ generate $L$.

Then how about $L$ is infinite dimensional? In [5], several results for some special cases are given:

Let $g(A)$ be a Kac-Moody algebra. For any given $h$ in a Cartan subalgebra, if $h$ is not in the center and not any real root vector $x_\beta$, then there exists an element $y$, such that $g'(A)$ is contained in the Lie subalgebra generated by $\{h, y\}$ or $\{x_\beta, y\}$.

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In [6], it is proved that:

If $A$ is a general Cartan matrix of type $X^{(1)}_l$, where $X = A, B, C$ or $D$, then for any imaginary root vector $x$, there exists $y \in g(A)$, such that $g'(A)$ is contained in the Lie subalgebra generated by $\{x, y\}$. But we can’t draw a good conclusion for general affine type using the methods in [6].

In this paper, we find a new method, with which we extend the results above to general affine Lie algebras.

§ 1. Main Lemmas

In order to get our result, we need the following two lemmas:

Lemma 1.1. Assume that $L$ is a complex simple Lie algebra of finite type, then for any non-zero element $h$ in a Cartan subalgebra $H$, there exists a prime root system $\Pi$, such that for arbitrary $\alpha_j, \alpha_k \in \Pi$, if

$$\alpha_j(h) + \alpha_k(h) = 0,$$

then

$$\alpha_j(h) = \alpha_k(h) = 0.$$

Proof: Assume $H_R$ is the $\mathbb{R}$-type real space of $H$, that is:

$$H \cong \mathbb{C} \otimes_R H_R, \quad H_R \subseteq H,$$

then $h$ has a unique decomposition:

$$h = h_1 + ih_2, h_1, h_2 \in H_R,$$

where $i = \sqrt{-1}$. Hence there exists a prime root system $\Pi'$, such that $\alpha_j(h_1) \geq 0$ for any $\alpha_j \in \Pi'$.

Set

$$\Pi'_1 = \{\alpha_j \mid \alpha_j(h_1) > 0\},$$

$$\Pi'_2 = \{\alpha_j \mid \alpha_j(h_1) = 0\},$$

$$W_2 = \langle r_j \mid \alpha_j \in \Pi'_2 \rangle,$$

then for arbitrary $w \in W_2$, $\alpha \in \Pi'_1$, $w(\alpha)(h_1) > 0$. Meanwhile, $W_2$ is the Weyl group of $L_2$ generated by $\{e_i, f_i \mid \alpha_i \in \Pi'_2\}$. Assume $H_2$ is the $\mathbb{R}$-type real space of the Cartan subalgebra of $L_2$, then

$$(H_2)^* = \bigoplus_{\alpha_i \in \Pi'_2} \mathbb{R} \alpha_i.$$  (1.5)

Since the Cartan matrix of $\Pi'_2$ is positive definite and its rank is finite, so the map:

$$f : H_2 \to ((H_2)^*)^*$$

$$f(h)(\alpha) = \alpha(h)$$

is bijective. In fact $h_2$ defines a function $\rho$ over $(H_2)^*$

$$\rho : \rho(\alpha) = \alpha(h_2),$$
so there must be a unique $h' \in H_2$ such that $\alpha(h_2 - h') = 0$ for any $\alpha \in (H_2)^*$. By the properties of semisimple Lie algebras we know that there exists $w \in W_2$ such that $h'$ is in the fundamental Weyl chamber determined by $w(\Pi'_2)$, that is, $w(\alpha)(h') \geq 0$ for all elements in $\Pi'_2$, hence

$$w(\alpha)(h_2) \geq 0, \forall \alpha \in \Pi'_2.$$  

Let $\Pi = w(\Pi')$, for all $w(\alpha_j), w(\alpha_k) \in \Pi$, if $w(\alpha_j)(h) + w(\alpha_k)(h) = 0$, then

$$w(\alpha_j)(h_1) + w(\alpha_k)(h_1) = 0,$$

$$w(\alpha_j)(h_2) + w(\alpha_k)(h_2) = 0,$$

from (1.6), $\alpha_j, \alpha_k \in \Pi'_2$, and from (1.7), we have $w(\alpha_j)(h) = w(\alpha_k)(h) = 0$, so $\Pi$ is the prime root system which is we need.

**Lemma 1.2.** Assume that $\{s_1, \cdots, s_m\}$ is a set of complex numbers, such that for all $(i, j)$,

$$s_i + s_j = 0 \text{ if and only if } s_i = s_j = 0,$$

matrix $B$ is defined as:

$$B = \begin{bmatrix}
1 & s_1 \cdots s_1 & s_1 \cdots s_1 & \cdots & s_1 \cdots s_1 \\
-1 & s_2 \cdots s_2 & s_2 \cdots s_2 & \cdots & s_2 \cdots s_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(-1)^{k+1} & s_{k+1} \cdots s_{k+1} & s_{k+1} \cdots s_{k+1} & \cdots & s_{k+1} \cdots s_{k+1} \\
(-1)^{q+1} & s_{q+1} \cdots s_{q+1} & s_{q+1} \cdots s_{q+1} & \cdots & s_{q+1} \cdots s_{q+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & s_m \cdots s_m & s_m \cdots s_m & \cdots & s_m \cdots s_m
\end{bmatrix},$$

where $p = 2q + 1$, $q = m(m - 1) + 1$. Then there exists a vector $v = (v_1, v_2, \cdots, v_q)$, such that $vB = (1, 0, 0, \cdots, 0)$.

**Proof:** We can directly suppose that

$$s_i \neq 0; s_i = s_j \text{ if and only if } i = j.$$  

Let matrix $D = (d_{kl})_{q \times q}$, where $d_{kl} = \begin{pmatrix} k-1 \\ l-1 \end{pmatrix}$, $k, l = 1, \cdots, q$. then

$$DB = \begin{bmatrix}
1 & a_{1,2;1} \cdots a_{1,m;1} & a_{1,2;3,1} \cdots a_{m-1,m;1} \\
0 & a_{1,2;2} \cdots a_{1,m;2} & a_{1,2;3,2} \cdots a_{m-1,m;2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a_{1,2;k} & a_{1,m;k} & a_{1,2;3,k} & a_{m-1,m;k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a_{1,2;q} & a_{1,m;q} & a_{1,2;3;q} & a_{m-1,m;q}
\end{bmatrix},$$

where $a_{i,j;kl} = (s_i + s_j)^{k-1}(s_i p^{-k}s_j - s_j p^{-k}s_i)$. Because $D$ is invertible, hence if the lemma holds for $DB$ so does for $B$. Assume

$$B' = \begin{bmatrix}
1 & a'_{1,2;1} \cdots a'_{1,m;1} & a'_{1,2;3,1} \cdots a'_{m-1,m;1} \\
0 & a'_{1,2;2} \cdots a'_{1,m;2} & a'_{1,2;3,2} \cdots a'_{m-1,m;2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a'_{1,2;k} & a'_{1,m;k} & a'_{1,2;3,k} & a'_{m-1,m;k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a'_{1,2;q} & a'_{1,m;q} & a'_{1,2;3;q} & a'_{m-1,m;q}
\end{bmatrix},$$
where $a'_{i,j;k} = (s_i + s_j)^{k-1}s_i^{p-k}s_j$, $a''_{i,j;k} = -(s_i + s_j)^{k-1}s_j^{p-k}s_i$. Obviously if there is a vector $v = (v_1, v_2, \ldots, v_q)$ such that

$$vB' = (1, 0, 0, \cdots, 0)_q,$$

then

$$vDB = (1, 0, 0, \cdots, 0).$$

Denote by

$$B' = (\gamma_0, \gamma_1, \cdots, \gamma_{q-1}).$$

If the rank of $(\gamma_1, \cdots, \gamma_{q-1})$ equals $R$, without losing generality, we can suppose that

$$\text{Rank}(\gamma_1, \cdots, \gamma_R) = R.$$ 

Denote by $B'' = (\gamma_0, \gamma_1, \cdots, \gamma_R)$. If the lemma is true for $B''$, it must be true for $B'$. So it is true for $B$. $B''$ is a matrix of the following form

$$\begin{bmatrix}
1 & z_1 & \cdots & z_R \\
0 & z_1\lambda_1 & \cdots & z_R\lambda_R \\
\vdots & \vdots & \ddots & \vdots \\
0 & z_1\lambda_{q-1} & \cdots & z_R\lambda_{q-1}
\end{bmatrix},$$

and all $z_i, \lambda_j$ are not 0. With $R < q$ and the properties of Vandermonde determinant, it is easy to know that there exists an element $v$ such that $vB'' = (1, 0, 0, \cdots, 0)$, so the lemma is true.

## § 2. Proof of non-twisted case

Assume $g$ is a finite dimensional complex simple Lie algebra of type $X_{li}$, then there is a realization of affine Kac-Moody algebra $g(A)$ of type $X_{li}^{(1)}$:

$$L(g) = C[t, t^{-1}] \otimes g + Cc + Cd. \quad (2.1)$$

Now $g'(A) = [L(g), L(g)] = C[t, t^{-1}] \otimes g + Cc$. Hence if $x \in L(g)$ is an imagine root vector, then $x = t^k \otimes h$, where $k$ is a non-zero integer, $h$ is a non-zero element in $H$, the Cartan subalgebra of $g$. From lemma 1, we can choose a good prime root system $\Pi$, such that for all $\alpha, \beta \in \Pi$,

$$\alpha(h) + \beta(h) = 0 \Leftrightarrow \alpha(h) = \beta(h) = 0.$$ 

Suppose $k > 0$, let

$$y = t^{-nk} \otimes \left( \sum_{i=1}^l a_i e_i \right) + \left( \sum_{i=1}^l f_i \right) + t \otimes h',$$

where $\{e_i, f_i \mid i = 1, \cdots, l\}$ is the Chevalley generator set of $g$, $h_i := [e_i, f_i]$, $h' \in H$, and for any $i$ one has $\alpha_i(h') \neq 0$, from the properties of semisimple Lie algebras we know the existence of such an $h'$. Non-zero complex coefficient $a_i$ is to be determined. Assume $n = 2l(l-1) + 3$, $q = l(l-1) + 1$, then $n = 2q + 1$. 


Note $y_j := (\text{ad}x)^j(y), j = 1, \ldots, n$, from (1.4) one has

$$y_j = \sum_{i=1}^{l} (a_i (\text{ad}x)^j(t^{-nk} \otimes e_i) + (\text{ad}x)^j(f_i)) + (\text{ad}x)^j(t \otimes h')$$

$$= \sum_{i=1}^{l} (a_i t^{jk-nk} \otimes (\text{ad}h)^j(e_i) + t^{jk} \otimes (\text{ad}h)^j(f_i))$$

$$= t^{jk-nk} \otimes \left( \sum_{i=1}^{l} \alpha_i (h)^j a_i e_i \right) + t^{jk} \otimes \left( \sum_{i=1}^{l} (\alpha_i(h))^{j} f_i \right),$$

then denote

$$Y_j := [y_j, y_{n-j}], \quad j = 1, \ldots, q.$$

Now we have

$$Y_j = \left[ t^{jk-nk} \otimes \left( \sum_{i=1}^{l} \alpha_i (h)^j a_i e_i \right), t^{-jk} \otimes \left( \sum_{i=1}^{l} \alpha_i (h)^{n-j} a_i e_i \right) \right]$$

$$+ \left[ t^{jk-nk} \otimes \left( \sum_{i=1}^{l} \alpha_i (h)^j a_i e_i \right), t^{jk} \otimes \left( \sum_{i=1}^{l} (\alpha_i(h))^{n-j} f_i \right) \right]$$

$$+ \left[ t^{jk} \otimes \left( \sum_{i=1}^{l} (\alpha_i(h))^{j} f_i \right), t^{-jk} \otimes \left( \sum_{i=1}^{l} \alpha_i (h)^{n-j} a_i e_i \right) \right]$$

$$+ \left[ t^{jk} \otimes \left( \sum_{i=1}^{l} (\alpha_i(h))^{j} f_i \right), t^{jk-nk} \otimes \left( \sum_{i=1}^{l} (\alpha_i(h))^{n-j} f_i \right) \right]$$

$$= \left( \sum_{1 \leq i < r \leq l} a_i a_r (\alpha_i(h)^j \alpha_r(h)^{n-j} - \alpha_i(h)^{n-j} \alpha_r(h)^j) t^{-nk} \otimes [e_i, e_r] \right)$$

$$+ \left( \sum_{i=1}^{l} \alpha_i(h)^n(-1)^{n-j} h_i \right) + \lambda_j c - \left( \sum_{i=1}^{l} \alpha_i(h)^n(-1)^j h_i \right)$$

$$+ \left( \sum_{1 \leq i < r \leq l} (-1)^n(\alpha_i(h)^j \alpha_r(h)^{n-j} - \alpha_i(h)^{n-j} \alpha_r(h)^j) t^{nk} \otimes [f_i, f_r] \right),$$

where $\lambda_j$ is some complex number correspond with $j$. And because $n$ is an odd integer, so

$$Y_j = \left( \sum_{i=1}^{l} 2(-1)^{j+1} \alpha_i(h)^n h_i \right) + \lambda_j c$$

$$+ \sum_{1 \leq i < r \leq l} (\alpha_i(h)^j \alpha_r(h)^{n-j} - \alpha_i(h)^{n-j} \alpha_r(h)^j)(a_i a_r t^{-nk} \otimes [e_i, e_r] - t^{nk} \otimes [f_i, f_r]).$$

Let

$$\lambda = 2 \sum_{i=1}^{l} \alpha_i(h)^n h_i,$$

$$M_{i,r,j} = \alpha_i(h)^j \alpha_r(h)^{n-j} - \alpha_i(h)^{n-j} \alpha_r(h)^j,$$

$$E_{i,r} = a_i a_r t^{-nk} \otimes [e_i, e_r] - t^{nk} \otimes [f_i, f_r],$$

then

$$Y_j = (-1)^{j+1} \lambda + \lambda_j c + \sum_{1 \leq i < r \leq l} M_{i,r,j} E_{i,r}.$$
From lemma 2, there exists a vector \((v_1, \cdots, v_q)\), such that
\[
\lambda = \sum_{j=1}^{q} v_j(Y_j - \lambda c),
\]
hence
\[
\lambda' := \lambda + \sum_{j=1}^{q} v_j \lambda c \in L(x, y).
\]
Note \(St_1 := \{i \mid \alpha_i(h) \neq 0\}\), and denote the Lie subalgebra generated by \((e_i, f_i) \mid i \in St_1\) by \(g(St_1)\), then \(g(St_1)\) a semisimple subalgebra of \(g\), and \(\lambda \in H \cap g(St_1)\), it is easy to know there exists non-zero complex numbers \(\{a_i \neq 0 \mid i \in St_1\}\) such that \(\alpha_i(\lambda) > 0\) \((i \in St_1)\) and every \(a_i\) is not equal to any else, so \(\alpha_i(\lambda') > 0(i \in St_1)\) and they are pairwise different. Determine these \(a_i\)'s, then one can compose \(t^{k-nk} \otimes e_i\) and \(t^k \otimes f_i\) linearly by
\[
(ad\lambda')^m(y_1), \ (m = 1, \cdots, 2|St_1|),
\]
where \(i\) runs through \(St_1\). And then we can get
\[
t^{qk-nk} \otimes e_i, t^qk \otimes f_i \in L(x, y), \ \forall q \in \mathbb{N},
\]
by the action of \(x\). In particular, \(h_i + kc = [t^{-k} \otimes e_i, t^k \otimes f_i]\), and \(h_i + 2kc = [t^{-2k} \otimes e_i, t^{2k} \otimes f_i]\) are both in \(L(x, y)\), so \(h_i, c \in L(x, y)\).

If \(St_1 \neq \{1, \cdots, l\}\), set \(x_1 := t^k \otimes h^*\), where
\[
h^* = \sum_{i \in St_1} m_i h_i, \ \ m_i > 0,
\]
and there exists \(i_0 \notin St_1, \ \alpha_{i_0}(h^*) \neq 0\), then
\[
[x_1, y] = t^{k-nk} \otimes \left( \sum_{i=1}^{1} \alpha_i(h^*) a_i e_i \right) + t^k \otimes \left( \sum_{i=1}^{1} (-\alpha_i(h^*)) f_i \right).
\]
But for any \(i \in St_1\), one has \(t^{k-nk} \otimes e_i \in L(x, y), t^k \otimes f_i \in L(x, y)\). Let \(St_2 := \{i \notin St_1 \mid \alpha_i(h^*) < 0\}\), if there is some \(i_0 \notin St_1 \cup St_2\), then \(\alpha_{i_0}(h^*) = 0\), so
\[
t^{k-nk} \otimes \left( \sum_{i \in St_2} \alpha_i(h^*) a_i e_i \right) + t^k \otimes \left( \sum_{i \in St_2} (-\alpha_i(h^*)) f_i \right) \in L(x, y),
\]
set
\[
y'_1 = t^{k-nk} \otimes \left( \sum_{i \in St_2} \alpha_i(h^*) a_i e_i \right) + t^k \otimes \left( \sum_{i \in St_2} (-\alpha_i(h^*)) f_i \right),
\]
define \(y'_{j+1} = [x_1, y'_j], \ j = 1, \cdots, n - 1, \) and \(Y'_j = [y'_j, y'_{j-1}], \ j = 1, \cdots, q\). In the same way one can determine each \(a_i, \ (i \in St_2)\), this follows
\[
h_i \in L(x, y), \ \forall i \in St_2.
\]
Repeat this process, since \(\Pi\) is finite, after finite times (denoted by \(m\)), one can get
\[
\bigcup_{r=1}^{m} St_r = \{1, \cdots, l\},
\]
so for any $i \in \{1, \cdots, l\}$, $h_i \in L(x, y)$, and then $f_i, t^{-nk} \otimes e_i, t \otimes h' \in L(x, y)$, for any $i$. Hence 

$$(\alpha_i(h'))^{nk}e_i = (\text{ad} t \otimes h')^{nk}(t^{-nk} \otimes e_i) \in L(x, y).$$

Because $(\alpha_i(h'))^{nk} \neq 0$, we have $e_i \in L(x, y), \forall i \in \{1, \cdots, l\}$, Thus

$$g = \langle e_i, f_i \mid i = 1, \cdots, l \rangle, \forall g, t \otimes g = \langle g, t \otimes h' \rangle \subseteq L(x, y).$$

and from $t^{-nk} \otimes e_i \in L(x, y)$, we have

$$t^{-nk} \otimes g \subseteq \langle g, t^{-nk} \otimes e_i \rangle \subseteq L(x, y).$$

So $t^{-1} \otimes g \subseteq \langle t \otimes g, t^{-nk} \otimes g \rangle \subseteq L(x, y)$, and

$$g'(A) \subseteq \langle t \otimes g, t^{-1} \otimes g \rangle \subseteq L(x, y).$$

(2.2)

When $k < 0$, the same conclusion can be obtained with similar method. So we have

**Theorem 2.1.** For any non-twisted affine Kac-Moody Lie algebra $g(A)$ and its arbitrary imaginary root vector $x$, there exists an element $y \in g(A)$ such that $g'(A)$ is contained in the subalgebra generated by $\{x, y\}$.

§ 3. CASE OF TWISTED NUMBER EQUALS TO 2

Recall the realization of affine Lie algebra of type $X_l^{(2)}$:

$$\mathbb{C}[t^2, t^{-2}] \otimes g_0 + t\mathbb{C}[t^2, t^{-2}] \otimes g_1 + \mathbb{C}c + Cd,$$

(3.1)

where $g_0 + g_1 = g$ is a finite dimensional complex simple Lie algebra of type $X_l$, $g_0$ is a simple subalgebra of $g$, and $g_1$ is a simple $g_0$-module. Now

$$g'(A) = \mathbb{C}[t^2, t^{-2}] \otimes g_0 + t\mathbb{C}[t^2, t^{-2}] \otimes g_1 + \mathbb{C}c.$$

If $x = t^{2k} \otimes h$ is a root vector, then $h$ lies in the Cartan subalgebra of $g_0$. If $g_0$ is of type $Y_l$, it is easy to prove that

$$L = \mathbb{C}[t^2, t^{-2}] \otimes g_0 + Cc + Cd$$

is an affine Lie algebra of type $Y_l^{(1)}$, and $x \in L$. From theorem 2.1, there is $y' \in L$ that $L^{(1)} \subseteq L(x, y')$. Let

$$y = y' + t \otimes h',$$

where $h' \in H \cap g_1, h' \neq 0$. Because $x$ is commutative with $t \otimes h'$, and by the proof above, we know

$$L^{(1)} \subseteq \sum_{j \geq 2} V^j,$$

where

$$V = \mathbb{C}x + C y' = V^1, \forall j \geq 2, V^j = \sum_{r+s=j} [V^r, V^s],$$

so for any $j \geq 2$, $V^j \subseteq L(x, y)$, hence $L^{(1)} \subseteq L(x, y)$, thus

$$t \otimes h' = y - y' \in L(x, y).$$
Since \( g_1 \) is a simple \( g_0 \)-module, then
\[
t^{-1} \otimes g_1 \subseteq \langle t \otimes h', t^{-2} \otimes g_0 \rangle,
\]
so
\[
g'(A) \subseteq \langle t^{-1} \otimes g_1, L^{(1)} \rangle \subseteq L(x, y). \tag{3.2}
\]

If \( x = t^{2k-1} \otimes h \), then \( h \) is in \( g_1 \cap H \), obviously the action of \((ad h)^2\) on \( g_0 \) is diagonal, so there is \( h_{\mu} \in g_0 \cap H \), so that \((ad h_{\mu} - (ad h)^2)(g_0) = 0\). In particular, one can let
\[
x_1 = t^{4k-2} \otimes h_{\mu},
\]
then
\[
[x_1, b] - [x, [x, b]] \in \mathbf{C}c,
\]
for all \( b \in L \), so there is an element \( y \in L \) satisfies \( L^{(1)} \subseteq L(x_1, y) \). Meanwhile,
\[
L^{(1)} \subseteq \sum_{j \geq 2} V^j,
\]
where
\[
V = Cx + Cy = V^1, V^j = \sum_{r+s \geq j} [V^r, V^s], \forall j \geq 2,
\]
and \( c \in L^{(1)} \), so \( L^{(1)} \subseteq L(x, y) \). Because \( x \in L(x, y) \), hence
\[
t^{2k-1} \otimes g_1 \subseteq \langle g, x \rangle \subseteq L(x, y),
\]
thus
\[
g'(A) \subseteq \langle L^{(1)}, t^{2k-1} \otimes g_1 \rangle \subseteq L(x, y). \tag{3.3}
\]

From (2.2) and (3.1) one can get

**Theorem 3.1.** For any affine Kac-Moody Lie algebra \( g(A) \) with twisted number \( k = 2 \) and its arbitrary imaginary root vector \( x \), there exists an element \( y \in g(A) \) such that \( g'(A) \) is contained in the subalgebra generated by \( \{x, y\} \).

§ 4. CASE OF TWISTED NUMBER EQUALS TO 3

If \( g(A) \) is an affine Lie algebra with twisted number 3, then it is only of type \( D_4^{(3)} \), it has a realization:
\[
\left( \sum_{i \in \mathbf{Z}} t^{3i} \otimes g_0 \right) + \left( \sum_{i \in \mathbf{Z}} t^{3i+1} \otimes g_1 \right) + \left( \sum_{i \in \mathbf{Z}} t^{3i-1} \otimes g_2 \right) + \mathbf{Cc} + \mathbf{Cd} \tag{4.1}
\]
where \( g_0 + g_1 + g_2 \) is a simple complex Lie algebra of type \( D_4 \), and \( g_0 \) is a simple subalgebra of type \( G_2 \) of \( g(A) \), \( g_1 \) and \( g_2 \) are isomorphic simple \( g_0 \)-modules. Similarly
\[
G = \left( \sum_{i \in \mathbf{Z}} t^{3i} \otimes g_0 \right) + \mathbf{Cc} + \mathbf{Cd},
\]
is a non-twisted affine Kac-Moody algebra and it is of type \( G_2^{(1)} \). Now
\[
g'(A) = \left( \sum_{i \in \mathbf{Z}} t^{3i} \otimes g_0 \right) + \left( \sum_{i \in \mathbf{Z}} t^{3i+1} \otimes g_1 \right) + \left( \sum_{i \in \mathbf{Z}} t^{3i-1} \otimes g_2 \right) + \mathbf{Cc}.
\]
If \( x \in G \), then there exists \( y' \in G \) such that \( G^{(1)} \subseteq L(x, y') \). Let \( h' \in g_1 \cap H \), and \( y = y' + t \otimes h' \). Since \( x \) and \( t \otimes h' \) are commutative, \( G^{(1)} \subseteq L(x, y) \). Hence \( t \otimes h' \in L(x, y) \). But \( g_1 \) is a simple \( g_0 \)-module, one has
\[
t^{3i+1} \otimes g_1 \subseteq L(x, y), \forall i \in \mathbb{Z},
\]
and then
\[
0 \neq [t^{3i-2} \otimes g_1, t \otimes g_1] \subseteq t^{3i-1} \otimes g_2.
\]
Since \( g_2 \) is also a simple \( g_0 \)-module, thus
\[
t^{3i-1} \otimes g_2 \subseteq L(x, y), \forall i \in \mathbb{Z}.
\]
Finally, we get
\[
g'(A) \subseteq L(x, y).
\]

If \( x = t^{3i+1} \otimes h \) for some \( i \), then \( h \in g_1 \cap H \). At first, we discuss the case \( i \geq 0 \). Because the Cartan subalgebra of \( g \) is 4-dimensional, the Cartan subalgebra of \( g_0 \) is 2-dimensional, and \( g_1 \) is isomorphic to \( g_2 \) as \( g_0 \)-modules, thus \( \dim(H \cap g_1) = 1 \). Assume \( \omega = e^{2\pi i/3} \), the Chevalley generators of \( g \) are
\[
\{e_i, f_i \mid i = 1, 2, 3, 4\},
\]
\[
h_i = [e_i, f_i], \quad i = 1, 2, 3, 4,
\]
and \( e_1, e_3, e_4 \) are pairwise commutative. Then we have \( h = K(h_1 + \omega h_3 + \omega^2 h_4) \), and we can suppose \( K = 1 \). Let
\[
y = e_2 + f_2 + t^{-9(3i+1)} \otimes (e_1 + e_3 + e_4) + (f_1 + f_3 + f_4) + t \otimes h,
\]
set \( y_j := (\text{ad}x)^j(y) \), then
\[
y_3 = 8t^{-6(3i+1)} \otimes (e_1 + e_3 + e_4) - t^{3(3i+1)} \otimes (f_1 + f_3 + f_4),
\]
\[
y_6 = 64t^{-3(3i+1)} \otimes (e_1 + e_3 + e_4) + t^6(3i+1) \otimes (f_1 + f_3 + f_4),
\]
\[
[y_3, y_6] = 72(h_1 + h_2 + h_3 + K' c) \subseteq L(x, y).
\]
So \( \lambda = h_1 + h_2 + h_3 + K' c \in L(x, y) \). Set
\[
y^j := (\text{ad}\lambda)^j(y),
\]
then all \( e_2, f_2, t^{-9(3i+1)} \otimes (e_1 + e_3 + e_4), (f_1 + f_3 + f_4) \) can be represented linearly by \( y^1, y^2, y^3 \) and \( y^4 \):
\[
\frac{1}{90} (12y^1 - 4y^2 - 3y^3 + y^4) = e_2,
\]
\[
-\frac{1}{90} (12y^1 + 4y^2 - 3y^3 - y^4) = f_2,
\]
\[
\frac{1}{40} (18y^1 + 9y^2 - 2y^3 - y^4) = t^{-9(3i+1)} \otimes (e_1 + e_3 + e_4),
\]
\[
-\frac{1}{40} (18y^1 - 4y^2 - 2y^3 + y^4) = f_1 + f_3 + f_4,
\]
that is, they all lie in \( L(x, y) \), thus \( t \otimes h \) also lies in \( L(x, y) \). And
\[
(\text{ad}x)^9(t^{-9(3i+1)} \otimes (e_1 + e_3 + e_4)) = 2^9(e_1 + e_3 + e_4),
\]
(\text{ad}(t \otimes h))^{9(3i+1)-1}(t^{-9(3i+1)} \otimes (e_1 + e_3 + e_4)) = 2^{9(3i+1)-1}t^{-1} \otimes (e_1 + \omega^2 e_3 + \omega e_4),

so

\begin{align*}
g_0 &= \langle e_1 + e_3 + e_4, f_1 + f_3 + f_4, e_2, f_2 \rangle \subseteq L(x, y), \\
t \otimes h &\in t \otimes g_1 \cap L(x, y) \neq 0, \\
t^{-1} \otimes (e_1 + \omega^2 e_3 + \omega e_4) &\in (t^{-1} \otimes g_2) \cap L(x, y) \neq 0,
\end{align*}

and then

\begin{align*}
t \otimes g_1, t^{-1} \otimes g_2 &\subseteq L(x, y).
\end{align*}

hence

\begin{align*}
g'(A) &\subseteq (g_0, t \otimes g_1, t^{-1} \otimes g_2) \subseteq L(x, y). \quad (4.2)
\end{align*}

When \( i < 0 \), we have the same conclusion. If \( x = t^{3i-1} \otimes h \), then \( h \in g_2 \), and the proof is similar as above. So we can get

**Theorem 4.1.** For an affine Kac-Moody Lie algebra \( g(A) \) with twisted number \( k = 3 \) and its arbitrary imaginary root vector \( x \), there exists an element \( y \in g(A) \) such that \( g'(A) \) is contained in the subalgebra generated by \( \{ x, y \} \).

Now combine theorem 2.1, 3.1 and 4.1, we get

**Theorem 4.2** For any affine Kac-Moody Lie algebra \( g(A) \) and its arbitrary imaginary root vector \( x \), there exists an element \( y \in g(A) \) such that \( g'(A) \) is contained in the subalgebra generated by \( \{ x, y \} \).

**REFERENCES**


