Hamiltonicity and Consecutive $L(2, 1)$-labelings

Changhong Lu†
Department of Mathematics
East China Normal University
Shanghai, 200062, P.R.China

Mingqing Zhai
Department of Mathematics and Computer Science
Chuzhou University, Chuzhou
Anhui, 239012, P.R. China

Abstract

For a given graph $G$ of order $n$, an $L(2, 1)$-labelling is defined as a function $f : V(G) \rightarrow \{0, 1, 2, \cdots \}$ such that $|f(u) - f(v)| \geq 2$ when $d_G(u, v) = 1$ and $|f(u) - f(v)| \geq 1$ when $d_G(u, v) = 2$, is the minimum length of a path between $u$ and $v$. A $k$-$L(2, 1)$-labelling is an $L(2, 1)$-labelling such that no label is greater than $k$. The $L(2, 1)$-labelling number of $G$, denoted by $\lambda(G)$, is the smallest number $k$ such that $G$ has a $k$-$L(2, 1)$-labelling. The consecutive $L(2, 1)$-labelling is a variation of $L(2, 1)$-labelling under the condition that the integers used are consecutive. The consecutive $L(2, 1)$-labelling number of $G$ is denoted by $\lambda'(G)$. Obviously, $\lambda(G) \leq \lambda'(G) \leq n - 1$ if $G$ admits a consecutive $L(2, 1)$-labelling. In this paper, we consider the graphs with $\lambda'(G) = n - 1$. The main results include: (1) For any two integers $n, m$ with $n \geq 3$ and $n - 2 \leq m \leq \frac{(n-1)(n-2)}{2}$, there exists a simple graph $G$ of order $n$ and size $m$ with $\lambda'(G) = n - 1$. And the graphs $G$ with $\lambda'(G) = n - 1$ and size $n - 2$ (or size $\frac{(n-1)(n-2)}{2}$) are completely determined; (2) For any two integers $n, m$ with $n \geq 4$ and $1 \leq m \leq \frac{n+1}{2}$, there exists a simple graph $G$ of order $n$ and $C(G) = m$ with $\lambda'(G) = n - 1$, where $C(G)$ denote the number of components of $G$. And the graphs $G$ with $\lambda'(G) = n - 1$ and $C(G) = \lceil \frac{n+1}{2} \rceil$ are completely determined; (3) Let $G$ be a connected graph of order $n \geq 6$ and its diameter $d$. If $\lambda'(G) = n - 1$, then $2 \leq d \leq \lceil \frac{n}{2} \rceil + 1$. Moreover for every two integers $n$ and $m$ with $n \geq 6$ and $2 \leq m \leq \lceil \frac{n+2}{2} \rceil$, there exists a simple graph $G$ of order $n$ and diameter $m$ with $\lambda'(G) = n - 1$. (4) For any integers $m, n$ with $2 \leq m \leq n - 1$, there exists a simple graph $G$ of order $n$ with $\lambda'(G) = m$.

Keywords: Channel assignment problems; Distance-two Labelling; Hamiltonian; $L(2, 1)$-labelling; No-hole coloring

*Supported in part by National Natural Science Foundation of China (No.10301010 ) and Science and Technology Commission of Shanghai Municipality (No. 04JC14031).
†Corresponding author. E-mail: chlu@math.ecnu.edu.cn
1 Introduction

The problem of vertex labelling with a condition at distance two, proposed by Griggs and Roberts [9], arose from a variation of channel assignment problem introduced by Hale [10]. Suppose a number of transmitters are given. We must assign a channel to each of the given transmitters such that the interference is avoided. In order to reduce the interference, any two “close” transmitters must receive different channels, and any two “very close” transmitters must receive channels at least two apart. One can construct an interference graph for this problem so that the transmitters are the vertices and there is an edge joining two “very close” transmitters. Two transmitters are defined as “close” if the corresponding vertices are of distance two.

Then, for a given graph $G$, an $L(2,1)$-labelling is defined as a function $f : V(G) \rightarrow \{0, 1, 2, \ldots \}$ such that $|f(u) - f(v)| \geq 2$ when $d_G(u, v) = 1$ and $|f(u) - f(v)| \geq 1$ when $d_G(u, v) = 2$, where $d_G(u, v)$, the distance between $u$ and $v$, is the minimum length of a path between $u$ and $v$. A $k$-$L(2,1)$-labelling is an $L(2,1)$-labelling such that no label is greater than $k$. The $L(2,1)$-labelling number of $G$, denoted by $\lambda(G)$, is the smallest number $k$ such that $G$ has a $k$-$L(2,1)$-labelling. The $L(2,1)$-labelling problem has been extensively studied during the past decade (see the references).

Another related interesting problem called consecutive 2-distant coloring of a graph was first introduced in [16] under the name “no-hole 2-distant coloring”. For a simple graph $G = (V, E)$, a consecutive 2-distant coloring of $G$ is an assignment $f : V \rightarrow \{0, 1, 2, \ldots \}$ such that $|f(u) - f(v)| \geq 2$ when $d_G(u, v) = 1$ and $\{f(v) : v \in V\}$ is a set of consecutive integers. We call $sp(G, f) = \max \{f(u) - f(v) : u, v \in V\}$ the span of $f$. If $G$ admits a consecutive 2-distant coloring, then define $csp(G) = \min sp(G, f)$ with minimum taking over all such colorings $f$. The reader is referred to [1, 2, 12, 16, 17, 18, 19] for recent concerning consecutive 2-distant colorings and the minimum span $csp(G)$.

Motivated by concepts of $L(2,1)$-labelling and consecutive 2-distant coloring of graph, in this paper we will focus on channel assignments under the following constraints: (a) neighbouring transmitters use channels that differ by at least 2; (b) transmitters with distance two use channels that differ by at least 1; (c) channels used consist of a set of consecutive integers. The consecutive $L(2,1)$-labelling is a variation of $L(2,1)$-labelling under the condition that the integers used are consecutive. The definition of the consecutive $L(2,1)$-labelling number $\lambda(G)$ is the same as that of the $L(2,1)$-labelling number except that the integers used are consecutive. The concept of consecutive $L(2,1)$-labelings of graphs was first introduced in [5] under the name “no-hole $L(2,1)$-colorings”. Some results on consecutive $L(2,1)$-labelings of graphs can be found in [5] and [6].

Observe, Many graphs don’t admit a consecutive $L(2,1)$-labelling. For example, any complete graph $K_n$ (with $n \geq 2$) doesn’t admit consecutive $L(2,1)$-labelling. First we concentrate our attention on the existence of consecutive $L(2,1)$-labelling of graphs. From prior works in [8], [16] and [5], the existence of consecutive $L(2,1)$-labelings of graphs can be established by the following theorem, which shows that it is closely related to the consecutive 2-distant coloring and the $L(2,1)$-labelings. This observation is also one of the motivations of studying consecutive $L(2,1)$-labelings and the parameter $\lambda$.

**Theorem 1** For any graph $G$ of order $n$, the following three statements are equivalent.

1. $G$ admits a consecutive $L(2,1)$-labelling;
2. $G$ admits a consecutive 2-distant coloring;
3. The complement graph $G^c$ has a Hamilton path;
4. The $L(2,1)$-labelling number of $G$ is no more than $n - 1$.

Let $G$ be a graph of order $n$. If $G$ admits a consecutive $L(2,1)$-labelling, then we easily know
\[ \lambda(G) \leq \overline{\lambda}(G) \leq n - 1. \] (1)

In [5] and [6], it is shown that \( \lambda(G) = \overline{\lambda}(G) \) for all connected \( \Delta = 2 \) graphs and all trees with \( \Delta \geq 3 \) except path \( P_3 \) and cycles \( C_3, C_4, C_6 \) and star \( K_{1,n-1} \). In this paper, we will try to characterize these graphs of order \( n \) with \( \overline{\lambda}(G) = n - 1 \).

2 Hamiltonicity of the Complement graph

In this section we first fix some notation and terminology, and then study the Hamiltonicity of the complement graph \( G^c \), which pay an important rule in the proof of main results.

In Here, please introduce some notations, included Union and join et.al. Some other notations and terminology not introduced in here can be found in [20].

The following lemma can be found in [4] and [14].

**Lemma 2** (i) ([4]) If \( G \) is a simple graph of order \( n \geq 3 \) and \( \delta(G) \geq \frac{n}{2} \), then \( G \) is Hamiltonian;
(ii) ([4]) If \( G \) is a simple graph with \( \delta(G) \geq \frac{n-1}{2} \), then \( G \) has a Hamilton path;
(iii) ([14]) Let \( G \) be a simple graph of order \( n \geq 3 \), if \( d_G(x) + d_G(y) \geq n \) for any two non-adjacent vertices \( x \) and \( y \), then \( G \) is Hamiltonian.

Since \( d_G(x) = n - 1 - d_G(x) \) for any \( x \in V(G) \), it is easy to know the following corresponding results for the complement graph \( G^c \).

**Lemma 3** (i) If \( G \) a simple graph of order \( n \geq 3 \) and \( \Delta(G) \leq \frac{n}{2} - 1 \), then \( G^c \) is Hamiltonian;
(ii) If \( G \) is a simple graph with \( \Delta(G) \leq \frac{n-1}{2} \), then \( G^c \) has a Hamilton path;
(iii) Let \( G \) is a simple graph of order \( n \geq 3 \). If \( d_G(x) + d_G(y) \leq n - 2 \) for any \( xy \in E(G) \), then \( G^c \) is Hamiltonian.

Now we will discuss other sufficient conditions for the Hamiltonicity of the complement graph.

**Theorem 4** If \( G \) is a simple graph of order \( n \geq 3 \) and size \( m < n - 2 \), then \( G^c \) is Hamiltonian; If \( m \leq n - 2 \), then \( G^c \) has a Hamilton path.

**Proof** If \( m < n - 2 \), then \( d_G(x) + d_G(y) \leq m - 1 + 2 \leq n - 2 \). Hence, \( G^c \) is Hamiltonian by Theorem 3. If \( m = n - 2 \), then \( G - e \) has \( n - 3 \) edges for any edge \( e \in E(G) \). Thus the complement graph \( (G - e)^c \) has a Hamilton cycle \( C \), which implies that \( G^c \) has a Hamilton path whether \( e \in C \) or not.

Let \( C(G) \) denote the number of components of a graph \( G \).

**Theorem 5** If \( G \) is a simple graph of order \( n \geq 3 \) and \( C(G) > \left\lfloor \frac{n+1}{2} \right\rfloor \), then \( G^c \) is Hamiltonian; if \( C(G) \geq \left\lceil \frac{n+1}{2} \right\rceil \), then \( G^c \) has a Hamilton path.

**Proof** Let \( H \) be a component with maximum order \( n_1 \). When \( C(G) > \left\lfloor \frac{n+1}{2} \right\rfloor \) and \( n \geq 3 \), we have \( \Delta(G) \leq n_1 - 1 \leq n - C(G) < n - \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \). Thus \( G^c \) is Hamiltonian by Lemma 3. When \( C(G) \geq \left\lceil \frac{n+1}{2} \right\rceil \), we have \( \Delta(G) \leq n - \left\lceil \frac{n+1}{2} \right\rceil = \left\lfloor \frac{n-1}{2} \right\rfloor \). Thus \( G^c \) has a Hamilton path by Lemma 3.

**Theorem 6** If \( G \) is a simple graph of order \( n \geq 3 \) and its diameter \( d > \left\lfloor \frac{n}{2} \right\rfloor + 1 \), then \( G^c \) is Hamiltonian. If \( d \geq \left\lceil \frac{n}{2} \right\rceil + 1 \), then \( G^c \) has a Hamilton path.
Proof Let $d_G(u,v) = d$, where $u,v \in V(G)$. We apply a Breadth-First Search (commonly abbreviated BFS) to $G$ rooted at $u$. Thus $V(G)$ can be partitioned into $V_0, V_1, \ldots, V_d$ with $d_G(u,x) = i$ for any vertex $x \in V_i$ for $i = 0, 1, \ldots, d$. Clearly $V_0 = \{u\}$ and $v \in V_d$, moreover $N_G(V_i) \subseteq V_{i-1} \cup V_i \cup V_{i+1}$ for $i = 1, 2, \ldots, d-1$, where $N_G(V_i)$ denote the unions of neighborhood of $x$ for $x \in V_i$. Hence, $\Delta(G) \leq n - d + 1 < n + 1 - \lceil \frac{n}{2} \rceil - 1 = \lceil \frac{n}{2} \rceil$ since $d > \lceil \frac{n}{2} \rceil + 1$. By lemma 3, $G^c$ is Hamiltonian if $n$ is even.

When $n$ is odd, we only need to consider the case of $\Delta(G) = \frac{n-1}{2}$ by Lemma 3. Notice that $\Delta(G) = \frac{n-1}{2}$ if and only if $d = \lceil \frac{n}{2} \rceil + 1 + 1 = \frac{n+3}{2}$ and there exists a $u-v$ path $P$ of length $d$ with an internal vertex $x$ in $P$, which degree is $\frac{n+3}{2}$. This implies that $x$ is adjacent to any vertex in $V(G) \setminus V(P)$. Let $V(P) = \{a_i| i = 0, 1, \ldots, \frac{n+3}{2}\}$, where $a_i \in V_i$, $V(G) \setminus V(P) = \{b_1, b_2, \ldots, b_{\frac{n}{2}}\}$. Suppose that $x = a_k (1 \leq k \leq d-1)$, then $\{a_{k-1}, a_k, a_{k+1}, b_1, b_2, \ldots, b_{\frac{n}{2}}\} = V_{k-1} \cup V_k \cup V_{k+1}$. (see Figure 1).

Case 1 There exists some $b_i$ with $d_G(b_i) < \frac{n-1}{2}$.

Let $H = G - b_i$ be the subgraph of $G$ obtained by deleting the vertex $b_i$. Obviously, $H$ is connected and the diameter of $H$ is no less than $d = \frac{n+3}{2}$. Note that $H$ has $n-1$ vertices and $n$ is even. We know $H^c$ is Hamiltonian by above proof. Let $C$ be a Hamilton cycle of $H^c$. Note that $|V(C)| = n - 1$ and $n-1$ is even. $d_G(b_i) < \frac{n-1}{2}$ implies that there exist two consecutive vertices $x, y \in V(C)$ such that $xb_i \notin E(G)$ and $yb_i \notin E(G)$. Thus $C - xy + xb_i + yb_i$ is a Hamilton cycle of $G^c$.

Case 2 $d_G(b_i) = \frac{n-1}{2}$ for any $i \in \{1, \ldots, \frac{n}{2}\}$.

In this case, $a_1, b_1, \ldots, b_{\frac{n}{2}}$ are pairwise adjacent in $G$ and $b_i$ has exactly three neighbors in path $P$ for any $i \in \{1, 2, \ldots, \frac{n}{2}\}$. Thus any $b_i (1 \leq i \leq \frac{n}{2})$ is not in $V_d$ since $b_i \notin V_d$ implies that $b_i$ has at most two neighbors in $P$. Obviously, either $\{a_{k-1}, a_k, b_1, b_2, \ldots, b_{\frac{n}{2}}\} = V_{k-1} \cup V_k$ or $\{a_k, a_{k+1}, b_1, b_2, \ldots, b_{\frac{n}{2}}\} = V_{k+1} \cup V_k$. Without loss of generality, we assume $\{a_{k-1}, a_k, b_1, b_2, \ldots, b_{\frac{n}{2}}\} = V_{k-1} \cup V_k$. When $k = d-1$, $G^c$ has a Hamilton cycle: $a_{\frac{n}{2}+1} a_{\frac{n}{2}+2} a_{\frac{n}{2}+3} a_{\frac{n}{2}+4} a_0 b_1 \cdots a_{\frac{n}{2}} b_{\frac{n}{2}} a_{\frac{n}{2}+1}$. When $1 \leq k \leq d-2$, $G^c$ has a Hamilton cycle: $a_k a_{k+2} a_{k+3} a_{k+4} a_{k+5} a_0 b_1 \cdots a_{\frac{n}{2}} b_{\frac{n}{2}} a_{\frac{n}{2}+1}$. Where $i_1, \ldots, i_{\frac{n}{2}}$ take distinct values from $\{0, 1, \ldots, \frac{n}{2}\}\{k-1, k, k+1, k+2\}$.

Now we prove that $G^c$ has a Hamilton path if the diameter of $G$ is no less than $\lceil \frac{n}{2} \rceil + 1$. Obviously, we only need to consider the case that $n$ is even and $d = \frac{n}{2} + 1$. Similarly, we have $\Delta(G) \leq n - d + 1 \leq n + 1 - \frac{n}{2}$. By Theorem 3, $G^c$ has a Hamilton path except for the case $\Delta(G) = \frac{n}{2}$. Notice that $\Delta(G) = \frac{n}{2}$ if and only if $d = \frac{n+2}{2}$ and there exists a path $P$ of length $d$ in $G$ with a internal vertex $x$ such that $V(G) \setminus V(P) \subset N(x)$. Now randomly select a vertex $y \in V(G) \setminus V(P)$. Let $H = G - y$ and suppose that the diameter of $H$ is $d'$. Since $H$ is connected, we have $d' \geq d = \frac{n+2}{2}$. Moreover $d = \frac{n+2}{2}$ implies $n \geq 4$, thus $H^c$ has a Hamilton cycle $C$ by Theorem 3. Since $d_G(y) \leq \Delta(G) = \frac{n}{2}$ and $|V((C))| = n - 1$, there exists at least a vertex $z$ in $C$ with $yz \notin E(G)$. Extending the cycle $C$ to $y$ at $z$, we obtain a Hamilton path of $G^c$.

Remark 7 The conditions proposed in the Theorem 2 ~ 6 can be weaken. The illustrations are not difficult and the main purpose in this paper is to consider consecutive $L(2,1)$-labelings, hence the illustrations are omitted in here.

Since graph $G$ admits a consecutive $L(2,1)$-labelling if and only if $G^c$ has a Hamilton path by Theorem 1. We know

Corollary 8 Let $G$ be a graph of order $n$ and size $m$.

(i) If $m \leq n - 2$, then $G$ admits a consecutive $L(2,1)$-labelling;
Figure 1: The graph $G$ with $d_G(a_k) = \frac{n+3}{2}$.

(ii) If $C(G) \geq \lceil \frac{n+1}{2} \rceil$, then $G$ admits a consecutive $L(2,1)$-labelling;

(iii) If $G$ is a connected graph and its diameter is no less than $\lceil \frac{n}{2} \rceil + 1$, then $G$ admits a consecutive $L(2,1)$-labelling.

3 Main Theorems

We have known that $\lambda(G) \leq \overline{\lambda}(G) \leq n - 1$. In this section we will study the graphs of $\overline{\lambda}(G) = n - 1$.

Lemma 9 If $G^c$ is Hamiltonian and $\overline{\lambda}(G) = n - 1$, then $G$ has at most two components.

Proof Let $C = v_0v_1 \cdots v_{n-1}v_0$ be a Hamilton cycle of $G^c$. We claim that either $d_G(v_i, v_{i+1}) = 2$ or $d_G(v_i, v_{i+2}) = 1$ for any $0 \leq i \leq n - 1$, where $i + j = i + j (\text{mod } n)$. Otherwise, for some integer $i (0 \leq i \leq n - 1)$, $d_G(v_i, v_{i+1}) > 2$ and $d_G(v_i, v_{i+2}) > 1$. We define a new labellings $f$ as follows: $f(v_i) = 0$ and $f(v_{i+j}) = j - 1$ for $j = 1, 2, \ldots, n - 1$. Obviously $f$ is a consecutive $L(2,1)$-labelling of $G$ while $\text{span}(f) = n - 2$. Hence $\overline{\lambda}(G) \leq n - 2$, a contradiction. This claim clearly implies $G$ has at most two components.

Theorem 10 Let $G$ be a simple graph of order $n$ and size $m$. If $\overline{\lambda}(G) = n - 1$, then $n - 2 \leq m \leq \frac{(n-1)(n-2)}{2}$. Moreover for any two integers $n, m$ with $n \geq 3$ and $n - 2 \leq m \leq \frac{(n-1)(n-2)}{2}$, there exists a graph $G$ of order $n$ and size $m$ with $\overline{\lambda}(G) = n - 1$.

Proof $\overline{\lambda}(G) = n - 1$ implies $G^c$ has a Hamilton Path by Theorem 1. Hence $G$ has at most $\frac{n(n-1)}{2} - (n-1)$ edges, i.e., $m \leq \frac{(n-1)(n-2)}{2}$. On the other hand, if $m \leq n - 3$, by Theorem 4, $G^c$ has a Hamilton cycle. Since $\overline{\lambda}(G) = n - 1$, by Lemma 9, we have $C(G) \leq 2$. But $C(G) \leq 2$ implies that $G$ has at least $n - 2$ edges. It’s a contradiction and hence $m \geq n - 2$.

We know $K_{1,n-2} \cup K_1$ is a graph of order $n$ and size $n - 2$ with $\overline{\lambda} = n - 1$. Now assigning a consecutive $L(2,1)$-labelling $f$ to $G$ and joining edges to those non-adjacent vertex pairs $(x, y)$ with $|f(x) - f(y)| \geq 2$. Suppose that the new graph gained by adding edges as above method is $H$. Clearly, $H$ admits a consecutive $L(2,1)$-labelling. Observe that $K_{1,n-2} \cup K_1$ is a spanning subgraph of $H$ and
$\overline{X}(K_{1,n-2} \cup K_1) = n - 1$, we know that $\overline{X}(H) = n - 1$. Therefore, we can obtain the desired graph of order $n$ and size $m$ with $\overline{X} = n - 1$, where $n - 2 \leq m \leq \frac{(n-1)(n-2)}{2}$.

Theorem 11 Let $G$ be a graph of order $n$ and size $m$ with $\overline{X}(G) = n - 1$. Then,

(i) $G \cong K_n - P_n$ when $m = \frac{(n-1)(n-2)}{2}$, where $P_n$ is a path of length $n$;

(ii) $G \cong P_2 \cup P_2$ or $K_3 \cup K_1 \cup K_1$ or $K_{1,n-2} \cup K_1$ when $m = n - 2$.

Proof

(i) Let $H$ denote $K_n - P_n$. Since $H$ has $\frac{(n-1)(n-2)}{2}$ edges and $H^c = P_n$, then $G$ must be $H$ if $m = \frac{(n-1)(n-2)}{2}$. Otherwise, $G^c$ hasn’t a Hamilton path and hence $G$ doesn’t admit a consecutive $L(2,1)$-labelling. It’s contradict to $\overline{X}(G) = n - 1$.

(ii) If $m = n - 2$, then $C(G) \geq 2$. Now we first assume $C(G) \geq 3$. By Lemma 9, $G^c$ isn’t Hamiltonian since $\overline{X}(G) = n - 1$. Since $G$ has size $n - 2$, then $d_G(x) + d_G(y) \leq n - 1$ for any two adjacent vertices of $G$. Moreover, $d_G(x) + d_G(y) = n - 1$ for some $x \in V(G)$ and $y \in V(G)$ if and only if for any edge $e \in E(G)$, $e$ is incident to either $x$ or $y$. Since $G^c$ isn’t Hamiltonian, by Theorem 2, there exist two adjacent vertices $x_1, x_2$ of $G$ such that every edge is incident to either of $x$ and $y$. Let $G_1$ be the component containing $x_1$ and $x_2$. Obviously the other components of $G$ are all isolated vertices. Hence $|V(G_1)| \leq n - 2$ since $C(G) \geq 3$. Since $|E(G_1)| = |E(G)| = n - 2$, we claim that $G_1$ isn’t a tree and $|V(G_1)| \geq 3$. Let $V(G_1) = \{x_1, x_2, \cdots, x_m\}$ and $V(G) \setminus V(G_1) = \{y_1, y_2, \cdots, y_{n-m}\}$, where $|V(G_1)| = n_1$.

If $n_1 \leq n - 3$, then $n - n_1 \geq 3$. Thus the cycle $x_1y_1x_2y_2x_3 \cdots x_{n_1}y_{n-n_1}x_1$ is a Hamilton cycle of $G^c$.

It is a contradiction to $G^c$ isn’t Hamiltonian. If $n_1 = n - 2 \geq 4$, either $d_G(x_1)$ or $d_G(x_2)$ is more than three. Hence, there exists at least one vertex of degree one in $G$, say $x_3$. Suppose that $x_3$ is adjacent to $x_2$, then the cycle $x_3x_1y_1x_2y_2x_4 \cdots x_{n-n_3}x_3$ is a Hamilton cycle of $G^c$. It is also a contradiction. So we know that $n_1 = n - 2 = 3$ and $G$ is $K_3 \cup K_1 \cup K_1$.

Now we consider the case $C(G) = 2$. Since $m = n - 2$ and $C(G) = 2$, the two components of $G$ are both trees. We denote them by $G_1$ and $G_2$, respectively. If neither $G_1$ nor $G_2$ is star, then they both admit consecutive $L(2,1)$-labelling since a tree admits a consecutive $L(2,1)$-labelling if and only it is not a star. (see[6]). Thus $\overline{X}(G) = \max\{\overline{X}(G_1), \overline{X}(G_2)\} < n - 1$, a contradiction. If there exists just one star between $G_1$ and $G_2$, say $G_1$, then $G_1$ admits a $\lambda(G_1)$- $L(2,1)$- labelling $f_1$ such that $1$ is the unique color not used. Since $G_2$ is not star, then $G_2$ admits a $\overline{X}(G_2)$-consecutive $L(2,1)$-labelling $f_2$. Since $G_2$ is not star, then $\overline{X}(G_2) \geq 3$. Combining $f_1$ and $f_2$, we obtain a consecutive $L(2,1)$- labelling of $G$ and $\overline{X}(G) \leq \{\lambda(G_1), \overline{X}(G_2)\} < n - 1$. It is also a contradiction. Thus we know $G_1$ and $G_2$ are both stars. Let $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$. If $n_1 \geq n_2 \geq 3$, then $G_1$ admits a $n_1 - L(2,1)$- labelling $f_1'$ such that 1 is the unique color not used by $f_1'$, and $G_2$ admits a $n_2 - L(2,1)$ - labelling $f_2'$ such that $n_2 - 1$ is the unique color not used by $f_2'$. Combining $f_1'$ and $f_2'$, we obtain a consecutive $L(2,1)$ - labelling of $G$ and $\overline{X}(G) \leq n(G_1) < n - 1$, a contradiction. Hence, we know $G_1$ and $G_2$ are both stars and one of them is of order at most 2. Without loss of generality, we assume $n_1 \leq 2$. If $n_1 = 1$, then $G$ must be $K_{1,n-2} \cup K_1$; if $n_1 = 2$, then $G_1$ is $P_2$ and $G_2$ is $P_2$ or $K_1$ since $\overline{X}(G) = n - 1$.

The proof is complete.

Theorem 12 If $2 \leq \overline{X}(G) \leq n - 2$, then there exists a simple graph $G'$ obtained by adjoining some edges to $G$ with $\overline{X}(G') = n - 1$.

Proof Let $f$ be a $\overline{X}(G)$- consecutive $L(2,1)$- labelling of $G$. Since $\overline{X}(G) < n - 1$, there exists a multiple color $i$ of $f$ such that $i \neq 0$ (If $0$ is the unique multiple color, we can define $f' = \overline{X}(G) - f$ and considerate
Suppose that \( u \) is a vertex labelled \( i \) and \( v \) labelled \( i - 1 \), we can join a edge \( e \) between \( u \) and \( v \), then define \( g \) as follows

\[
g(x) = \begin{cases} 
  f(x) + 1 & \text{if } f(x) > i \\
  i + 1 & \text{if } x = u \\
  f(x) & \text{if } f(x) \leq i \text{ and } x \neq u
\end{cases}
\]

Clearly \( g \) is a \( \overline{\lambda}(G) + 1 \) - consecutive \( L(2,1) \) - labelling of \( G + e \). Thus \( \lambda(G) \leq \overline{\lambda}(G) + 1 \). Since the edge number can’t be added infinitely and \( \lambda \) is raised at most one as one edge is adjoined, by induction, \( \overline{\lambda} \) can be raised at \( n - 1 \) after some steps.

Theorem 12 and its proof also implies that:

**Theorem 13** For any two integers \( m, n \) with \( 2 \leq m \leq n - 1 \), there exists a graph \( G \) with order \( n \) and \( \overline{\lambda}(G) = m \).

**Lemma 14** If \( G \) is neither \( K_n \) nor \( K_n - e \), then \( \lambda(G) \leq 2n - 4 \). Moreover, \( \lambda(G) = 2n - 3 \) if and only if \( G \cong K_n - e \).

**Proof** When \( n = 3 \), we have \( |E(G)| \leq 1 \) and hence \( \lambda(G) \leq 2 = 2n - 4 \). So we assume that \( n \geq 4 \). We have \( \lambda(G) \leq 2n - 3 \) since \( \lambda(K_n - e) = 2n - 3 \) and \( G \) is a subgraph of \( K_n - e \). Suppose that \( \lambda(G) = 2n - 3 \), we define a graph \( G' \) as the union of \( G \) and \( n - 2 \) isolated vertices. Since every \( 2n - 3 - L(2,1) \)-labelling of \( G \) has at most \( n - 2 \) colors not used, we claim \( G' \) admits a consecutive \( L(2,1) \)-labelling and \( \overline{\lambda}(G') \leq 2n - 3 \). On the other hand, if \( \overline{\lambda}(G') = k < 2n - 3 \), then \( G \) admits a \( k - L(2,1) \)-labelling. It is a contradiction to the hypothesis \( \lambda(G) = 2n - 3 \). Thus \( \overline{\lambda}(G') = 2n - 3 = |V(G')| - 1 \). Let \( V(G) = \{x_1, x_2, \ldots, x_n\} \) and \( V(G') = \{y_1, \ldots, y_{n-2}\} \). Without loss of generality, we first assume that \( x_1 x_{n-1} \notin E(G) \) and \( x_1 x_n \notin E(G) \). Then \( G^{\text{inc}} \) has a Hamilton cycle \( x_1 x_n y_1 x_2 y_2 x_3 \cdots y_{n-2} x_{n-1} x_1 \). If \( x_1 x_n \notin E(G) \) and \( x_2 x_{n-1} \notin E(G) \), then \( G^{\text{inc}} \) also has a Hamilton cycle \( x_1 x_n y_1 x_2 x_{n-1} y_2 x_3 \cdots y_{n-3} x_{n-2} x_{n-2} x_1 \). Thus, for any case \( G^{\text{inc}} \) has certainly a Hamilton cycle. By Lemma 3, we have \( C(G') \leq 2 \). But \( n \geq 4 \) and hence \( C(G') = n - 1 \geq 3 \). It’s a contradiction. So \( \lambda(G) \leq 2n - 4 \). Now we give another main theorem.

**Theorem 15** If \( \overline{\lambda}(G) = n - 1 \), then \( C(G) \leq \lceil \frac{n+1}{2} \rceil \). Moreover for every two integers \( n \) and \( m \) with \( n \geq 4 \) and \( 1 \leq m \leq \lceil \frac{n+1}{2} \rceil \), there exists a simple graph \( G \) of order \( n \) and \( C(G) = m \) with \( \overline{\lambda}(G) = n - 1 \).

**Proof** Notice that \( \overline{\lambda}(G) = n - 1 \) implies \( n \geq 3 \). If \( C(G) > \lceil \frac{n+1}{2} \rceil \), by Theorem 5, \( G^{\text{inc}} \) has a Hamilton cycle. However, we have \( C(G) \leq 2 \) by Lemma 9. It’s a contradiction.

For even \( n \geq 4 \), it is easy to check that the union of \( K_{\frac{n+1}{2} - 1} - e \) and \( \frac{n}{2} - 1 \) isolated vertices is a graph of order \( n \) and \( C(G) = \lceil \frac{n+1}{2} \rceil \) with \( \overline{\lambda}(G) = n - 1 \). For odd \( n \geq 5 \), the union of \( K_{\frac{n+1}{2}} \) and \( \frac{n-1}{2} \) isolated vertices is a graph of order \( n \) and \( C(G) = \lceil \frac{n+1}{2} \rceil \) with \( \overline{\lambda}(G) = n - 1 \). Assigning them consecutive \( L(2,1) \)-labelling \( f_1, f_2 \) respectively and joining edges to those non-adjacent vertex pairs \((x,y)\) with \( |f_i(x) - f_i(y)| > 1 \) \((i = 1 \text{ or } 2)\), we can obtain the desired graphs.

**Theorem 16** Let \( G \) be a graph of order \( n \) and \( C(G) = \lceil \frac{n+1}{2} \rceil \). If \( \overline{\lambda}(G) = n - 1 \), then

(i) For \( n = 4 \), \( G \cong K_2 \cup K_2 \) or \( P_3 \cup K_1 \);
(ii) For even $n \geq 6$, $G$ is the union of $K_{\frac{n+1}{2}+1} - e$ and $\frac{n}{2} - 1$ isolated vertices.

(iii) For odd $n \geq 3$, $G$ is the union of $K_{\frac{n+1}{2}}$ and $\frac{n-1}{2}$ isolated vertices.

**Proof**

(i) Obviously.

(ii) For even $n \geq 6$, $C(G) = \lfloor \frac{n+1}{2} \rfloor = \frac{n}{2} > 2$. By Lemma 9, $G^c$ isn’t Hamiltonian since $\overline{X}(G) = n - 1$. We claim $G$ has only one nontrivial component. Otherwise, $d_G(x) + d_G(y) \leq 2(n_1 - 1) < n - 2$ for any two adjacent vertices $x$ and $y$ of $G$, where $n_1$ denote the order of the maximum component of $G$. By Theorem 5, $G^c$ is Hamiltonian, a contradiction. Let $G_1$ be the unique nontrivial component and $|V(G_1)| = n_1$. We have $n_1 = \frac{n}{2} + 1$ since $C(G) = \frac{n}{2}$ and $G_1$ is the only nontrivial component.

If $G_1 \cong K_{\frac{n}{2} + 1}$, then $G$ doesn’t admit a consecutive $L(2, 1)$-labelling since $G$ has only $\frac{n}{2} - 1$ isolated vertices. If $G_1$ is a graph obtained by omitting more than one edge of $K_{\frac{n}{2} + 1}$, by Lemma 14, we have $\lambda(G_1) \leq 2\left(\frac{n}{2} + 1\right) - 4 = n - 2$. Note that every $\lambda(G_1) - L(2, 1)$-labelling $f$ of $G_1$ has at most $\frac{\lambda(G_1)}{2}$ colors not used. However $G$ has $\frac{n}{2} - 1$ isolated vertices, so $G$ admits a consecutive $L(2, 1)$-labelling and $\overline{X}(G) = \lambda(G_1) \leq n - 2$. It is a contradiction to $\overline{X}(G) = n - 1$. Thus, $G$ can only be the union of $K_{\frac{n}{2} + 1} - e$ and $\frac{n}{2} - 1$ isolated vertices.

(iii) For odd $n \geq 3$, $C(G) = \lfloor \frac{n+1}{2} \rfloor = \frac{n+1}{2} \geq 2$. If $C(G) = 2$, then $n = 3$. It is clear that $G$ is the union of $K_2$ and one isolated vertex. If $C(G) \geq 3$, Similarly discussed as above, $G$ has only one nontrivial component $G_1$ and $\frac{n}{2} - 1$ isolated vertices, where $|V(G_1)| = n_1 = \frac{n+1}{2}$. If $G_1$ isn’t $K_{\frac{n+1}{2}}$, then $\lambda(G_1) < 2\left(\frac{n+1}{2} - 1\right) = n - 1$. However $G$ has $\frac{n-1}{2}$ isolated vertices, so $G$ admits a consecutive and $\overline{X}(G) = \lambda(G_1) < n - 1$. It’s a contradiction. Hence $G$ is the union of $K_{\frac{n+1}{2}}$ and $\frac{n-1}{2}$ isolated vertices.

Let $G$ be a connected graph of order $n$ and its diameter $d$ with $\overline{X}(G) = n - 1$. Obviously $n \geq 4$. If $n = 4$, it is easy to know $G \cong P_4$ and its diameter is 3; If $n = 5$, we have $2 \leq d \leq 4$. Moreover $d = 2$ if and only if $G \cong C_5$; $d = 3$ if and only if $G$ is a tree of maximum degree 3; $d = 4$ if and only if $G \cong P_5$. In general, we have the following theorem.

**Theorem 17** Let $G$ be a connected graph of order $n \geq 6$ and its diameter $d$. If $\overline{X}(G) = n - 1$, then $2 \leq d \leq \lfloor \frac{n}{2} \rfloor + 1$. Moreover for every two integers $n$ and $d$ with $n \geq 6$ and $2 \leq d \leq \lfloor \frac{n}{2} \rfloor + 1$, there exists a simple graph $G$ of order $n$ and diameter $d$ with $\overline{X}(G) = n - 1$.

**Proof** If $n \geq 6$ and $d \geq \lfloor \frac{n}{2} \rfloor + 2$, we first claim that there exists a vertex $x$ of $G$ such that it is forbidden at most $n - 2$ colors under any $L(2, 1)$-labelling of $G$.

Let $d_G(u, v) = d$ for $u, v \in V(G)$. We apply Breadth-First Search (commonly abbreviated BFS) to graph $G$ rooted at $u$ and partition $V(G)$ into $V_0, V_1, \cdots, V_d$ such that $d_G(u, w) = i$ for any $w \in V_i (i = 0, 1, \cdots, d)$. Clearly $V_0 = \{u\}$ and $V \in V_d$. Let $P$ be a path of length $d$ between $u$ and $v$, we have $|V(P)| = d + 1 \geq \lfloor \frac{n}{2} \rfloor + 3$ and $|V(G) \setminus V(P)| = n - (d + 1) \leq \lfloor \frac{n-5}{2} \rfloor$.

If $|V_1| \leq \lfloor \frac{n-3}{2} \rfloor$, then $|V_2| \leq \lfloor \frac{n-5}{2} \rfloor - (|V_1| - 1) + 1 = \lfloor \frac{n-3}{2} \rfloor - |V_1|$. Thus $u$ is forbidden at most $3|V_1| + |V_2| \leq 2|V_1| + \lfloor \frac{n-1}{2} \rfloor \leq n - 2$ colors. On the other hand, if $|V_1| \geq \lfloor \frac{n-3}{4} \rfloor + 1$, note that $n \geq 6$ and $d \geq \lfloor \frac{n}{2} \rfloor + 2 \geq 5$, thus the distance of $v$ and $v_1$ is more than 3 for any $v_1 \in V_1$. Hence, $v$ is forbidden at most $3(|V(G) \setminus V(P) \cup V_1| + 1) \leq 3\left(\lfloor \frac{n-5}{2} \rfloor - \lfloor \frac{n-3}{4} \rfloor \right) + 4$ colors. Denote $3\left(\lfloor \frac{n-5}{2} \rfloor - \lfloor \frac{n-3}{4} \rfloor \right) + 4$ by $k$. For even $n \geq 6$, we have $k = 3\left(\frac{n-5}{2} - \frac{n-6}{4}\right) + 4 = \frac{3}{4}n - \frac{1}{2} \leq n - 2$; For odd $n \geq 9$, we have $k = 3\left(\frac{n-5}{2} - \frac{n-3}{4}\right) + 4 = \frac{3}{4}n + \frac{1}{4} \leq n - 2$. For $n = 7$, we have $k = 4 < 5$. Hence, either $u$ or $v$ is the vertex $x$ in our claim.
By Theorem 10, \( G^c \) has a Hamilton cycle since we assume that \( d > \lfloor \frac{n}{2} \rfloor + 1 \). Certainly \( G^c \) has a Hamilton path \( P \) with endpoint \( x \). Now label the vertices along path \( P - x \) as \( 0, 1, \cdots, n - 2 \) in turn. At last we can select a color in \( \{0, 1, \cdots, n - 2\} \) to label \( x \) since \( x \) is forbidden at most \( n - 2 \) colors by our claim. Thus \( \overline{X}(G) \leq n - 2 \). It’s a contradiction. Hence, \( d \leq \lfloor \frac{n}{2} \rfloor + 1 \).

Now we build the desired graphs of order \( n \) and diameter \( d \) with \( n \geq 6 \) and \( 2 \leq d \leq \lfloor \frac{n}{2} \rfloor + 1 \).

For even \( n \geq 6 \), let \( V(K_{\frac{n}{2}+1}) = \{x_1, x_2, \cdots, x_{\frac{n}{2}+1}\} \) and \( P_{\frac{n}{2}-1} \) be the path \( y_1y_2 \cdots y_{\frac{n}{2}-1} \). We first build \( G_1 \) by deleting edge \( x_1x_2 \) and adding an edge \( x_1y_1 \). Then \( G_1 \) is connected and its diameter is \( \frac{n}{2} + 1 \).

Now we prove that \( \overline{X}(G_1) = n - 1 \). Obviously \( x_1x_2y_1x_3y_2x_4y_3 \cdots x_{\frac{n}{2}}y_{\frac{n}{2}-1}x_{\frac{n}{2}+1} \) is a Hamilton path of \( G_1^c \). Then \( \overline{X}(G_1) \leq n - 1 \). If \( \overline{X}(G_1) = k \leq n - 2 \), then \( \lambda(K_{\frac{n}{2}+1}) - e \leq k \leq n - 2 \) since \( K_{\frac{n}{2}+1} - e \) is a induced subgraph of \( G_1 \), where \( e \) is the edge \( x_1x_2 \). It is a contradiction to Lemma 14. Hence \( \overline{X}(G_1) = n - 1 \).

Define a function \( f \) as follows: \( f(x_i) = 0 \); \( f(x_i) = 2i - 3 \) for \( 1 \leq i \leq \frac{n}{2} + 1 \); \( f(y_1) = 2i \) for \( 1 \leq i \leq \frac{n}{2} + 1 \). It is easy to check that \( f \) is a \( n - 1 \)-consecutive \( L(2, 1) \)-labelling of \( G_1 \). Adding an edge \( x_1y_j \) if \( j \neq i + 1 \) and \( j \neq i + 2 \) or an edge \( y_iy_j \) if \( |j - i| \geq 2 \), the new graph gained from \( G_1 \) also has \( \overline{X} = n - 1 \) and its diameter may be shorter. Thus we can easily build the desired graph \( G \) of even order \( n \geq 6 \) with \( \overline{X}(G) = n - 1 \) by adjoining some suitable edges \( x_1y_j \) or \( y_iy_j \) to \( G_1 \) such that its diameter \( d \) with \( 2 \leq d \leq \frac{n}{2} + 1 \).

For odd \( n \geq 7 \), let \( V(K_{\frac{n+1}{2}}) = \{x_1, x_2, \cdots, x_{\frac{n+1}{2}}\} \) and \( P_{\frac{n+1}{2}-1} \) be the path \( y_1y_2 \cdots y_{\frac{n+1}{2}} \). We can build \( G_2 \) by adding one edge \( x_1y_1 \) to connect \( K_{\frac{n+1}{2}} \) and \( P_{\frac{n+1}{2}-1} \). Then \( G_2 \) is connected and its diameter is \( \frac{n+1}{2} \).

Similarly, we can prove that \( \overline{X}(G_2) = n - 1 \) and adjoining some suitable edges to \( G_2 \), we can build the desired graphs for odd order \( n \geq 7 \). The detail is left to readers.

**Remark 18** Different from Theorem 12 and 15, it may be difficult to determine which graph has diameter \( \lfloor \frac{n}{2} \rfloor + 1 \) and \( \overline{X} = n - 1 \). For example, these graphs in Fig 2 show that there are at least three graphs with diameter \( d = \lfloor \frac{n}{2} \rfloor + 1 \) and \( \overline{X} = n - 1 \) even if \( n = 7 \).

![Figure 2: The graphs of order 7 and diameter 4 with \( \overline{X} = 6 \).](image)

**Remark 19** Contrasting Theorem 4 and 10, Theorem 5 and 12, Theorem 6 and 17 respectively, we find that \( \overline{X}(G) < n - 1 \) once the conditions in Theorem 4, 5 and 6 can ensure the existence of Hamilton cycle of \( G^c \). Is it true that \( \overline{X}(G) = n - 1 \) implies the nonexistence of Hamilton cycle of \( G^c \) when the order of \( G \) is large enough? The answer is negative. For example, look at the cartesian product \( K_a \Box K_b \) \( (a > b \geq 2) \). By Theorem 3, its complement graph has a Hamilton cycle. But we can check that \( \overline{X}(K_a \Box K_b) = ab - 1 = n(K_a \Box K_b) - 1 \) except for \( K_2 \Box K_2 \).

**References**