Infinite propagation speed for a shallow water equation

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Abstract: In this paper, we prove that the propagation speed for a shallow water equation speed is infinite. Moreover, we show that either the solution itself or its first derivative can not, or neither of them can decay faster than $e^{-|x|}$ at infinite uniformly in any small time interval $[0, \epsilon]$, although the initial datum $u_0(x)$ has compact support.

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In this paper we consider the following shallow water equation

\[
\begin{cases}
  u_t - u_{xxx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, & t > 0, x \in \mathbb{R} \\
  u(x, 0) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\]

(1)

This equation, also known as the Camassa-Holm equation, models wave motion in shallow water with $u$ denoting the height of the water above a flat bottom, cf. [1].

The above equation was found earlier by Fokas and Fuchssteiner (see [8]) as a bi-Hamiltonian generalization of KdV. Recently, the alternative derivations of the Camassa-Holm equation as a model for water waves and a motion of plane curves were presented by Johnson [9] and Chou and Qu [2] respectively. Many satisfactory results have been obtained for this shallow water equation recently. Local well-posedness for the initial datum $u_0(x) \in H^s$ with $s > 3/2$ was proved.
by several authors, see [10, 12, 14]. Using the solitons, it can be easily seen that
the Camassa-Holm equation does not make sense in $H^s$ for $s < \frac{3}{2}$, but it is quite
interesting to ask what is the minimum Sobolev index $s$ for the Camassa-Holm
equation to be well-posed (see the comments in Molinet’s paper [13]). Moreover,
wave breaking for a large class of initial data has been established in [3, 4, 10,
11, 17]. On the other hand, in [15, 16], global existence of weak solutions is
proved but uniqueness is obtained only under an assumption that is known to
hold only for initial data $u_0(x) \in H^1$ such that $u_0 - u_{0xx}$ is a sign-definite Radon
measure (under this condition, global existence and uniqueness was shown in [5]
also). Very recently, in [7], the existence and uniqueness are established with low
regularity initial datum for the periodic case. The solitary waves of the Camassa-
Holm equation are peaked solitons [1] with $u(x, t) = e^{-|x-ct|}$, where $c \in \mathbb{R}$ is the
wave speed. The orbital stability of the peakons was shown by Constantin and
Strauss [6] (see also Zhou’s paper [18], where the same result is proved using an
alternative method).

In this paper, we want to establish a fine property for the Camassa-Holm
equation: infinite propagation speed. The theorem reads:

**Theorem 1** If the initial condition $u_0(x)$ of the Camassa-Holm equation has
compact support, then the solution $u(x, t)$ does not have compact $x-$support im-
mediately for $t > 0$.

**Remark 1** From the equation, it is not clear whether (1) has the infinite propa-
gation speed or not.

On the other hand, (1) can be rewritten as

$$u_t + uu_x + \partial_x G \ast \left( u^2 + \frac{1}{2}u_x^2 \right) = 0, \quad (2)$$

where $G(x) = \frac{1}{2}e^{x}$ is the Green’s function of $(1-\partial_x^2)$, while $\ast$ denotes convolution.
In this form (2) is Burger’s equation with a “pressure”. It is well-known that
although the propagation speed depends on the solution itself, the speed is finite
in general for 1-D conservation laws. Of course, the Camassa-Holm equation
is not Burger’s equation. However, this is a quite interesting property for the
Camassa-Holm equation.
Proof: Suppose not, that is, there exists a $t_0$ such that $u(x, t)$ has compact support with respect to the space variable for $0 \leq t \leq t_0$.

Now we let

$$E(t) = \int_{\mathbb{R}} e^x y(x, t) dx,$$

where $y(x, t) = (1 - \partial_x^2) u(x, t)$. Hence $y(x, t)$ has compact support for $0 \leq t \leq t_0$ and consequently $E(t)$ is well-defined.

Integration by parts, for $0 \leq t \leq t_0$, yields

$$E(t) = \int_{\mathbb{R}} e^x y(x, t) dx = \int_{\mathbb{R}} e^x u(x, t) dx - \int_{\mathbb{R}} e^x u_{xx}(x, t) dx$$

$$= \int_{\mathbb{R}} e^x u(x, t) dx + \int_{\mathbb{R}} e^x u_x(x, t) dx = 0,$$

since $u(x, t)$ has compact support.

On the other hand,

$$\frac{dE(t)}{dt} = \int_{\mathbb{R}} e^x y_t(x, t) dx.$$

Now we want to derive an equation for $y_t$. Differentiating twice equation (2), we get

$$0 = u_{xxt} + (uu_x)_{xx} + \partial_x \partial_x^2 G \ast \left( u^2 + \frac{1}{2} u_x^2 \right)$$

$$= u_{xxt} + (uu_x)_{xx} - \partial_x \left( 1 - \partial_x^2 \right) G \ast \left( u^2 + \frac{1}{2} u_x^2 \right) + \partial_x G \ast \left( u^2 + \frac{1}{2} u_x^2 \right)$$

$$= u_{xxt} + (uu_x)_{xx} - \partial_x \left( u^2 + \frac{1}{2} u_x^2 \right) + \partial_x G \ast \left( u^2 + \frac{1}{2} u_x^2 \right). \quad (3)$$

Combining (2) and (3), we obtain

$$y_t = (1 - \partial_x^2) u_t = -uu_x + (uu_x)_{xx} - \partial_x \left( u^2 + \frac{1}{2} u_x^2 \right).$$

Therefore, by integration by parts again,

$$\frac{dE(t)}{dt} = \int_{\mathbb{R}} e^x y_t(x, t) dx = \int_{\mathbb{R}} e^x (1 - \partial_x^2) u_t(x, t) dx$$

$$= -\int_{\mathbb{R}} e^x uu_x(x, t) dx + \int_{\mathbb{R}} e^x (uu_x)_{xx}(x, t) dx - \int_{\mathbb{R}} e^x \partial_x \left( u^2 + \frac{1}{2} u_x^2 \right)(x, t) dx$$

$$= \int_{\mathbb{R}} e^x \left( u^2 + \frac{1}{2} u_x^2 \right)(x, t) dx.$$
Now we have two equalities
\[ E(t) = 0 \quad \text{and} \quad \frac{dE(t)}{dt} = \int_{\mathbb{R}} e^x \left( u^2 + \frac{1}{2} u_x^2 \right) (x, t) dx \geq 0, \quad (4) \]
for \( 0 \leq t \leq t_0 \). Therefore \( u \equiv 0 \) for all \( 0 \leq t \leq t_0 \), which is a contradiction. \( \square \)

**Remark 2** From the proof, it is clear that actually we show that either the strong solution itself or its first order derivative cannot, or neither of them can decay faster than \( e^{-x} \) at infinite uniformly in any small time interval \([0, \epsilon]\), although the nonzero initial datum \( u_0(x) \) belongs to \( C^\infty_0(\mathbb{R}) \) (or \( u_0 \in C^\infty \) and decays faster than \( e^{-x} \) at infinite).

In fact, if both the strong solution itself and its first derivative decay faster than \( e^{-x} \) in some time interval, that is, \( |u|, |u_x| \sim \omega e^{-x} \), as \( x \to \infty \), then the above computations in the proof still hold. In particular \((4)\) is true, then consequently we obtain a contradiction that the solution \( u(x, t) \) is zero with nonzero initial datum \( u_0 \).

How about the behavior of the solution at \(-\infty\)? Can it (itself and its first order derivative) decays faster than \( e^x \) at negative infinite? The answer is NO.

In fact, let
\[ F(t) = \int_{\mathbb{R}} e^{-x} y(x, t), \]
with \( y(x, t) = (1 - \partial_x^2) u(x, t) \). For \( u_0 \in C^\infty_0(\mathbb{R}) \), if the solution itself and its first derivative decays faster than \( e^x \) at negative infinity in some time interval, say \([0, t_1]\). By the similar computation, we obtain
\[ F(t) = 0 \quad \text{and} \quad \frac{dF(t)}{dt} = -\int_{\mathbb{R}} e^{-x} \left( u^2 + \frac{1}{2} u_x^2 \right) (x, t) dx \leq 0, \]
for \( 0 \leq t \leq t_1 \). Hence we get a contradiction that the solution is zero with nonzero initial datum.

**Remark 3** How about the behavior as time goes on? We would like to make a conjecture here: the solution \( u(x, t) \) can not have compact \( x \)-support any longer in its lifespan even with compact supported initial datum. Although we can not prove it at present, from the above proof, it is clear that the collection of times at which the solution \( u(x, t) \) has compact \( x \)-support only could be a finite set.
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