A Reverse Isoperimetric Inequality For Closed Strictly Convex Plane Curves *

Shengliang Pan       Hong Zhang

Department of Mathematics, East China Normal University, Shanghai, 200062, P. R. China
email: slpan@math.ecnu.edu.cn

May 26, 2006

Abstract In this note we present a reverse isoperimetric inequality for closed convex curves, which states that if $\gamma$ is a closed strictly convex plane curve with length $L$ and enclosing an area $A$, then one gets

$$L^2 \leq 4\pi(A + \tilde{A}),$$

where $|\tilde{A}|$ denotes the absolute area of the domain enclosed by the locus of curvature centers of $\gamma$. The equality holds if and only if $\gamma$ is a circle.

Mathematics Subject Classification : 52A38, 52A40

Key words convex curves, Minkowski’s support function, locus of centers of curvature, integral of radius of curvature, reverse isoperimetric inequality.

1 Introduction

The classical isoperimetric inequality in the Euclidean plane $\mathbb{R}^2$ states that for a simple closed curve $\gamma$ of length $L$, enclosing a region of area $A$, one gets

$$L^2 - 4\pi A \geq 0,$$  \hspace{1cm} (1.1)

and the equality holds if and only if $\gamma$ is a circle. This fact was known to the ancient Greeks, the first mathematical proofs were only given in the 19th century by Steiner [13]. There are many proofs, sharpened forms, generalizations, and applications of this inequality, see for instance [1], [2], [3], [4], [7], [8], [9], [10], [11], [12], [14], [15], etc., and the literature therein.

In [5], there is a reverse isoperimetric inequality for the plane curves under some assumption on curvature. Now, in the present note, we establish a reverse isoperimetric inequality for convex curves, which states that if $\gamma$ is a closed strictly convex curve in the plane $\mathbb{R}^2$ with length $L$ and enclosing an area $A$, then we get

$$L^2 \leq 4\pi(A + \tilde{A}),$$  \hspace{1cm} (1.2)

*This work is supported in part by the National Science Foundation of China (10371039), the Shanghai Science and Technology Committee Program and the Shanghai Priority Academic Discipline
where $|\mathcal{A}|$ denotes the absolute area of the domain enclosed by the locus of curvature centers of $\gamma$, and the equality holds if and only if $\gamma$ is a circle.

In this note, we first recall some facts about Minkowski’s support function of closed convex plane curves and then give some properties of the locus of curvature centers of closed strictly convex plane curves, and finally present the above reverse isoperimetric inequality.

Remarks. Since we get the above reverse isoperimetric inequality by the integral of radius of curvature, our curves must be strictly convex. We wonder if this reverse isoperimetric inequality holds for any simple closed curves in the plane. And furthermore, it would be interesting to generalize this reverse isoperimetric inequality to the higher dimensional cases.

Acknowledgements. The authors are grateful to Professors Yu Zheng and Xuecheng Pang, and other colleagues of our geometric inequalities seminar for many useful and stimulating discussions. And especially, the authors would like to thank Professors Xingbin Pan and Yibing Shen for reading the manuscript of this note and giving us some suggestion and invaluable comments.

2 Minkowski’s Support function for Convex Plane Curves

From now on, without loss of generality, suppose that $\gamma$ is a smooth regular positively oriented and closed strictly convex curve in the plane, and take a point $O$ inside $\gamma$ as the origin of our frame. Let $p$ be the oriented perpendicular distance from $O$ to the tangent line $\Gamma$ at a point on $\gamma$, and $\theta$ be the oriented angle from the positive $x_1$-axis to this perpendicular ray. Clearly, $p$ is a single-valued periodic function of $\theta$ with period $2\pi$. The equation of the tangent line $\Gamma$ can be written in the form

$$x_1 \cos \theta + x_2 \sin \theta = p(\theta). \tag{2.1}$$

$\gamma$ can be thought of as the envelope of all its tangent lines. To parameterize $\gamma$, it suffices to determine the point $(\gamma_1, \gamma_2)$ of tangency of tangent line (2.1) with $\gamma$. According to the method of determining the envelope we differentiate (2.1) with respect to $\theta$ to get

$$-x_1 \sin \theta + x_2 \cos \theta = p'(\theta),$$

giving us the parametrization of $\gamma$ in terms of $\theta$ as follows

$$\gamma(\theta) = (\gamma_1(\theta), \gamma_2(\theta)) = \left(p(\theta) \cos \theta - p'(\theta) \sin \theta, p(\theta) \sin \theta + p'(\theta) \cos \theta\right). \tag{2.2}$$

Thus, if $\theta$ and $p(\theta)$ are given, one can determine a unique point $(\gamma_1(\theta), \gamma_2(\theta))$ on the curve $\gamma$, and vice versa. The couple $(\theta, p(\theta))$ is usually called the polar tangential coordinate on $\gamma$, and $p(\theta)$ Minkowski’s support function of $\gamma$.

Differentiating (2.2) with respect to $\theta$ yields

$$\gamma'(\theta) = (p(\theta) + p''(\theta))(-\sin \theta, \cos \theta). \tag{2.3}$$

Now, the unit tangent vector of $\gamma$ can be expressed as $T = (-\sin \theta, \cos \theta) = (d\gamma_1/ds, d\gamma_2/ds)$, and one also has $k = d\theta/ds > 0$. Thus, one gets

$$-\sin \theta = \frac{d\gamma_1}{ds} = \frac{d\gamma_1}{d\theta} \frac{d\theta}{ds} = -
\left[(p + p'') \sin \theta\right] k, \quad \cos \theta = \frac{d\gamma_2}{ds} = \frac{d\gamma_2}{d\theta} \frac{d\theta}{ds} = \left[(p + p'') \cos \theta\right] k.$$
Therefore, the curvature \( k \) and the radius of curvature \( \rho \) of \( \gamma \) can be calculated by

\[
k(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta) + p''(\theta)} > 0, \quad \rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta) > 0. \tag{2.4}
\]

Now, denote \( L \) and \( A \) the length of \( \gamma \) and the area it bounds, respectively. Then one can get

\[
L = \int_{\gamma} ds = \int_{0}^{2\pi} \rho(\theta)d\theta = \int_{0}^{2\pi} \left[p(\theta) + p''(\theta)\right] d\theta = \int_{0}^{2\pi} p(\theta)d\theta, \tag{2.5}
\]

and

\[
A = \frac{1}{2} \int_{\gamma} p(\theta)ds = \frac{1}{2} \int_{0}^{2\pi} p(\theta)\left[p(\theta) + p''(\theta)\right] d\theta = \frac{1}{2} \int_{0}^{2\pi} \left[p^2(\theta) - p^2(\theta)\right] d\theta. \tag{2.6}
\]

(2.5) and (2.6) are known as Cauchy’s formula and Blaschke’s formula, respectively.

### 3 Some Properties of the Locus of Curvature Centers

Now, we turn to studying the properties of the locus of curvature centers of a closed strictly convex plane curve \( \gamma \) which is given by (2.2). Let \( \beta \) represent the locus of centers of curvature of \( \gamma \). Then \( \beta(\theta) = (\beta_1(\theta), \beta_2(\theta)) \) can be given by

\[
\beta(\theta) = \gamma(\theta) + \rho(\theta)N(\theta) = (-p'(\theta)\sin \theta - p''(\theta)\cos \theta, p'(\theta)\cos \theta - p''(\theta)\sin \theta), \tag{3.1}
\]

where \( N(\theta) = (-\cos \theta, -\sin \theta) \) is the unit inward normal vector field along \( \gamma \). Thus we calculate

\[
\beta'(\theta) = -(p'(\theta) + p''(\theta))(\cos \theta, \sin \theta) \tag{3.2}
\]

and

\[
\beta''(\theta) = -(p'' + p^{(4)})(\cos \theta, \sin \theta) - (p' + p''')(\cos \theta, \sin \theta)
\]
\[
= (-p'' + p^{(4)})\cos \theta + (p' + p''')\sin \theta, -p'' + p^{(4)}\sin \theta - (p' + p''')\cos \theta. \tag{3.3}
\]

From the above calculations we get

**Proposition 3.1.** The locus of curvature centers of a closed strictly convex plane curve has at least four cusps at which the curvature is infinite.

**Proof.** From the well-known four-vertex theorem, a closed convex plane curve has at least four vertices, each of which is defined to be a point where the curvature \( k \) of the curve has a relative extremum; that is to say, at which one gets \( k' = 0 \). Now, the curvature, denoted by \( \tilde{k} \), of \( \beta \) can be calculated by

\[
\tilde{k} = \frac{\beta_1\beta_2'' - \beta_1'\beta_2'''}{\left(\beta_1^2 + \beta_2^2\right)^{3/2}} = \frac{1}{|p' + p'''|} = \frac{k^2}{|k|}, \tag{3.4}
\]

from which one can easily arrive at the claimed conclusion. \( \square \)

**Proposition 3.2** The oriented area of the domain enclosed by \( \beta \) is nonpositive. And moreover, if \( \beta \) is simple, then orientation of \( \beta \) is the reverse direction against that of the original curve \( \gamma \) and the total curvature of \( \beta \) is equal to \(-2\pi\).

**Proof.** To get the claimed results, we calculate the oriented area, denoted by \( \hat{A} \), of \( \beta \) by Green’s formula. Since

\[
\beta_1 d\beta_2 - \beta_2 d\beta_1 = p'(\theta)(p'(\theta) + p''(\theta)) d\theta,
\]

3
the oriented area of the domain enclosed by $\beta$ is given by
\[ \tilde{A} = \frac{1}{2} \int_{\gamma} \beta_1 d\beta_2 - \beta_2 d\beta_1 = \frac{1}{2} \int_0^{2\pi} p'(\theta)(p'(\theta) + p'')d\theta = \frac{1}{2} \int_0^{2\pi} (p'(\theta) - p''')d\theta. \] (3.5)

Using the Wirtinger inequality for $2\pi$-periodic $C^2$ real functions gives us $\tilde{A} \leq 0$. If $\beta$ is simple, then, from Green’s formula and the fact that $\tilde{A} \leq 0$, it follows that orientation of $\beta$ is the reverse direction against that of $\gamma$ and the total curvature of $\beta$ is equal to $-2\pi$. \hfill \Box

The following result is essential to the proof of the main result of this note.

**Proposition 3.3.** Let $\gamma$ be a $C^2$ closed and strictly convex curve in the plane, $\rho$ the radius of curvature of $\gamma$, $A$ the area enclosed by $\gamma$ and $|\tilde{A}|$ the area enclosed by $\beta$. Then we have
\[ \int_0^{2\pi} \rho^2 d\theta = 2(A + |\tilde{A}|). \] (3.6)

**Proof.** From (2.4), we have $p'' = \rho - p$, and thus,
\[ p''^2 = \rho^2 - 2pp' + p^2 = \rho^2 - 2(p + p') + p^2 = \rho^2 - 2pp'' - p^2. \]

Now, according to (3.5), the absolute area $|\tilde{A}|$ can be rewritten as
\[
|\tilde{A}| = \frac{1}{2} \int_0^{2\pi} (\rho^2 - 2pp'' - p^2 - p'^2)d\theta \\
= \frac{1}{2} \int_0^{2\pi} \rho^2 - \int_0^{2\pi} pp''d\theta - \frac{1}{2} \int_0^{2\pi} (p^2 + p'^2)d\theta \\
= \frac{1}{2} \int_0^{2\pi} \rho^2 d\theta - \frac{1}{2} \int_0^{2\pi} (p'^2 + p''^2)d\theta \\
= \frac{1}{2} \int_0^{2\pi} \rho^2 d\theta - \frac{1}{2} \int_0^{2\pi} (p^2 - p'^2)d\theta \\
= \frac{1}{2} \int_0^{2\pi} \rho^2 d\theta - A,
\]
which completes the proof. \hfill \Box

### 4 A Reverse Isoperimetric Inequality

**Lemma 4.1.** Let $\gamma$ be a smooth closed and strictly convex curve in the plane, $\rho$ be the radius of curvature of $\gamma$, and $L$ be the length of $\gamma$. We have
\[ \int_\gamma \rho ds \geq \frac{L^2}{2\pi}. \] (4.1)

Furthermore, the equality in (4.1) holds if and only if $\gamma$ is a circle.

**Proof.** Note that $\int_\gamma \rho ds = \int_0^{2\pi} \rho^2(\theta)d\theta$. Now, from the Cauchy-Schwartz inequality, we get
\[ 2\pi \int_0^{2\pi} \rho^2(\theta)d\theta \geq \left( \int_0^{2\pi} \rho(\theta)d\theta \right)^2 = \left( \int_\gamma ds \right)^2 = L^2. \] (4.2)
And furthermore, the equality in (4.2) holds if and only if $\rho$ is a constant which means that $\gamma$ is a circle, because a simple closed plane curve with constant curvature must be a circle. \hfill \Box

Now, from the above lemma and Proposition 3.3, one can easily get our main result.

**Theorem 4.2.** (A Reverse Isoperimetric Inequality) If $\gamma$ is a closed strictly convex plane curve with length $L$ and enclosing an area $A$, let $|\tilde{A}|$ denote the absolute area of its locus of centers of curvature, then we get

$$L^2 \leq 4\pi(A + |\tilde{A}|),$$

(4.3)

where the equality holds if and only if $\gamma$ is a circle. \hfill \Box

**Corollary 4.3.** Let $\beta$ be the locus of curvature centers of a closed strictly convex plane curve $\gamma$. Then the area $\tilde{A}$ of $\beta$ is zero if and only if $\gamma$ is a circle and thus $\beta$ is a point which is the center of $\gamma$.

**Proof.** It is obvious that if $\gamma$ is a circle, then $\beta$ is a point, the center of $\gamma$, and thus its area $\tilde{A} = 0$. Conversely, if $\tilde{A} = 0$, then, from the classical isoperimetric inequality (1.1) and the reverse isoperimetric inequality (4.3), it follows that the area $A$ and the length $L$ of $\gamma$ satisfies $L^2 = 4\pi A$, which implies that $\gamma$ is a circle and, therefore, $\beta$ is a point. \hfill \Box

**References**


